$y = \sin x$ explicit

$x^2 + y^2 = c$ implicit

Ex $y'' + 5y' + 6y = e^{-2t}$

① $y_h'' + 5y_h' + 6y_h = 0$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3 \Rightarrow y_h(t) = c_1 \underline{e^{-2t}} + c_2 e^{-3t}$

② $y_p'' + 5y_p' + 6y_p = \underline{e^{-2t}}$

Normally $y_p = \underline{Ae^{-2t}} \Rightarrow 0 = e^{-2t}$ oops!

Modified Method of Undetermined Coeffs.

$y_p = Ate^{-2t}$ use to get A

<u>not on test</u>

Systems w/ complex Roots:

2 sol. $\vec{v}_1 e^{(\alpha + i\beta)t} = \vec{x}_1$

$\vec{v}_2 e^{(\alpha - i\beta)t} = \vec{x}_2$

Use $\underline{Re(\vec{x}_1)}, \underline{Im(\vec{x}_1)}$

to give $\vec{x} = c_1 \underline{Re(\vec{x}_1)} + c_2 \underline{Im(\vec{x}_1)}$

$A\vec{v}_1 = (\alpha + i\beta)\vec{v}_1$, A is real

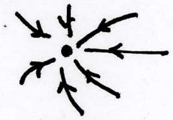
Conjugate $\overline{A\vec{v}_1} = \overline{(\alpha + i\beta)\vec{v}_1}$

$A\bar{\vec{v}}_1 = (\alpha - i\beta)\bar{\vec{v}}_1$

$\Rightarrow \vec{v}_2 = \bar{\vec{v}}_1$

<u>Classifications of Eq. Sol's</u>

nodes



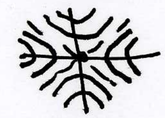
Real λ's

$\lambda_1, \lambda_2 > 0$ unstable

or $\lambda_1, \lambda_2 < 0$ Stable.

Stable sink vs unstable source

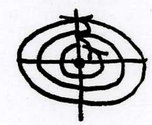
Saddles



Real λs

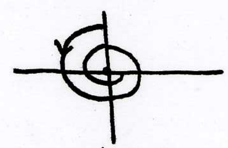
$\lambda_1 < 0, \lambda_2 > 0$

Center



λ's are pure imaginary

Spiral focus



λ's are complex

$\lambda = \alpha \pm i\beta$

Stable vs unstable.

$\alpha < 0$ $\alpha > 0$

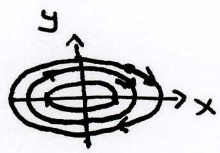
$e^{\alpha t} \cos \beta t$

$x' = 5y$
$y' = -2x$

$$\frac{dy}{dx} = \frac{-2x}{5y}$$

$$5\int y\,dy = -2\int x\,dx$$

$$\frac{5}{2}y^2 = -x^2 + C$$



$$\frac{5}{2}y^2 + x^2 = C$$

$$\frac{x^2}{1^2} + \frac{y^2}{\left(\sqrt{\frac{2}{5}}\right)^2} = C$$

$$A = \begin{pmatrix} 0 & 5 \\ -2 & 0 \end{pmatrix}$$

$$(-\lambda)^2 + 10 = 0$$

$$\lambda^2 + 10 = 0 \Rightarrow \lambda = \pm\sqrt{10}\,i$$

<u>1.c.</u> $x^2y'' + 3xy' + y = 3x^6$

a) $x^2y_h'' + 3xy_h' + y_h = 0, \quad y = x^r$

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0 \Rightarrow r = -1.$$

$$y_h = (c_1 + c_2 \ln|x|)\,x^{-1}$$

b) $$\left.\begin{aligned} y &= Ax^6 \\ y' &= 6Ax^5 \\ y'' &= 30Ax^4 \end{aligned}\right\} \quad (30A + 18A + A)x^6 = 3x^6$$

$$49A = 3$$

$$A = \frac{3}{49}$$

$$y(x) = (c_1 + c_2 \ln|x|)\,x^{-1} + \frac{3}{49}x^6$$

<u>1b.</u> $xy' - y = x^3e^x$

$$y' - \frac{1}{x}y = x^2e^x$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\left(\frac{1}{x}y\right)' = xe^x$$

$$\frac{1}{x}y = \int xe^x\,dx$$

$$= xe^x - e^x + C$$

$$\boxed{y = x^2e^x - xe^x + Cx}$$

MAT 463/563 Midterm Name ______________________

6. (12 pts) Consider the system $\begin{array}{l} x' = -2x + y \\ y' = 4x + y \end{array}$.

a. Find the eigenvalues of the coefficient matrix.

$$A = \begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix} \qquad (-2-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0 \Rightarrow \lambda = -3, 2$$

b. Find the eigenvectors of the coefficient matrix.

$\underline{\lambda = -3}$ $\quad (A - \lambda I)\vec{v} = 0$

$\underline{-2-(-3)}$ $\quad \begin{pmatrix} +1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{array}{c} v_1 + v_2 = 0 \\ v_1 = -v_2 \end{array} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{\lambda = 2}$ $\quad \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{array}{c} -4v_1 + v_2 = 0 \\ v_2 = 4v_1 \end{array} \qquad \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{2t}$$

c. Construct the Fundamental matrix.

$$\Phi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & 4e^{2t} \end{pmatrix}$$

d. Write the general solution.

$$\Phi(t)\vec{C} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -c_1 e^{-3t} + 4c_2 e^{2t} \end{pmatrix} \begin{array}{l} - x \\ - y \end{array}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$$

$$= \begin{pmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & 4e^{2t} \end{pmatrix}\begin{pmatrix} 4e^{2t_0} & -e^{2t_0} \\ +e^{-3t_0} & e^{-3t_0} \end{pmatrix} \frac{1}{4e^{-t_0} + e^{-t_0}}$$

$$\underline{\Psi(t_0) = I}$$