**MAT 518 Final Exam** **Name ________________________**

**Instructions:**

- Place your name on all of the pages.
- Do all of your work in this booklet. Do not tear off any sheets.
- Show all of your steps in the problems for full credit.
- Be clear and neat in your work. Any illegible work, or scribbling in the margins, will not be graded.
- Put a box around your answers when appropriate.
- If you need more space, you may use the back of a page and write *On back of page #* in the problem space or the attached blank sheet. **No other scratch paper is allowed.**

**Try to answer as many problems as possible**. Provide as much information as possible. Show sufficient work or rationale for full credit. Remember that some problems may require less work than brute force methods.

**If you are stuck**, or running out of time, indicate as completely as possible, the methods and steps you would take to tackle the problem. Also, indicate any relevant information that you would use. Do not spend too much time on one problem. **Pace yourself.**

**Pay attention to the point distribution**. Not all problems have the same weight.

| Page | Pts | Score |
|---|---|---|
| 1 | 17 | |
| 2 | 13 | |
| 3 | 15 | |
| 4 | 12 | |
| 5 | 17 | |
| 6 | 12 | |
| 7 | 14 | |
| Total | 100 | |

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} f(\phi)\, d\phi$$

## Have a good summer!

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1. (4 pts) Let $L$ be a self-adjoint operator, $< u, Lv >=< Lu, v >$. Prove that eigenfunctions corresponding to different eigenvalues, $L\phi_n = \lambda_n \phi_n$, are orthogonal.

2. (3 pts) On January 8, 1730, Euler gave the integral $\int_0^1 (-\ln s)^n \, ds$, which Gauss later wrote as $\Pi(n)$ for $n > -1$. Rewrite $\Pi(n)$ as a Gamma function using a substitution.

3. (4 pts) Show that the vibrating string solution, $u(x,t) = \sum_{n=1}^{\infty} c_n \cos 3nx \sin 150nt$, is the sum of a left and right traveling wave. What is the wave speed? _____________

4. (6 pts) Given the following initial conditions for a vibrating string with two fixed ends, write down the solution $u(x,t)$. Let the wave speed be $c = 3$.
   a. $u(x,0) = \sin(\pi x) - 0.5\cos(5\pi x)$, $u_t(x,0) = 0$, $x \in [0,1]$.

   b. $u(x,0) = 2\cos(3\pi x)$, $u_t(x,0) = 2\pi \sin(3\pi x)$, $x \in [0,2]$.

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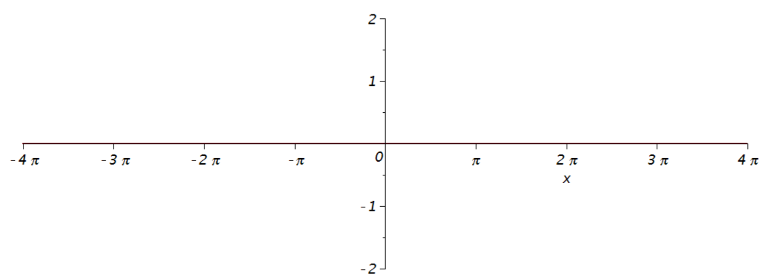
5. (6 pts) Consider the equation $xu_x + yu_y = xe^{-u}$
   a. Find the general solution.

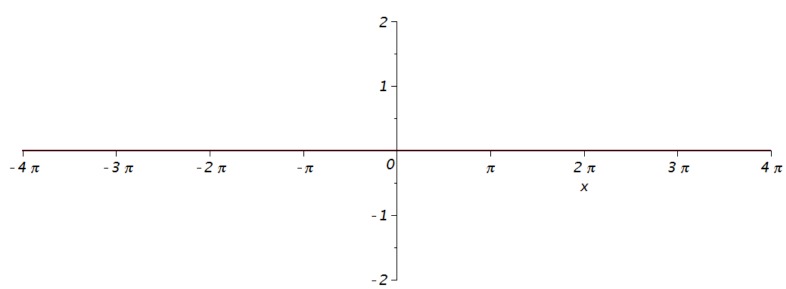
   b. Find the particular solution satisfying $u(x, x^2) = 0$.

6. (7 pts) Let $f(x) = \begin{cases} \dfrac{x}{\pi}, & 0 \le x \le \pi, \\ 2 - \dfrac{x}{\pi}, & \pi \le x \le 2\pi. \end{cases}$

   a. Sketch <u>several</u> periods of the following and <u>state the period</u>:
      i. The periodic extension. Period = _________




      ii. The odd extension. Period = _________




   b. Find the Fourier series representation of this function.

7. (5 pts) The Dirichlet problem on the unit disk is
   $\nabla^2 u = 0,\ 0 < r < 1,\ 0 < \theta < 2\pi,\ u(r,\theta) = f(\theta).$

   a. In the general solution, $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R_n(r)\left[a_n \cos n\theta + b_n \sin n\theta\right]$, what is $R_n(r)$? __________

   b. Let $f(\theta) = \cos^2 2\theta$. What is $u(r,\theta)$? [Hint: There are a finite number of terms]

   c. What is the value of the solution at the origin?

8. (4 pts) Consider the ODE: $y'' - 2xy' + \lambda y = 0,\ -\infty \le x < \infty$.

   a. Put this equation in self-adjoint form.

   b. Write down the appropriate orthogonality condition for any two eigenfunctions of this ODE, $y_n(x)$ and $y_m(x)$.

9. (4 pts) Knowing $P_0(x)$ and $P_1(x)$, find $P_3(x)$ using the three term recursion formula,
   $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$

10. (2 pts) What is $Y_{32}(\theta,\phi)$?

11. (3 pts) Solve the initial value problem: $xu_x + u_t + 2u = 0,\ u(x,0) = \cos(x).$

12. (5 pts) A thin $2 \times 2$ cm$^2$ plate is heated such that the temperatures on the edges are kept at 0°C.
    a. What are the product solutions, $u(x, y, t) = X(x)Y(y)T(t)$?

    b. The plate is initially is at temperature $50^{\circ}$C. If $k = 1/3$ cm$^2$/s, find the temperature at the center of the plate after one minute.

13. (3 pts) Assume that you know the solution of the eigenvalue problem $L\phi_n = -\lambda_n \sigma \phi_n,\ n = 1, 2, \ldots,$ where $L$ is a Sturm-Liouville operator. Derive a formal solution to the equation $Ly(x) = f(x),$ using the method of eigenfunction expansions.

14. (1 pt) For a Sturm-Liouville Problem, the boundary conditions are $y(0) = 0,\ y'(\pi) = 0.$ What is a good test function to use to get a bound on the lowest eigenvalue?

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15. (4 pts) Consider Laplace's equation, $u_{xx} + u_{yy} = 0,\ 0 < x < \pi,\ 0 < y < \pi,$ satisfying the boundary conditions: $u_y(x,0) = 0,\ u(0,y) = 0 = u(\pi, y).$ Carry out the needed separation of variables and write down <u>the product solutions</u> for this problem.

16. (5 pts) Consider a sphere of radius $r = 2$ with boundary condition $u(2,\theta,\phi) = \sin^2 3\theta.$
   a. Exploiting azimuthal symmetry, write the general solution of Laplace's equation on the sphere.

   b. Obtain the solution satisfying the boundary condition.

17. (8 pts) Give typical product solutions for the following using Dirichlet conditions:

| **Equation** | **Coordinates** | **Product solutions** |
|---|---|---|
| Laplace's Equation in 2D | Polar | |
| Heat Equation in 3D | Cylindrical | |
| Wave Equation in 3D | Spherical | |
| Helmholtz Equation in 2D | Cartesian | |

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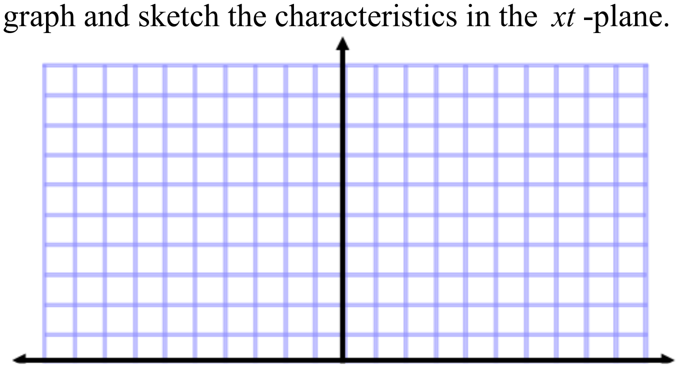
18. (4 pts) A hot dog initially at temperature 5ºC is put into boiling water at 100ºC. Assume the hot dog is 12.0 cm long, has a radius of 2.00 cm, and the heat constant is $2.0\times10^{-5}$ cm$^2$/s. Find the general solution for the temperature and indicate how one might proceed with the remaining information in order to determine when the hot dog is cooked; i.e., when the center temperature is 80ºC.

19. (8 pts) Consider the problem $u_t + uu_x = 0,\ u(x,0) = \begin{cases} 0, & x < -1, \\ 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

a. Use the Method of Characteristics to obtain the general solution

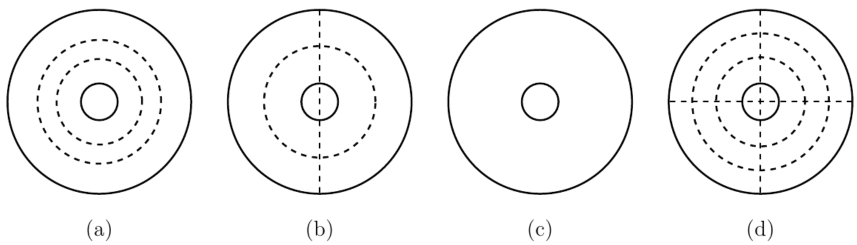
b. Label the graph and sketch the characteristics in the $xt$-plane.



c. Does a shock form? If so, then when does it form? $t =$ ___________

20. (2 pts) Put $u_t + 6u^2 u_x + u_{xxx} = 0$ in conservation law form.

21. (12 pts) Below are the sketches of nodal curves for an annular vibrating membrane, where the radii of the annulus are $a = 1$ and $b = 5$. The product solutions are $u(r,\theta,t) = \left(aJ_m(\sqrt{\lambda}r) + bN_m(\sqrt{\lambda}r)\right)\cos m\theta \cos(\omega t)$. The zeros of the eigenvalue equation, $J_m(\sqrt{\lambda})N_m(5\sqrt{\lambda}) - J_m(5\sqrt{\lambda})N_m(\sqrt{\lambda}) = 0$, are needed to find nodal curves. For each membrane, enter the mode $(m,n)$ and determine the frequencies assuming a unit wave speed.




| Mode | m | n | $\omega$ |
|---|---|---|---|
| **a** | | | |
| **b** | | | |
| **c** | | | |
| **d** | | | |

Zeros of $J_m(z)N_m(5z) - J_m(5z)N_m(z)$

| n | m=0 | m=1 | m=2 | m=3 |
|---|---|---|---|---|
| **1** | 0.763 | 0.847 | 1.044 | 1.279 |
| **2** | 1.557 | 1.611 | 1.761 | 1.975 |
| **3** | 2.346 | 2.385 | 2.499 | 2.676 |

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**Extra**