MAT 463/563 Final Exam **Name ________________________**

Instructions:

- Place your name on all of the pages.
- Do all of your work in this booklet. Do not tear off any sheets.
- Show all of your steps in the problems for full credit.
- Be clear and neat in your work. Any illegible work, or scribbling in the margins, will not be graded.
- Put a box around your answers when appropriate..
- If you need more space, you may use the back of a page and write *On back of page #* in the problem space or the attached blank sheet. **No other scratch paper is allowed.**

Try to answer as many problems as possible. Provide as much information as possible. Show sufficient work or rationale for full credit. Remember that some problems may require less work than brute force methods.

If you are stuck, or running out of time, indicate as completely as possible, the methods and steps you would take to tackle the problem. Also, indicate any relevant information that you would use. Do not spend too much time on one problem. **Pace yourself**

Pay attention to the point distribution. Not all problems have the same weight.

Page	**Pts**	**Score**
1	**18**	
2	**17**	
3	**24**	
4	**21**	
5	**20**	
Total	**100**	

Some possible relations that you might need are:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x), \quad n = 1,2,\ldots;$$

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}, \; n = 0,1,2,\ldots \qquad g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x), \quad \frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x), \quad J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$\sinh(A+B) = \sinh A\cosh B + \sinh B\cosh A, \; \cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$$

MAT 463/563 Final Exam **Name** ________________________

1. (5 pts) Solve the boundary value problem:
$x^2 y''(x) - xy'(x) + 2y(x) = 0,\ y(1) = 0, y(e) = 1.$

2. (5 pts) Consider the eigenvalue problem: $y'' + \lambda y = 0,\ y(0) = 0,\ y'(L) = 0.$ Find the eigenvalues λ_n and corresponding eigenfunctions, $y_n(x).$

3. (6 pts) Consider the integral $I = \int_0^\infty x^{2k} e^{-ax^2}\, dx.$

   a. Write this in terms of a Gamma function.

   b. Prove that $I = \frac{(2k-1)!!}{2^{k+1} a^k} \sqrt{\frac{\pi}{a}}.$

4. (2 pts) Knowing that $|x| = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2 \pi^2} \cos\left((2k-1)\pi x\right), |x| \le 1,$

   sum the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$

5. (12 pts) Let $f(x)=\begin{cases} x, & 0\le x<1 \\ 1, & 1\le x\le 2 \end{cases}$.
   a. Find the Fourier trigonometric series representation of $f(x)$.

   b. Sketch the odd periodic extension of $f(x)$ that you would expect on the interval $x\in[-2,4]$. What is the period? _________

   c. Find the Fourier Sine series representation of $f(x)$ over $0<x<1$.

6. (5 pts) Use the generating function for Legendre polynomials to show that $P_n'(1)=\frac{1}{2}n(n+1)$.

7. (5 pts) Consider the Sturm-Liouville problem $L[\phi] \equiv \frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi = -\lambda\sigma\phi,$ with boundary conditions $\phi'(a) = 0,\ \phi(b) = 0.$ Prove that the eigenvalues are real.

8. (9 pts) Evaluate:

   a. $\int_0^\infty \delta(x-5)(x^2-4)dx =$

   b. $\int_0^\infty \delta(x^2-4)(x-5)dx =$

   c. $\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(4)\Gamma\left(\frac{7}{4}\right)} =$

9. (6 Pts) Consider the set of functions $\phi_n(x) = A\cos n\pi x,\ n = 0,1,\ldots$ defined on $0 \le x \le 1.$

   a. Show that this set is orthogonal.

   b. Determine A to make the ϕ_n's orthonormal.

10. (4 pts) Without integrating, find the Fourier coefficients of $f(x) = 2 + 4\sin^2 3x - \cos 2x$ on the interval $x \in [-\pi, \pi].$

11. (5 pts) Expand the polynomial $f(x) = 7x^3 - 3x + 1$ in terms of Legendre polynomials.

12. (7 pts) Consider the boundary value problem $x^2 y'' + xy' + y = 1$. $y(1) = 0,\ y(e) = 0$.

    a. Rewrite this in Sturm-Liouville (self-adjoint) form, $L[y] = f$.

    b. Find the eigenfunctions and eigenvalues of the corresponding eigenvalue problem, $L[\phi] + \lambda\sigma\phi = 0,\ \ \phi(1) = 0,\ \phi(e) = 0$.

13. (6 pts) Use the Method of Variation of Parameters to get the general solution of the problem: $y'' + 2y' - 3y = xe^x$.

14. (3 pts) Consider the operator $L = x\dfrac{d^3}{dx^3} - 2x^2\dfrac{d^2}{dx^2} + 5\dfrac{d}{dx} - \sin x$. What is the (formal) adjoint operator, $L^\dagger$?

15. (20 pts) Consider the nonhomogeneous boundary value problem:

$$\frac{d^2 y}{dx^2} - y = 2\sinh 1, 0 < x < 1, \quad y(0) = 0, y(1) = 0.$$

a. Determine the closed form Green's function for this problem. [Hint: Your answer should involve hyperbolic functions.]

b. Use the Green's function to solve the nonhomogeneous problem.

c. What are the eigenvalues and eigenfunctions of the corresponding eigenvalue problem, $y'' - y = -\lambda y,\ y(0) = 0, y(1) = 0$? [You should be able to practically write these down without much work!]

d. Use the above eigenfunctions to solve the nonhomogeneous problem; i.e., assume that $y(x) = \sum c_n \phi_n(x)$ and determine the expansion coefficients.

MAT 463/563 Final Exam **Name** ________________________

Bonus Problems

A. Show that $\int_0^1 x^3 J_0(ax)dx = \left(1 - \frac{4}{a}\right) J_1(a)$ if $J_0(a) = 0$.

B. Consider the fourth order operator, $L = \frac{d^4}{dx^4}$.

a. Show that $uL(v) - vL(u)$ is an exact differential.

b. Verify $\int_0^1 [uL(v) - vL(u)]\, dx = 0$ for any u and v satisfying the boundary conditions $\phi(0) = 0, \phi'(0) = 0, \phi(1) = 0, \phi'(1) = 0$.

c. For the eigenvalue problem $\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$ and satisfying the conditions in part b, prove that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

Extra Space