

$y = \sin x$ explicit

$x^2 + y^2 = C$ implicit

Ex $y'' + 5y' + 6y = e^{-2t}$

① $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3 \Rightarrow y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}$

② $y'' + 5y' + 6y = e^{-2t}$

Normally $y_p = A e^{-2t} \Rightarrow 0 = e^{-2t}$
oops!

Modified Method of Undetermined Coeffs.

$y_p = A t e^{-2t}$ use to get A
not on test

Systems w/ complex roots:

2 sol. $\vec{v}_1 e^{(\alpha+i\beta)t} = \vec{x}_1$
 $\vec{v}_2 e^{(\alpha-i\beta)t} = \vec{x}_2$

Use $\text{Re}(\vec{x}_1), \text{Im}(\vec{x}_1)$

to give $\vec{x} = c_1 \text{Re}(\vec{x}_1) + c_2 \text{Im}(\vec{x}_1)$

$A \vec{v}_1 = (\alpha + i\beta) \vec{v}_1$, A is real

Conjugate $A \vec{v}_1 = (\alpha + i\beta) \vec{v}_1$

$A \vec{v}_1 = (\alpha - i\beta) \vec{v}_1$

$\Rightarrow \vec{v}_2 = \vec{v}_1$

Classifications of Eq. Sol's

nodes



Stable Sink
unstable Source

Real λ 's
 $\lambda_1, \lambda_2 > 0$ unstable
or $\lambda_1, \lambda_2 < 0$ Stable.

Saddles



Real λ 's
 $\lambda_1 < 0, \lambda_2 > 0$

Center



λ 's are pure imaginary

spiral focus



Stable $\alpha < 0$
unstable $\alpha > 0$

λ 's are complex
 $\lambda = \alpha + i\beta$
unstable $\alpha > 0$

$e^{\alpha t} \cos \beta t$

$$\begin{aligned} x' &= 5y \\ y' &= -2x \end{aligned}$$



$$\begin{aligned} \frac{dy}{dx} &= \frac{-2x}{5y} \\ 5y \, dy &= -2x \, dx \\ \frac{5}{2}y^2 &= -x^2 + C \end{aligned}$$

$$\begin{aligned} \frac{5}{2}y^2 + x^2 &= C \\ \frac{x^2}{1^2} + \frac{y^2}{(\sqrt{\frac{2}{5}})^2} &= C \end{aligned}$$

$$A = \begin{pmatrix} 0 & 5 \\ -2 & 0 \end{pmatrix}$$

$$(-\lambda)^2 + 10 = 0$$

$$\lambda^2 + 10 = 0 \Rightarrow \lambda = \pm\sqrt{10}i$$

l.c. $x^2 y'' + 3x y' + y = 3x^6$

a) $x^2 y_h'' + 3x y_h' + y_h = 0, \quad y = x^r$

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0 \Rightarrow r = -1$$

$$y_h = (c_1 + c_2 \ln|x|) x^{-1}$$

b)
$$\begin{cases} y = Ax^6 \\ y' = 6Ax^5 \\ y'' = 30Ax^4 \end{cases} \quad \begin{cases} (30A + 18A + A)x^6 = 3x^6 \\ 49A = 3 \\ A = \frac{3}{49} \end{cases}$$

$$y(x) = (c_1 + c_2 \ln|x|) x^{-1} + \frac{3}{49} x^6$$

1b. $x y' - y = x^3 e^x$

$$y' - \frac{1}{x} y = x^2 e^x$$

$$\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\left(\frac{1}{x} y\right)' = x e^x$$

$$\frac{1}{x} y = \int x e^x dx$$

$$= x e^x - e^x + C$$

$$\boxed{y = x^2 e^x - x e^x + Cx}$$

6. (12 pts) Consider the system
- $$\begin{aligned} x' &= -2x + y \\ y' &= 4x + y \end{aligned}$$

a. Find the eigenvalues of the coefficient matrix.

$$A = \begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix} \quad (-2-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0 \Rightarrow \lambda = -3, 2$$

b. Find the eigenvectors of the coefficient matrix.

$$\lambda = -3 \quad (A - \lambda I)\vec{v} = 0$$

$$\begin{pmatrix} -2 - (-3) & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \begin{aligned} v_1 + v_2 &= 0 \\ v_1 &= -v_2 \end{aligned} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \begin{aligned} -4v_1 + v_2 &= 0 \\ v_2 &= 4v_1 \end{aligned} \quad \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

2. (5 pts) Two large tanks, each holding 10 liters of a saltwater solution are interconnected by pipes. Fresh water flows into tank A at a rate of 5 liters per minute, and fluid is drained out of tank B at the same rate. The tanks have two pipes connecting them which allow for exchange of fluid at the following rates: from A to B at 7 liters per minute and from B to A at 3 liters per minute. The solution in tank A is initially 1 liter of salt per liter of water. Model the system of salt concentrations in the tanks over time.

c. Construct the Fundamental matrix.

$$\Phi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & 4e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{2t}$$

d. Write the general solution.

$$\begin{aligned} \Phi(t)\vec{c} &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -c_1 e^{-3t} + 4c_2 e^{2t} \end{pmatrix} \begin{matrix} x \\ y \end{matrix} \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$$

$$= \begin{pmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & 4e^{2t} \end{pmatrix} \begin{pmatrix} 4e^{2t_0} - e^{2t_0} \\ +e^{-3t_0} & e^{-3t_0} \end{pmatrix} \frac{1}{4e^{-t_0} + e^{-t_0}}$$

$$\underline{\Psi(t_0) = I}$$