3

Nonlinear Systems

3.1 Introduction

Most of your studies of differential equations to date have been the study linear differential equations and common methods for solving them. However, the real world is very nonlinear. So, why study linear equations? Because they are more readily solved. As you may recall, we can use the property of linear superposition of solutions of linear differential equations to obtain general solutions. We will see that we can sometimes approximate the solutions of nonlinear systems with linear systems in small regions of phase space.

In general, nonlinear equations cannot be solved obtaining general solutions. However, we can often investigate the behavior of the solutions without actually being able to find simple expressions in terms of elementary functions. When we want to follow the evolution of these solutions, we resort to numerically solving our differential equations. Such numerical methods need to be executed with care and there are many techniques that can be used. We will not go into these techniques in this course. However, we can make use of computer algebra systems, or computer programs, already developed for obtaining such solutions.

Nonlinear problems occur naturally. We will see problems from many of the same fields we explored in Section 2.9. One example is that of population dynamics. Typically, we have a certain population, $y(t)$, and the differential equation governing the growth behavior of this population is developed in a manner similar to that used previously for mixing problems. We note that the rate of change of the population is given by the *Rate In* minus the *Rate Out*. The Rate In is given by the number of the species born per unit time. The Rate Out is given by the number that die per unit time.

A simple population model can be obtained if one assumes that these rates are linear in the population. Thus, we assume that the Rate In $= by$ and the Rate Out $= my$. Here we have denoted the birth rate as b and the mortality rate as $m,$. This gives the rate of change of population as

$$\frac{dy}{dt} = by - my. \tag{3.1}$$

Generally, these rates could depend upon time. In the case that they are both constant rates, we can define $k = b - m$ and we obtain the familiar exponential model:

$$\frac{dy}{dt} = ky.$$

This is easily solved and one obtains exponential growth ($k > 0$) or decay ($k < 0$). This model has been named after Malthus[1], a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

However, when populations get large enough, there is competition for resources, such as space and food, which can lead to a higher mortality rate. Thus, the mortality rate may be a function of the population size, $m = m(y)$. The simplest model would be a linear dependence, $m = \tilde{m} + cy$. Then, the previous exponential model takes the form

$$\frac{dy}{dt} = ky - cy^2. \tag{3.2}$$

This is known as the *logistic model* of population growth. Typically, c is small and the added nonlinear term does not really kick in until the population gets large enough.

While one can solve this particular equation, it is instructive to study the qualitative behavior of the solutions without actually writing down the explicit solutions. Such methods are useful for more difficult nonlinear equations. We will investigate some simple first order equations in the next section. In the following section we present the analytic solution for completeness.

We will resume our studies of systems of equations and various applications throughout the rest of this chapter. We will see that we can get quite a bit of information about the behavior of solutions by using some of our earlier methods for linear systems.

3.2 Autonomous First Order Equations

In this section we will review the techniques for studying the stability of nonlinear first order autonomous equations. We will then extend this study to looking at families of first order equations which are connected through a parameter.

Recall that a first order autonomous equation is given in the form

[1] Malthus, Thomas Robert. An Essay on the Principle of Population. Library of Economics and Liberty. Retrieved August 2, 2007 from the World Wide Web: http://www.econlib.org/library/Malthus/malPop1.html

$$\frac{dy}{dt} = f(y).$$

We will assume that f and $\frac{\partial f}{\partial y}$ are continuous functions of y, so that we know that solutions of initial value problems exist and are unique.

We will recall the qualitative methods for studying autonomous equations by considering the example

$$\frac{dy}{dt} = y - y^2. \tag{3.3}$$

This is just an example of a logistic equation.

First, one determines the equilibrium, or constant, solutions given by $y' = 0$. For this case, we have $y - y^2 = 0$. So, the equilibrium solutions are $y = 0$ and $y = 1$. Sketching these solutions, we divide the ty-plane into three regions. Solutions that originate in one of these regions at $t = t_0$ will remain in that region for all $t > t_0$ since solutions cannot intersect. [Note that if two solutions intersect then they have common values y_1 at time t_1. Using this information, we could set up an initial value problem for which the initial condition is $y(t_1) = y_1$. Since the two different solutions intersect at this point in the phase plane, we would have an initial value problem with two different solutions corresponding to the same initial condition. This contradicts the uniqueness assumption stated above. We will leave the reader to explore this further in the homework.]

Next, we determine the behavior of solutions in the three regions. Noting that dy/dt gives the slope of any solution in the plane, then we find that the solutions are monotonic in each region. Namely, in regions where $dy/dt > 0$, we have monotonically increasing functions. We determine this from the right side of our equation.

For example, in this problem $y - y^2 > 0$ only for the middle region and $y - y^2 < 0$ for the other two regions. Thus, the slope is positive in the middle region, giving a rising solution as shown in Figure 3.1. Note that this solution does not cross the equilibrium solutions. Similar statements can be made about the solutions in the other regions.

We further note that the solutions on either side of $y = 1$ tend to approach this equilibrium solution for large values of t. In fact, no matter how close one is to $y = 1$, eventually one will approach this solution as $t \to \infty$. So, the equilibrium solution is a *stable solution*. Similarly, we see that $y = 0$ is an *unstable equilibrium solution*.

If we are only interested in the behavior of the equilibrium solutions, we could just construct a *phase line*. In Figure 3.2 we place a vertical line to the right of the ty-plane plot. On this line one first places dots at the corresponding equilibrium solutions and labels the solutions. These points at the equilibrium solutions are end points for three intervals. In each interval one then places arrows pointing upward (downward) indicating solutions with positive (negative) slopes. Looking at the phase line one can now determine if a given equilibrium is stable (arrows pointing towards the point) or unstable

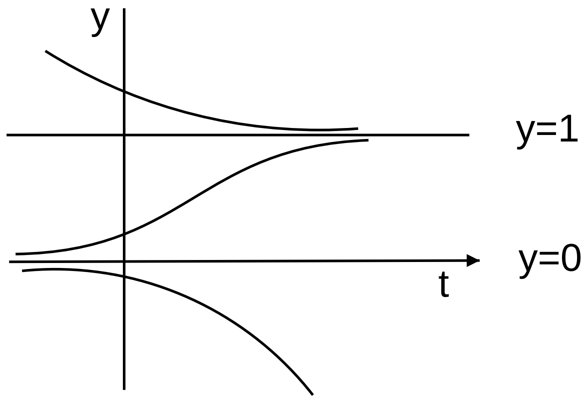

Fig. 3.1. Representative solution behavior for $y' = y - y^2$.

(arrows pointing away from the point). In Figure 3.3 we draw the final phase line by itself.

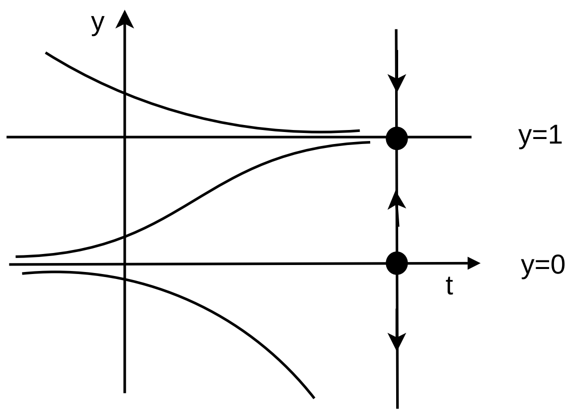

Fig. 3.2. Representative solution behavior and phase line for $y' = y - y^2$.

3.3 Solution of the Logistic Equation

We have seen that one does not need an explicit solution of the logistic equation (3.2) in order to study the behavior of its solutions. However, the logistic equation is an example of a nonlinear first order equation that is solvable. It is an example of a Riccati equation.

The general form of the *Riccati equation* is

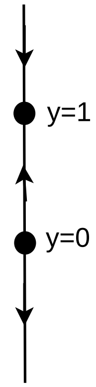

Fig. 3.3. Phase line for $y' = y - y^2$.

$$\frac{dy}{dt} = a(t) + b(t)y + c(t)y^2. \tag{3.4}$$

As long as $c(t) \neq 0$, this equation can be reduced to a second order linear differential equation through the transformation

$$y(t) = -\frac{1}{c(t)}\frac{\dot{x}(t)}{x(t)}.$$

We will demonstrate this using the simple case of the logistic equation,

$$\frac{dy}{dt} = ky - cy^2. \tag{3.5}$$

We let

$$y(t) = \frac{1}{c}\frac{\dot{x}}{x}.$$

Then

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{c}\left[\frac{\ddot{x}}{x} - \left(\frac{\dot{x}}{x}\right)^2\right] \\ &= \frac{1}{c}\left[\frac{\ddot{x}}{x} - (cy)^2\right] \\ &= \frac{1}{c}\frac{\ddot{x}}{x} - cy^2. \end{aligned} \tag{3.6}$$

Inserting this into the logistic equation (3.5), we have

$$\frac{1}{c}\frac{\ddot{x}}{x} - cy^2 = k\frac{1}{c}\left(\frac{\dot{x}}{x}\right) - cy^2,$$

or

$$\ddot{x} = k\dot{x}.$$

This equation is readily solved to give

$$x(t) = A + Be^{kt}.$$

Therefore, we have the solution to the logistic equation is

$$y(t) = \frac{1}{c}\frac{\dot{x}}{x} = \frac{kBe^{kt}}{c(A + Be^{kt})}.$$

It appears that we have two arbitrary constants. But, we started out with a first order differential equation and expect only one arbitrary constant. However, we can resolve this by dividing the numerator and denominator by kBe^{kt} and defining $C = \frac{A}{B}$. Then we have

$$y(t) = \frac{k/c}{1 + Ce^{-kt}}, \tag{3.7}$$

showing that there really is only one arbitrary constant in the solution.

We should note that this is not the only way to obtain the solution to the logistic equation, though it does provide an introduction to Riccati equations. A more direct approach would be to use separation of variables on the logistic equation. The reader should verify this.

3.4 Bifurcations for First Order Equations

In this section we introduce families of first order differential equations of the form

$$\frac{dy}{dt} = f(y; \mu).$$

Here μ is a parameter that we can change and then observe the resulting effects on the behaviors of the solutions of the differential equation. When a small change in the parameter leads to large changes in the behavior of the solution, then the system is said to undergo a *bifurcation.* We will turn to some generic examples, leading to special bifurcations of first order autonomous differential equations.

Example 3.1. $y' = y^2 - \mu$.

First note that equilibrium solutions occur for $y^2 = \mu$. In this problem, there are three cases to consider.

1. $\mu > 0$.
   In this case there are two real solutions, $y = \pm\sqrt{\mu}$. Note that $y^2 - \mu < 0$ for $|y| < \sqrt{\mu}$. So, we have the left phase line in Figure 3.4.

2. $\mu = 0$.
   There is only one equilibrium point at $y = 0$. The equation becomes $y' = y^2$. It is obvious that the right side of this equation is never negative. So, the phase line is shown as the middle line in Figure 3.4.
3. $\mu < 0$.
   In this case there are no equilibrium solutions. Since $y^2 - \mu > 0$, the slopes for all solutions are positive as indicated by the last phase line in Figure 3.4.

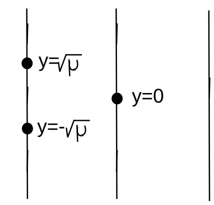

Fig. 3.4. Phase lines for $y' = y^2 - \mu$. On the left $\mu > 0$ and on the right $\mu < 0$.

We can combine these results into one diagram known as a bifurcation diagram. We plot the equilibrium solutions y vs μ. We begin by lining up the phase lines for various μ's. We display these in Figure 3.5. Note the pattern of equilibrium points satisfies $y = \mu^2$ as it should. This is easily seen to be a parabolic curve. The upper branch of this curve is a collection of unstable equilibria and the bottom is a stable branch. So, we can dispose of the phase lines and just keep the equilibria. However, we will draw the unstable branch as a dashed line and the stable branch as a solid line.

The bifurcation diagram is displayed in Figure 3.6. This type of bifurcation is called a *saddle-node bifurcation*. The point $\mu = 0$ at which the behavior changes is called the *bifurcation point*. As μ goes from negative to positive, we go from having no equilibria to having one stable and one unstable equilibrium point.

Example 3.2. $y' = y^2 - \mu y$.

In this example we have two equilibrium points, $y = 0$ and $y = \mu$. The behavior of the solutions depends upon the sign of $y^2 - \mu y = y(y - \mu)$. This leads to four cases with the indicated signs of the derivative.

1. $y > 0,\ y - \mu > 0 \Rightarrow y' > 0$.
2. $y < 0,\ y - \mu > 0 \Rightarrow y' < 0$.
3. $y > 0,\ y - \mu < 0 \Rightarrow y' < 0$.
4. $y < 0,\ y - \mu < 0 \Rightarrow y' > 0$.

The corresponding phase lines and superimposed bifurcation diagram are shown in 3.7. The bifurcation diagram is in Figure 3.8 and this is called a *transcritical bifurcation*.

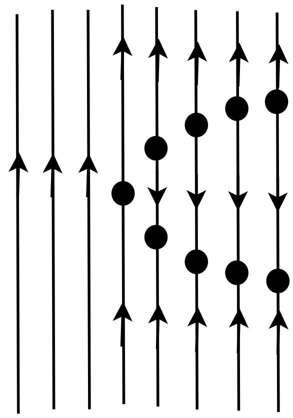

Fig. 3.5. The typical phase lines for $y' = y^2 - \mu$.

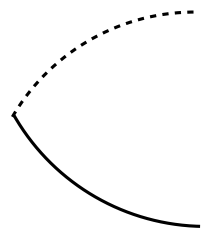

Fig. 3.6. Bifurcation diagram for $y' = y^2 - \mu$. This is an example of a saddle-node bifurcation.

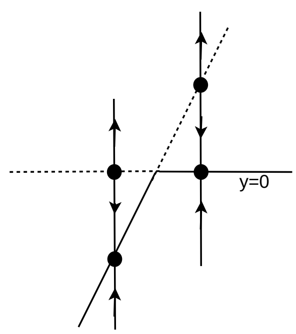

Fig. 3.7. Collection of phase lines for $y' = y^2 - \mu y$.

Example 3.3. $y' = y^3 - \mu y$.

For this last example, we find from $y^3 - \mu y = y(y^2 - \mu) = 0$ that there are two cases.

1. $\mu < 0$ In this case there is only one equilibrium point at $y = 0$. For positive values of y we have that $y' > 0$ and for negative values of y we have that $y' < 0$. Therefore, this is an unstable equilibrium point.

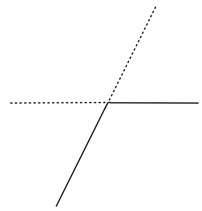

Fig. 3.8. Bifurcation diagram for $y' = y^2 - \mu y$. This is an example of a transcritical bifurcation.

2. $\mu > 0$ Here we have three equilibria, $x = 0, \pm\sqrt{\mu}$. A careful investigation shows that $x = 0$. is a stable equilibrium point and that the other two equilibria are unstable.

In Figure 3.9 we show the phase lines for these two cases. The corresponding bifurcation diagram is then sketched in Figure 3.10. For obvious reasons this has been labeled a *pitchfork bifurcation*.

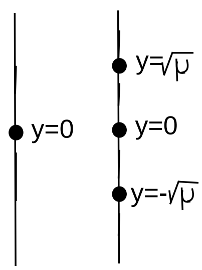

Fig. 3.9. The phase lines for $y' = y^3 - \mu y$. The left one corresponds to $\mu < 0$ and the right phase line is for $\mu > 0$.

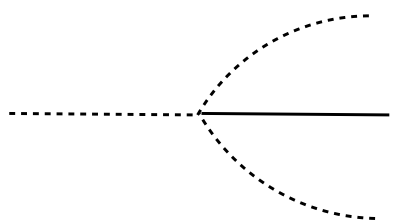

Fig. 3.10. Bifurcation diagram for $y' = y^3 - \mu y$. This is an example of a pitchfork bifurcation.

3.5 Nonlinear Pendulum

In this section we will introduce the nonlinear pendulum as our first example of periodic motion in a nonlinear system. Oscillations are important in many areas of physics. We have already seen the motion of a mass on a spring, leading to simple, damped, and forced harmonic motions. Later we will explore these effects on a simple nonlinear system. In this section we will introduce the nonlinear pendulum and determine its period of oscillation.

We begin by deriving the pendulum equation. The simple pendulum consists of a point mass m hanging on a string of length L from some support. [See Figure 3.11.] One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time, $\theta(t)$.

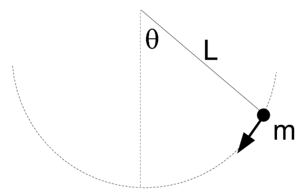

Fig. 3.11. A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

There are a couple of derivations possible. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque. We will use the former only to limit the amount of physics background needed.

There are two forces acting on the point mass, the weight and the tension in the string. The weight points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. At the surface of the earth we can take this to be 9.8 m/s^2 or 32.2 ft/s^2. In Figure 3.12 we show both the weight and the tension acting on the mass. The net force is also shown.

The tension balances the projection of the weight vector, leaving an unbalanced component of the weight in the direction of the motion. Thus, the magnitude of the sum of the forces is easily found from this unbalanced component as $F = mg\sin\theta$.

Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by our point mass. The arclength is related to the angle, provided the angle is measured in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

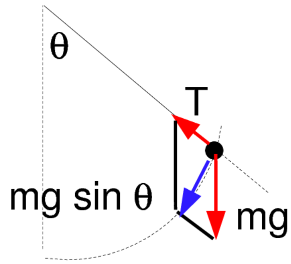

Fig. 3.12. There are two forces acting on the mass, the weight mg and the tension T. The magnitude of the net force is found to be $F = mg\sin\theta$.

$$mL\ddot{\theta} = -mg\sin\theta.$$

Canceling the masses, leads to the *nonlinear pendulum equation*

$$L\ddot{\theta} + g\sin\theta = 0. \tag{3.8}$$

There are several variations of Equation (3.8) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (3.8) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{3.9}$$

We can also make the system more realistic by adding damping. This could be due to energy loss in the way the string is attached to the support or due to the drag on the mass, etc. Assuming that the damping is proportional to the angular velocity, we have equations for the damped nonlinear and damped linear pendula:

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0. \tag{3.10}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{3.11}$$

Finally, we can add forcing. Imagine that the support is attached to a device to make the system oscillate horizontally at some frequency. Then we could have equations such as

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t. \tag{3.12}$$

We will look at these and other oscillation problems later in the exercises. These are summarized in the table below.

Equations for Pendulum Motion

1. Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0$.
2. Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0$.
3. Linear Pendulum: $L\ddot{\theta} + g\theta = 0$.
4. Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0$.
5. Forced Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t$.
6. Forced Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t$.

3.5.1 In Search of Solutions

Before returning to studying the equilibrium solutions of the nonlinear pendulum, we will look at how far we can get at obtaining analytical solutions. First, we investigate the simple linear pendulum.

The linear pendulum equation (3.9) is a constant coefficient second order linear differential equation. The roots of the characteristic equations are $r = \pm\sqrt{\frac{g}{L}}i$. Thus, the general solution takes the form

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right). \tag{3.13}$$

We note that this is usually simplified by introducing the *angular frequency*

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{3.14}$$

One consequence of this solution, which is used often in introductory physics, is an expression for the *period* of oscillation of a simple pendulum. REcall that the period is the time it takes to complete one cycle of the oscillation. The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \tag{3.15}$$

This value for the period of a simple pendulum is based on the linear pendulum equation, which was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle? We recall the Taylor series approximation of $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots. \tag{3.16}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \left| \frac{\sin\theta - \theta}{\sin\theta} \right| \times 100\%.$$

A plot of the relative error is given in Figure 3.13. We note that a one percent relative error corresponds to about 0.24 radians, which is less that fourteen degrees. Further discussion on this is provided at the end of this section.

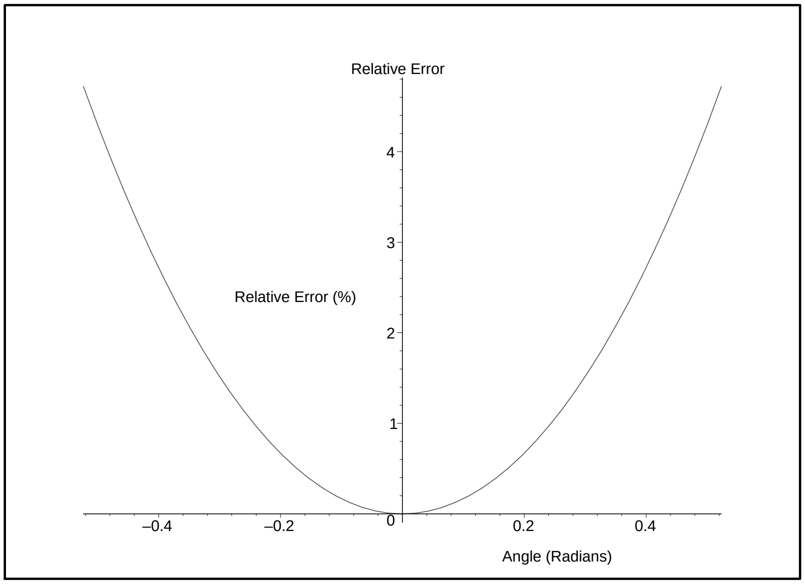

Fig. 3.13. The relative error in percent when approximating $\sin\theta$ by θ.

We now turn to the nonlinear pendulum. We first rewrite Equation (3.8) in the simpler form

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{3.17}$$

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = \left[\ddot{\theta} + F(\theta)\right]\dot{\theta}.$$

For our problem, we multiply Equation (3.17) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{3.18}$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2\cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2\cos\theta)}}. \tag{3.19}$$

Of course, one needs to be able to do the integral. When one gets a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral. In the appendix to this chapter we show that this solution can be written in terms of elliptic integrals and derive corrections to formula for the period of a pendulum.

3.6 The Stability of Fixed Points in Nonlinear Systems

We are now interested in studying the stability of the equilibrium solutions of the nonlinear pendulum. Along the way we will develop some basic methods for studying the stability of equilibria in nonlinear systems.

We begin with the linear differential equation for damped oscillations as given earlier in Equation (3.9). In this case, we have a second order equation of the form

$$x'' + bx' + \omega^2 x.$$

Using the methods of Chapter 2, this second order equation can be written as a system of two first order equations:

$$\begin{aligned} x' &= y \\ y' &= -by - \omega^2 x. \end{aligned} \tag{3.20}$$

This system has only one equilibrium solution, $x = 0$, $y = 0$.

Turning to the damped nonlinear pendulum, we have the system

$$\begin{aligned} x' &= y \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.21}$$

This system also has the equilibrium solution, $x = 0$, $y = 0$. However, there are actually an infinite number of solutions. The equilibria are determined from $y = 0$ and $-by - \omega^2 \sin x = 0$. This implies that $\sin x = 0$. There are an infinite number of solutions: $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So, we have an infinite number of equilibria, $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

Next, we need to determine their stability. To do this we need a more general theory for nonlinear systems. We begin with the $n-$dimensional system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n. \tag{3.22}$$

Here $\mathbf{f} : \mathrm{R}^n \to \mathrm{R}^n$. We define *fixed points,* or equilibrium solutions, of this system as points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

The stability in the neighborhood of fixed points can now be determined. We are interested in what happens to solutions of our system with initial conditions starting near a fixed point. We can represent a point near a fixed point in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$, where the length of $\boldsymbol{\xi}$ gives an indication of how close we are to the fixed point. So, we consider that initially, $|\boldsymbol{\xi}| \ll 1$.

As the system evolves, $\boldsymbol{\xi}$ will change. The change of $\boldsymbol{\xi}$ in time is in turn governed by a system of equations. We can approximate this evolution as follows. First, we note that

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

We can expand the right side about the fixed point using a multidimensional version of Taylor's Theorem. Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

Here $D\mathbf{f}$ is the *Jacobian matrix*, defined as

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Noting that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we then have that system (3.22) becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi}. \tag{3.23}$$

It is this equation which describes the behavior of the system near the fixed point. We say that system (3.22) has been *linearized* or that Equation (3.23) is the *linearization* of system (3.22).

Example 3.4. As an example of the application of this linearization, we look at the system

$$\begin{aligned} x' &= -2x - 3xy \\ y' &= 3y - y^2 \end{aligned} \tag{3.24}$$

We first determine the fixed points:

$$\begin{aligned} 0 &= -2x - 3xy = -x(2 + 3y) \\ 0 &= 3y - y^2 = y(3 - y) \end{aligned} \tag{3.25}$$

From the second equation, we have that either $y = 0$ or $y = 3$. The first equation then gives $x = 0$ in either case. So, there are two fixed points: $(0, 0)$ and $(0, 3)$.

Next, we linearize about each fixed point separately. First, we write down the Jacobian matrix.

$$D\mathbf{f}(x, y) = \begin{pmatrix} -2 - 3y & -3x \\ 0 & 3 - 2y \end{pmatrix}. \tag{3.26}$$

1. Case I $(0, 0)$.
   In this case we find that
   $$D\mathbf{f}(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3.27}$$
   Therefore, the linearized equation becomes
   $$\xi' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{\xi}. \tag{3.28}$$
   This is equivalently written out as the system
   $$\begin{aligned} \xi_1' &= -2\xi_1 \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{3.29}$$
   This is the linearized system about the origin. Note the similarity with the original system. We emphasize that the linearized equations are constant coefficient equations and we can use earlier matrix methods to determine the nature of the equilibrium point. The eigenvalues of the system are obviously $\lambda = -2, 3$. Therefore, we have that the origin is a saddle point.
2. Case II $(0, 3)$.
   In this case we proceed as before. We write down the Jacobian matrix and look at its eigenvalues to determine the type of fixed point. So, we have that the Jacobian matrix is
   $$D\mathbf{f}(0, 3) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}. \tag{3.30}$$
   Here, we have the eigenvalues $\lambda = -2, -3$. So, this fixed point is a stable node.

This analysis has given us a saddle and a stable node. We know what the behavior is like near each fixed point, but we have to resort to other means to say anything about the behavior far from these points. The phase portrait for this system is given in Figure 3.14. You should be able to find the saddle point and the node. Notice how solutions behave in regions far from these points.

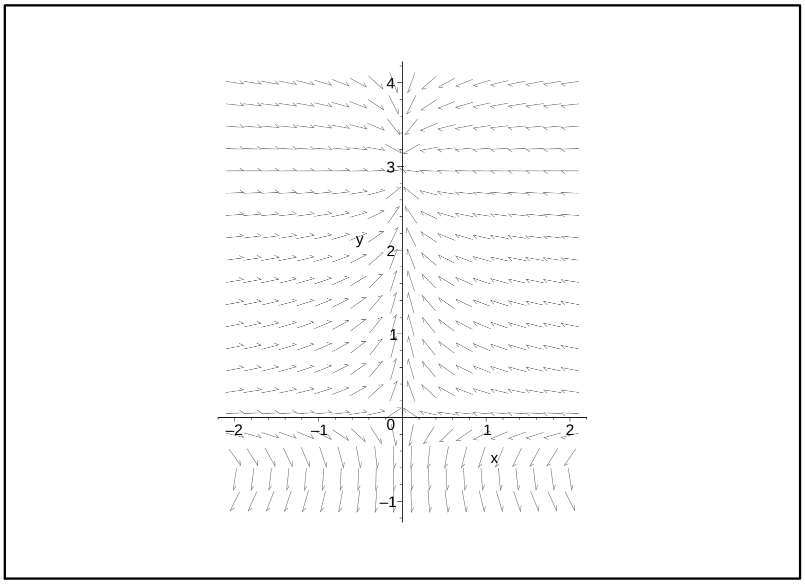

Fig. 3.14. Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

We can expect to be able to perform a linearization under general conditions. These are given in the *Hartman-Großman Theorem*:

Theorem 3.5. *A continuous map exists between the linear and nonlinear systems when* $D\mathbf{f}(\mathbf{x}^*)$ *does not have any eigenvalues with zero real part.*

Generally, there are several types of behavior that one can see in nonlinear systems. One can see sinks or sources, hyperbolic (saddle) points, elliptic points (centers) or foci. We have defined some of these for planar systems. In general, if at least two eigenvalues have real parts with opposite signs, then the fixed point is a *hyperbolic point.* If the real part of a nonzero eigenvalue is zero, then we have a center, or *elliptic point.*

Example 3.6. **Return to the Nonlinear Pendulum**

We are now ready to establish the behavior of the fixed points of the damped nonlinear pendulum in Equation (3.21). The system was

$$\begin{aligned} x' &= y \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.31}$$

We found that there are an infinite number of fixed points at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

We note that the Jacobian matrix is

$$D\mathbf{f}(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}. \tag{3.32}$$

Evaluating this at the fixed points, we find that

$$D\mathbf{f}(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 (-1)^{n+1} & -b \end{pmatrix}. \tag{3.33}$$

There are two cases to consider: n even and n odd. For the first case, we find the eigenvalue equation

$$\lambda^2 + b\lambda + \omega^2 = 0.$$

This has the roots

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even n values. If there is no damping, then we obtain centers.

In the second case, n odd, we have that

$$\lambda^2 + b\lambda - \omega^2 = 0.$$

In this case we find

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have hyperbolic points, or saddles.

In Figure (3.15) we show the phase plane for the undamped nonlinear pendulum. We see that we have a mixture of centers and saddles. There are orbits for which there is periodic motion. At $\theta = \pi$ the behavior is unstable. This is because it is difficult to keep the mass vertical. This would be appropriate if we were to replace the string by a massless rod. There are also unbounded orbits, going through all of the angles. These correspond to the mass spinning around the pivot in one direction forever. We have indicated in the figure solution curves with the initial conditions $(x_0, y_0) = (0, 3), (0, 2), (0, 1), (5, 1)$.

When there is damping, we see that we can have a variety of other behaviors as seen in Figure (3.16). In particular, energy loss leads to the mass settling around one of the stable fixed points. This leads to an understanding as to why there are an infinite number of equilibria, even though physically the mass traces out a bound set of Cartesian points. We have indicated in the Figure (3.16) solution curves with the initial conditions $(x_0, y_0) = (0, 3), (0, 2), (0, 1), (5, 1)$.

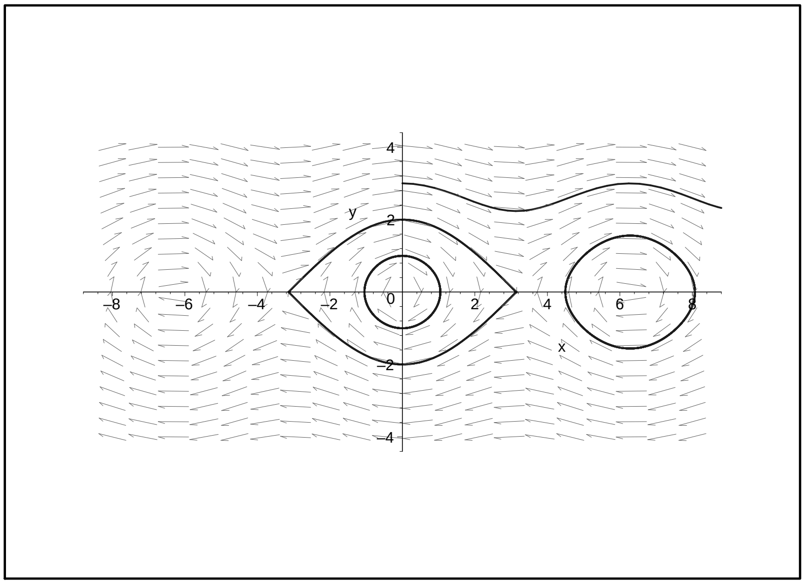

Fig. 3.15. Phase plane for the undamped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0, 3), (0, 2), (0, 1), (5, 1)$.

3.7 Nonlinear Population Models

We have already encountered several models of population dynamics. Of course, one could dream up several other examples. There are two standard types of models: Predator-prey and competing species. In the predator-prey model, one typically has one species, the predator, feeding on the other, the prey. We will look at the standard Lotka-Volterra model in this section. The competing species model looks similar, except there are a few sign changes, since one species is not feeding on the other. Also, we can build in logistic terms into our model. We will save this latter type of model for the homework.

The Lotka-Volterra model takes the form

$$\begin{aligned}\dot{x} &= ax - bxy,\\ \dot{y} &= -dy + cxy.\end{aligned} \tag{3.34}$$

In this case, we can think of x as the population of rabbits (prey) and y is the population of foxes (predators). Choosing all constants to be positive, we can describe the terms.

- ax: When left alone, the rabbit population will grow. Thus a is the natural growth rate without predators.
- $-dy$: When there are no rabbits, the fox population should decay. Thus, the coefficient needs to be negative.

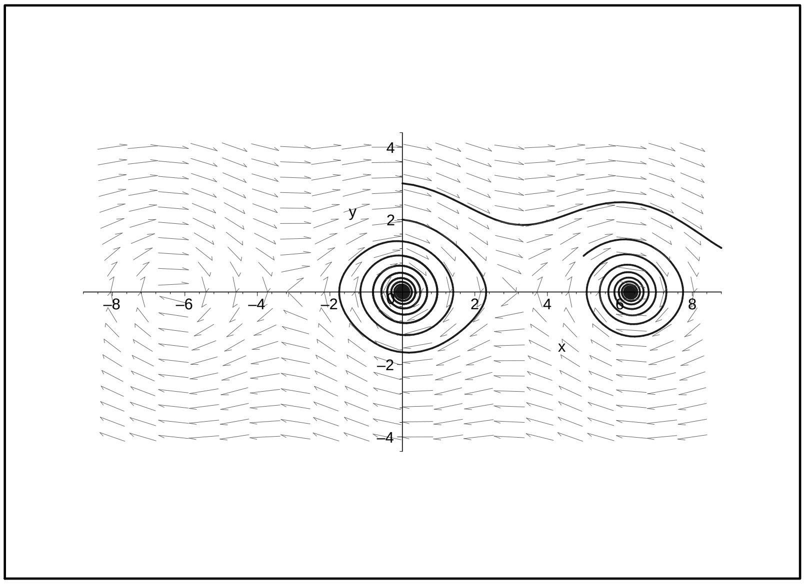

Fig. 3.16. Phase plane for the damped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

- $-bxy$: We add a nonlinear term corresponding to the depletion of the rabbits when the foxes are around.
- cxy: The more rabbits there are, the more food for the foxes. So, we add a nonlinear term giving rise to an increase in fox population.

The analysis of the Lotka-Volterra model begins with determining the fixed points. So, we have from Equation (3.34)

$$\begin{aligned} x(a - by) &= 0, \\ y(-d + cx) &= 0. \end{aligned} \tag{3.35}$$

Therefore, the origin and $(\frac{d}{c}\,\frac{a}{b})$ are the fixed points.

Next, we determine their stability, by linearization about the fixed points. We can use the Jacobian matrix, or we could just expand the right hand side of each equation in (3.34). The Jacobian matrix is $Df(x,y) = \begin{pmatrix} a - by & -bx \\ cy & -d + cx \end{pmatrix}$. Evaluating at each fixed point, we have

$$Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.36}$$

$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}. \tag{3.37}$$

The eigenvalues of (3.36) are $\lambda = a, -d$. So, the origin is a saddle point. The eigenvalues of (3.37) satisfy $\lambda^2 + ad = 0$. So, the other point is a center. In Figure 3.17 we show a sample direction field for the Lotka-Volterra system.

Another way to linearize is to expand the equations about the fixed points. Even though this is equivalent to computing the Jacobian matrix, it sometimes might be faster.

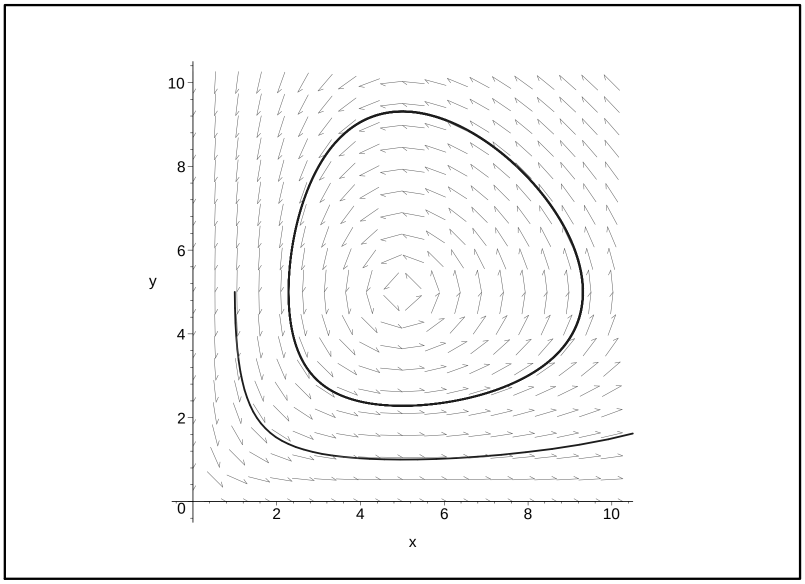

Fig. 3.17. Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$. Solution curves are shown for initial conditions $(x_0, y_0) = (8, 3), (1, 5)$.

3.8 Limit Cycles

So far we have just been concerned with equilibrium solutions and their behavior. However, asymptotically stable fixed points are not the only attractors. There are other types of solutions, known as limit cycles, towards which a solution may tend. In this section we will look at some examples of these periodic solutions.

Such solutions are common in nature. Rayleigh investigated the problem

$$x'' + c\left(\frac{1}{3}(x')^2 - 1\right)x' + x = 0 \tag{3.38}$$

in the study of the vibrations of a violin string. Van der Pol studied an electrical circuit, modelling this behavior. Others have looked into biological systems, such as neural systems, chemical reactions, such as Michaelis-Menton kinetics or systems leading to chemical oscillations. One of the most important models in the historical study of dynamical systems is that of planetary motion and investigating the stability of planetary orbits. As is well known, these orbits are periodic.

Limit cycles are isolated periodic solutions towards which neighboring states might tend when stable. A key example exhibiting a limit cycle is given by the system

$$\begin{aligned} x' &= \mu x - y - x(x^2 + y^2) \\ y' &= x + \mu y - y(x^2 + y^2). \end{aligned} \tag{3.39}$$

It is clear that the origin is a fixed point. The Jacobian matrix is given as

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}. \tag{3.40}$$

The eigenvalues are found to be $\lambda = \mu \pm i$. For $\mu = 0$ we have a center. For $\mu < 0$ we have a stable spiral and for $\mu > 0$ we have an unstable spiral. However, this spiral does not wander off to infinity. We see in Figure 3.18 that equilibrium point is a spiral. However, in Figure 3.19 it is clear that the solution does not spiral out to infinity. It is bounded by a circle.

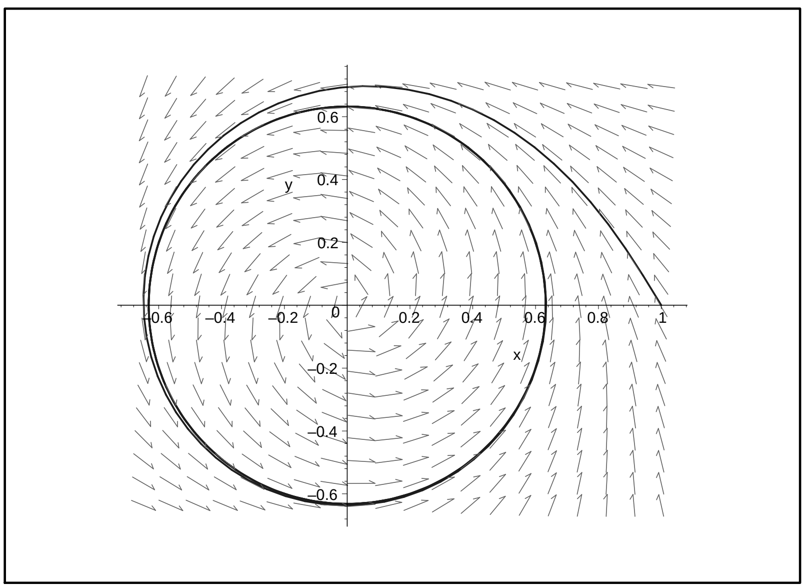

Fig. 3.18. Phase plane for system (3.39) with $\mu = 0.4$.

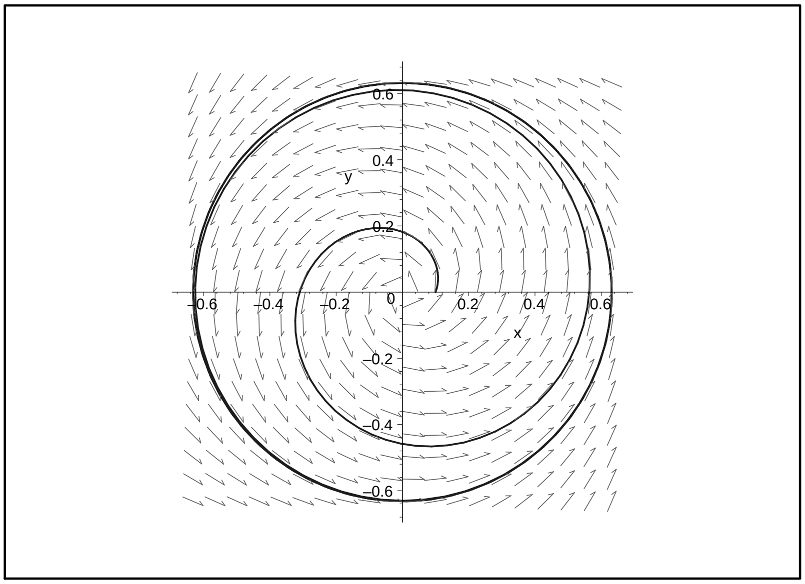

Fig. 3.19. Phase plane for system (3.39) with $\mu = 0.4$ showing that the inner spiral is bounded by a limit cycle.

One can actually find the radius of this circle. This requires rewriting the system in polar form. Recall from Chapter 2 that this is done using

$$rr' = xx' + yy', \tag{3.41}$$

$$r^2\theta' = xy' - yx'. \tag{3.42}$$

Inserting the system (3.39) into these expressions, we have

$$rr' = \mu r^2 - r^4, \qquad r^2\theta' = r^2,$$

or

$$r' = \mu r - r^3, \theta' = 1. \tag{3.43}$$

Of course, for a circle $r = \text{const}$, therefore we need to look at the equilibrium solutions of Equation (3.43). This amounts to solving $\mu r - r^3 = 0$ for r. The solutions of this equation are $r = 0, \pm\sqrt{\mu}$. We need only keep the one positive radius solution, $r = \sqrt{\mu}$. In Figures 3.18-3.19 $\mu = 0.4$, so we expect a circle with $r = \sqrt{0.4} \approx 0.63$. The θ equation just tells us that we follow the limit cycle in a counterclockwise direction.

Limit cycles are not always circles. In Figures 3.20-3.21 we show the behavior of the Rayleigh system (3.38) for $c = 0.4$ and $c = 2.0$. In this case we see that solutions tend towards a noncircular limit cycle.

A slight change of the Rayleigh system leads to the van der Pol equation:

$$x'' + c(x^2 - 1)x' + x = 0 \tag{3.44}$$

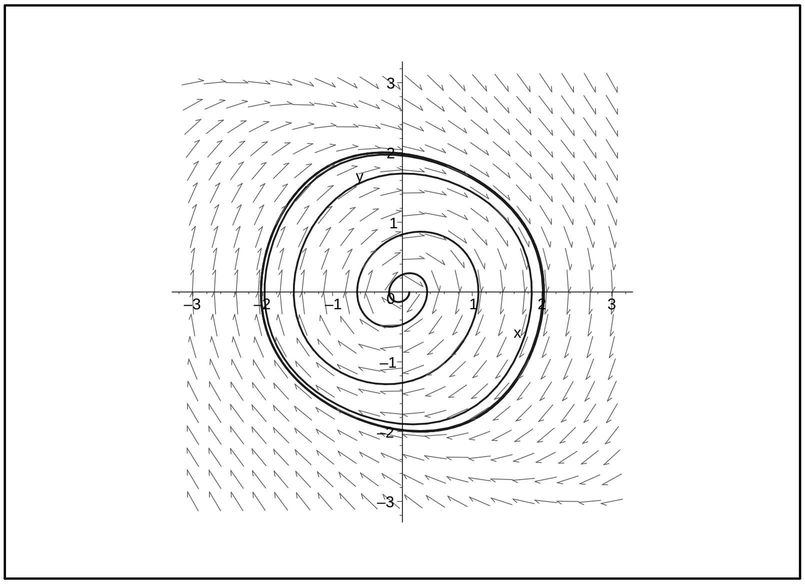

Fig. 3.20. Phase plane for the Rayleigh system (3.38) with $c = 0.4$.

The limit cycle for $c = 2.0$ is shown in Figure 3.22.

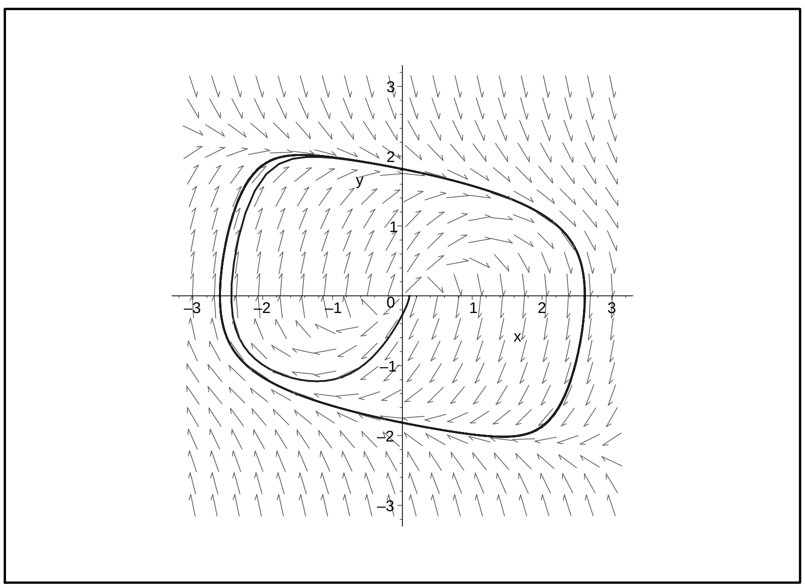

Fig. 3.21. Phase plane for the Rayleigh system (3.38) with $c = 2.0$.

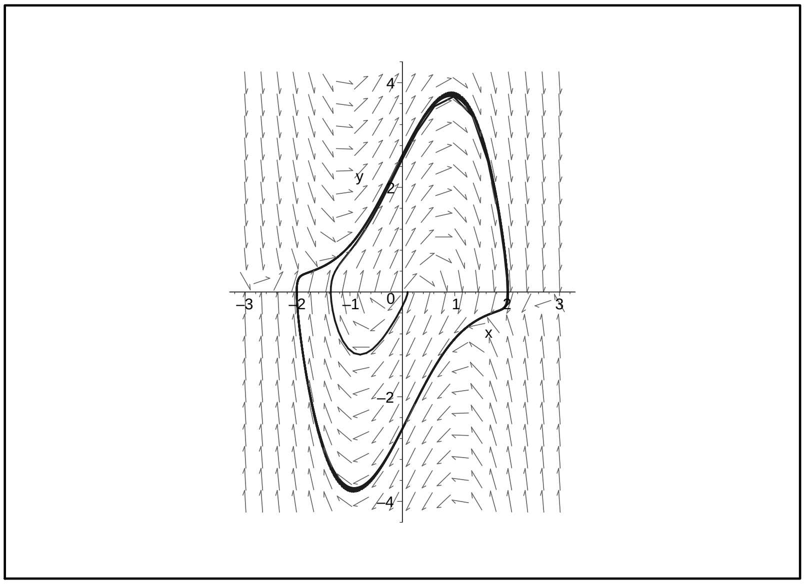

Fig. 3.22. Phase plane for the Rayleigh system (3.44) with $c = 0.4$.

Can one determine ahead of time if a given nonlinear system will have a limit cycle? In order to answer this question, we will introduce some definitions.

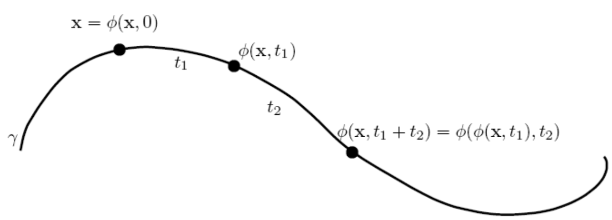

Fig. 3.23. A sketch depicting the idea of trajectory, or orbit, passing through $\mathbf{x}$.

We first describe different trajectories and families of trajectories. A *flow* on R^2 is a function ϕ that satisfies the following

1. $\phi(\mathbf{x}, t)$ is continuous in both arguments.
2. $\phi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in R^2$.
3. $\phi(\phi(\mathbf{x}, t_1), t_2) = \phi(\mathbf{x}, t_1 + t_2)$.

The *orbit*, or *trajectory*, through $\mathbf{x}$ is defined as $\gamma = \{\phi(\mathbf{x}, t) | t \in I\}$. In Figure 3.23 we demonstrate these properties. For $t = 0$, $\phi(\mathbf{x}, 0) = \mathbf{x}$. Increasing t, one follows the trajectory until one reaches the point $\phi(\mathbf{x}, t_1)$. Continuing t_2 further, one is then at $\phi(\phi(\mathbf{x}, t_1), t_2)$. By the third property, this is the same as going from $\mathbf{x}$ to $\phi(\mathbf{x}, t_1 + t_2)$ for $t = t_1 + t_2$.

Having defined the orbits, we need to define the asymptotic behavior of the orbit for both positive and negative large times. We define the *positive semiorbit* through $\mathbf{x}$ as $\gamma^+ = \{\phi(\mathbf{x}, t) | t > 0\}$. The *negative semiorbit* through $\mathbf{x}$ is defined as $\gamma^- = \{\phi(\mathbf{x}, t) | t < 0\}$. Thus, we have $\gamma = \gamma^+ \cup \gamma^-$.

The *positive limit set*, or *ω-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^+ = \{\mathbf{y} | \text{ there exists a sequence of } t_n \to \infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}.$$

The $\mathbf{y}$'s are referred to as *ω-limit points*. This is shown in Figure 3.24.

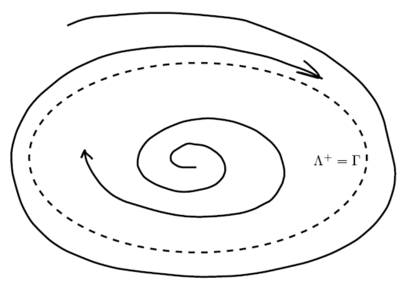

Fig. 3.24. A sketch depicting an ω-limit set. Note that the orbits tends towards the set as t increases.

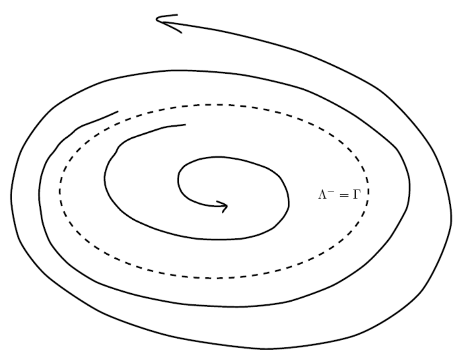

Fig. 3.25. A sketch depicting an α-limit set. Note that the orbits tends away from the set as t increases.

Similarly, we define the *negative limit set*, or it *alpha*-limit sets, of point $\mathbf{x}$ is defined as

$$\Lambda^- = \{\mathbf{y}|\text{ there exists a sequences of } t_n \to -\infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}$$

and the corresponding $\mathbf{y}$'s are *α-limit points*. This is shown in Figure 3.25.

There are several types of orbits that a system might possess. A *cycle* or *periodic orbit* is any closed orbit which is not an equilibrium point. A periodic orbit is stable if for every neighborhood of the orbit such that all nearby orbits stay inside the neighborhood. Otherwise, it is unstable. The orbit is asymptotically stable if all nearby orbits converge to the periodic orbit.

A limit cycle is a cycle which is the α or ω-limit set of some trajectory other than the limit cycle. A limit cycle Γ is stable if $\Lambda^+ = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. A limit cycle Γ is unstable if $\Lambda^- = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. Finally, a limits cycle is semistable if it is attracting on one side and repelling on the other side. In the previous examples, we saw limit cycles that were stable. Figures 3.24 and 3.25 depict stable and unstable limit cycles, respectively.

We now state a theorem which describes the type of orbits we might find in our system.

Theorem 3.7. Poincaré-Bendixon Theorem *Let γ^+ be contained in a bounded region in which there are finitely many critical points. Then Λ^+ is either*

1. *a single critical point;*
2. *a single closed orbit;*
3. *a set of critical points joined by heteroclinic orbits. [Compare Figures 3.27 and* **??***.]*

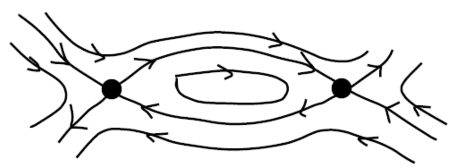

Fig. 3.26. A heteroclinic orbit connecting two critical points.

We are interested in determining when limit cycles may, or may not, exist. A consequence of the Poincaré-Bendixon Theorem is given by the following corollary.

Corollary Let D be a bounded closed set containing no critical points and suppose that $\gamma^+ \subset D$. Then there exists a limit cycle contained in D.

More specific criteria allow us to determine if there is a limit cycle in a given region. These are given by Dulac's Criteria and Bendixon's Criteria.

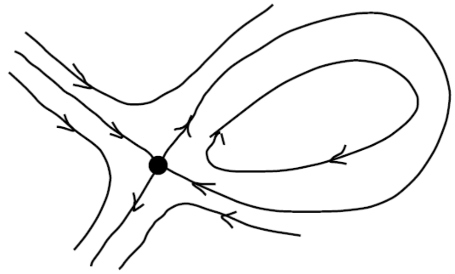

Fig. 3.27. A homoclinic orbit returning to the point it left.

Dulac's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

and a continuously differentiable function ψ defined on an annular region D contained in some open set. If

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g)$$

does not change sign in D, then there is at most one limit cycle contained entirely in D.

Bendixon's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

defined on a simply connected domain D such that

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g) \neq 0$$

in D. Then there are no limit cycles entirely in D.

These are easily proved using Green's Theorem in the plane. We prove Bendixon's Criteria. Let $\mathbf{f} = (f, g)$. Assume that Γ is a closed orbit lying in D. Let S be the interior of Γ. Then

$$\begin{aligned} \int_S \nabla \cdot \mathbf{f}\, dxdy &= \oint_\Gamma (f\, dy - g\, dx) \\ &= \int_0^T (f\dot{y} - g\dot{x})dt \\ &= \int_0^T (fg - gf)dt = 0. \end{aligned} \tag{3.45}$$

So, if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in S, then from the continuity of $\nabla \cdot \mathbf{f}$ in S we have that the right side above is either positive or negative. Thus, we have a contradiction and there is no closed orbit lying in D

Example 3.8. Consider the earlier example in (3.39) with $\mu = 1$.

$$\begin{aligned} x' &= x - y - x(x^2 + y^2) \\ y' &= x + y - y(x^2 + y^2). \end{aligned} \tag{3.46}$$

We already know that a limit cycle exists at $x^2+y^2 = 1$. A simple computation gives that

$$\nabla \cdot \mathbf{f} = 2 - 4x^2 - 4y^2.$$

For an arbitrary annulus $a < x^2 + y^2 < b$, we have

$$2 - 4b < \nabla \cdot \mathbf{f} < 2 - 4a.$$

For $a = 3/4$ and $b = 5/4$, $-3 < \nabla \cdot \mathbf{f} < -1$. Thus, $\nabla \cdot \mathbf{f} < 0$ in the annulus $3/4 < x^2 + y^2 < 5/4$. Therefore, by Dulac's Criteria there is at most one limit cycle in this annulus.

Example 3.9. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -ax - by + cx^2 + dy^2. \end{aligned} \tag{3.47}$$

Let $\psi(x, y) = e^{-2dx}$. Then,

$$\frac{\partial}{\partial x}(\psi y) + \frac{\partial}{\partial y}(\psi(-ax - by + cx^2 + dy^2)) = -be^{-2dx} \neq 0.$$

We conclude by Bendixon's Criteria that there are no limit cycles for this system.

3.9 Nonautonomous Nonlinear Systems

In this section we discuss nonautonomous systems. Recall that an autonomous system is one in which there is no explicit time dependence. A simple example is the forced nonlinear pendulum given by the nonhomogeneous equation

$$\ddot{x} + \omega^2 \sin x = f(t). \tag{3.48}$$

We can set this up as a system of two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(t). \end{aligned} \tag{3.49}$$

This system is not in a form for which we could use the earlier methods. Namely, it is a nonautonomous system. However, we introduce a new variable $z(t) = t$ and turn it into an autonomous system in one more dimension. The new system takes the form

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(z). \\ \dot{z} &= 1.\end{aligned} \tag{3.50}$$

This system is a three dimensional autonomous, possibly nonlinear, system and can be explored using our earlier methods.

A more interesting model is provided by the Duffing Equation. This equation models hard spring and soft spring oscillations. It also models a periodically forced beam as shown in Figure 3.28. It is of interest because it is a simple system which exhibits chaotic dynamics and will motivate us towards using new visualization methods for nonautonomous systems.

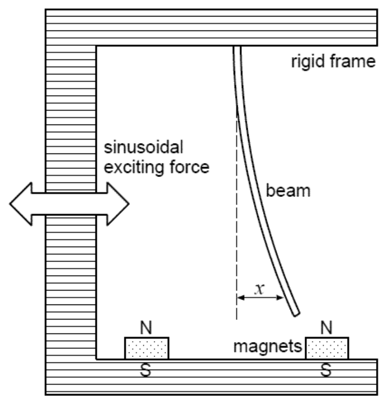

Fig. 3.28. One model of the Duffing equation describes a periodically forced beam which interacts with two magnets.

The most general form of Duffing's equation is given by

$$\ddot{x} + k\dot{x} + (\beta x^3 \pm \omega_0^2 x) = \Gamma \cos(\omega t + \phi). \tag{3.51}$$

This equation models hard spring ($\beta > 0$) and soft spring ($\beta < 0$) oscillations. However, we will use a simpler version of the Duffing equation:

$$\ddot{x} + k\dot{x} + x^3 - x = \Gamma \cos \omega t. \tag{3.52}$$

Let's first look at the behavior of some of the orbits of the system as we vary the parameters. In Figures 3.29-3.31 we show some typical solution plots superimposed on the direction field.

We start with the the undamped ($k = 0$) and unforced ($\Gamma = 0$) Duffing equation,

$$\ddot{x} + x^3 - x == 0.$$

We can write this second order equation as the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2). \end{aligned} \tag{3.53}$$

We see there are three equilibrium points at (0,0), (±1,0). In Figure 3.29 we plot several orbits for We see that the three equilibrium points consist of two centers and a saddle.

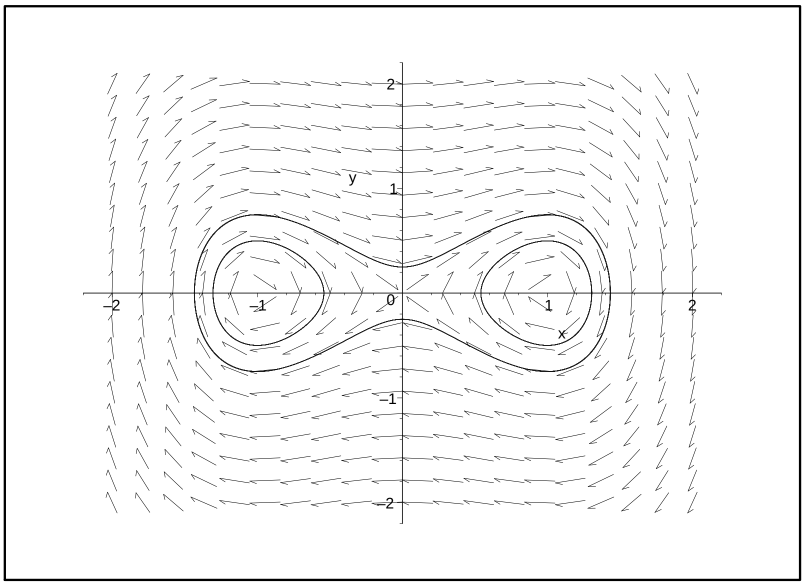

Fig. 3.29. Phase plane for the undamped, unforced Duffing equation ($k = 0, \Gamma = 0$).

We now turn on the damping. The system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky + x(1 - x^2). \end{aligned} \tag{3.54}$$

In Figure 3.30 we show what happens when $k = 0.1$ These plots are reminiscent of the plots for the nonlinear pendulum; however, there are fewer equilibria. The centers become stable spirals.

Next we turn on the forcing to obtain a damped, forced Duffing equation. The system is now nonautonomous.

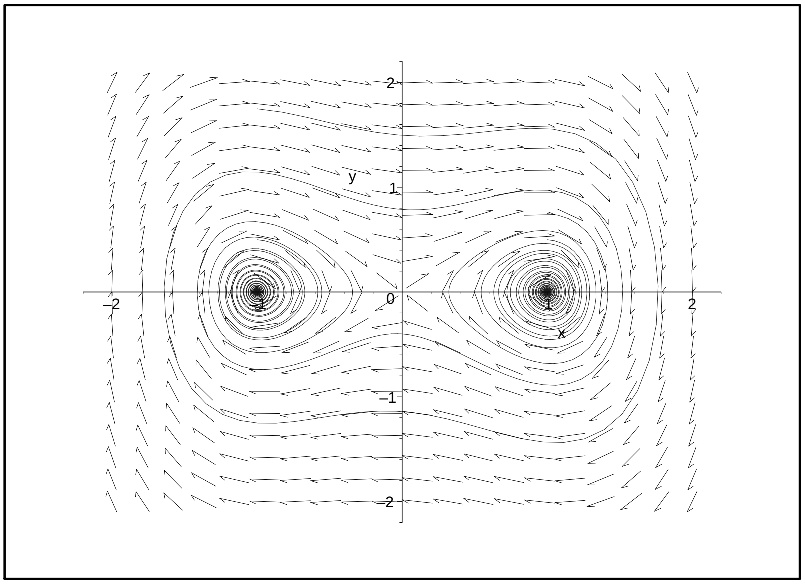

Fig. 3.30. Phase plane for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2) + \Gamma \cos \omega t. \end{aligned} \tag{3.55}$$

In Figure 3.31 we only show one orbit with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. The solution intersects itself and look a bit messy. We can imagine what we would get if we added any more orbits. For completeness, we show in Figure 3.32 an example with four different orbits.

In cases for which one has periodic orbits such as the Duffing equation, Poincaré introduced the notion of *surfaces of section*. One embeds the orbit in a higher dimensional space so that there are no self intersections, like we saw in Figures 3.31 and 3.32. In Figure 3.33 we show an example where a simple orbit is shown as it periodically pierces a given surface.

In order to simplify the resulting pictures, one only plots the points at which the orbit pierces the surface as sketched in Figure 3.34. In practice, there is a natural frequency, such as ω in the forced Duffing equation. Then, one plots points at times that are multiples of the period, $T = \frac{2\pi}{\omega}$. In Figure 3.35 we show what the plot for one orbit would look like for the damped, unforced Duffing equation.

The more interesting case, is when there is forcing and damping. In this case the surface of section plot is given in Figure 3.36. While this is not as busy as the solution plot in Figure 3.31, it still provides some interesting behavior. What one finds is what is called a strange attractor. Plotting many orbits, we find that after a long time, all of the orbits are attracted to a small region in the plane, much like a stable node attracts nearby orbits. However, this

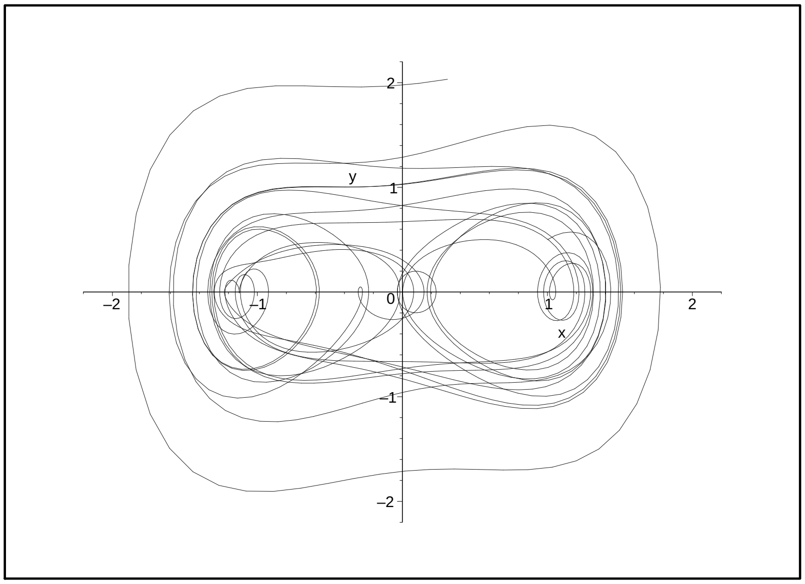

Fig. 3.31. Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case we show only one orbit which was generated from the initial condition $(x_0 = 1.0, \quad y_0 = 0.5)$.

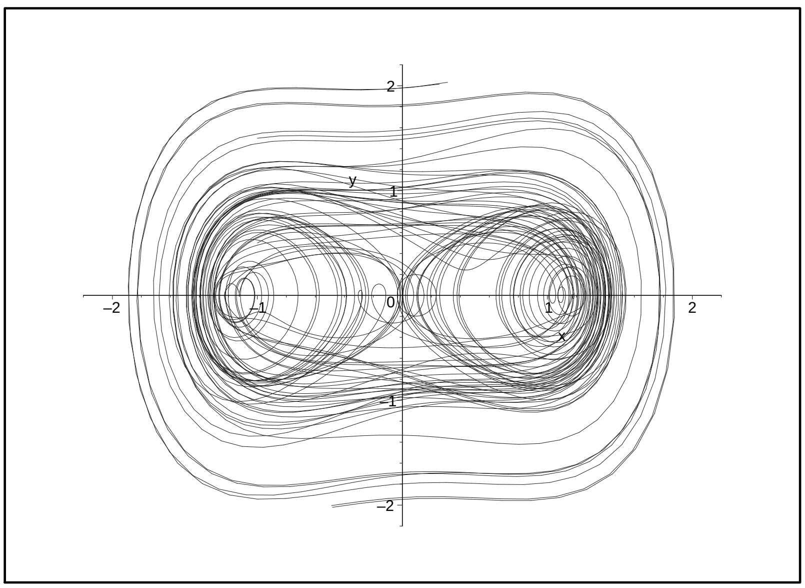

Fig. 3.32. Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case four initial conditions were used to generate four orbits.

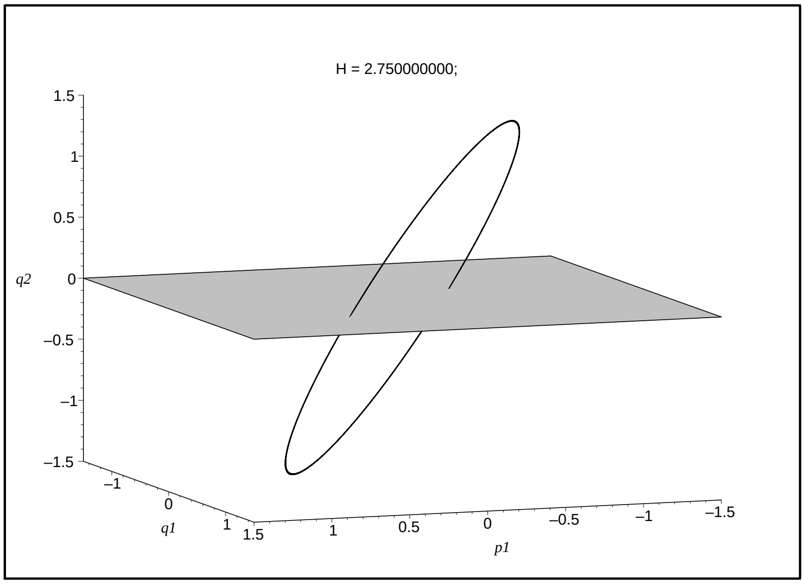

Fig. 3.33. Poincaré's surface of section. One notes each time the orbit pierces the surface.

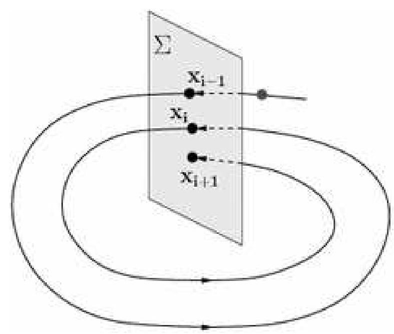

Fig. 3.34. As an orbit pierces the surface of section, one plots the point of intersection in that plane to produce the surface of section plot.

set consists of more than one point. Also, the flow on the attractor is chaotic in nature. Thus, points wander in an irregular way throughout the attractor. This is one of the interesting topics in chaos theory and this whole theory of dynamical systems has only been touched in this text leaving the reader to wander of into further depth into this fascinating field.

3.9.1 Maple Code for Phase Plane Plots

For reference, the plots in Figures 3.29 and 3.30 were generated in Maple using the following commands:

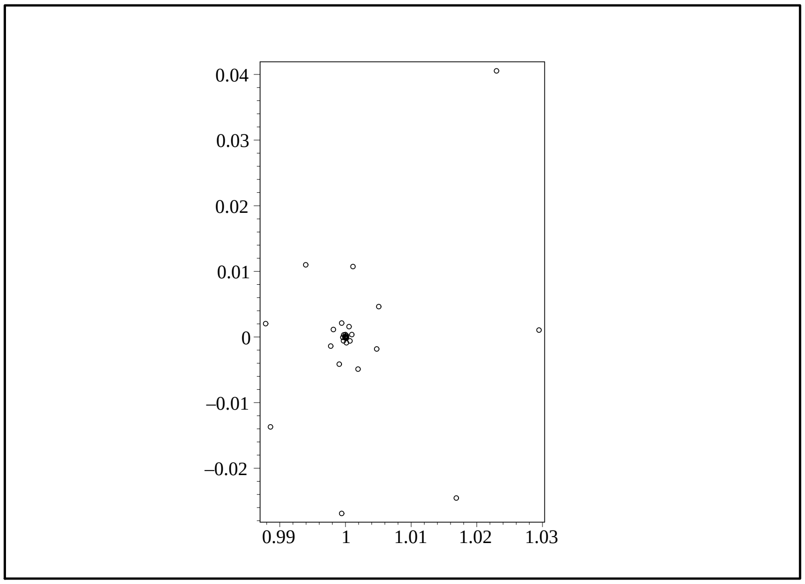

Fig. 3.35. Poincaré's surface of section plot for the damped, unforced Duffing equation.

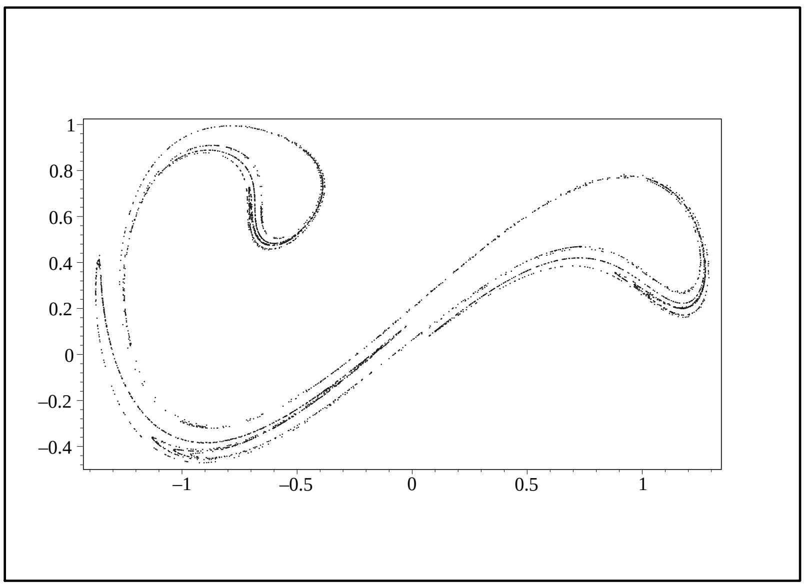

Fig. 3.36. Poincaré's surface of section plot for the damped, forced Duffing equation. This leads to what is known as a strange attractor.

```
> with(DEtools):
> Gamma:=0.5:omega:=1.25:k:=0.1:
> DEplot([diff(x(t),t)=y(t), diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t)], [x(t),y(t)],t=0..500,[[x(0)=1,y(0)=0.5],
  [x(0)=-1,y(0)=0.5], [x(0)=1,y(0)=0.75], [x(0)=-1,y(0)=1.5]],
  x=-2..2,y=-2..2, stepsize=0.1, linecolor=blue, thickness=1,
  color=black);
```

The surface of section plots at the end of the last section were obtained using code from S. Lynch's book *Dynamical Systems with Applications Using Maple*. The Maple code is given by

```
> Gamma:=0:omega:=1.25:k:=0.1:
> f:=
  dsolve({diff(x(t),t)=y(t),diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t),x(0)=1,y(0)=0.5},{x(t),y(t)},
   type=numeric,method=classical,output=procedurelist):
> pt:=array(0..10000):x1:=array(0..10000):y1:=array(0..10000):
> imax:=4000:
> for i from 0 to imax do
>   x1[i]:=eval(x(t),f(i*2*Pi/omega)):
>   y1[i]:=eval(y(t),f(i*2*Pi/omega)):
> end do:
> pts:=[[x1[n],y1[n]]\$ n=10..imax]:
> # Plot the points on the Poincare section #
> pointplot(pts,style=point,symbol=circle,symbolsize=10,
  color=black,axes=BOXED,scaling=CONSTRAINED,
  font=[TIMES,ROMAN,15]);
```

3.10 Appendix: Period of the Nonlinear Pendulum

In Section 3.5.1 we saw that the solution of the nonlinear pendulum problem can be found up to quadrature. In fact, the integral in Equation (3.19) can be transformed into what is know as an elliptic integral of the first kind. We will rewrite our result and then use it to obtain an approximation to the period of oscillation of our nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (3.18). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. Noting that $v = L\dot{\theta}$, the kinetic energy of the mass on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1 - \cos\theta)$. This gives the potential energy as

$$U = mgL(1 - \cos\theta).$$

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\theta'^2 + mgL(1 - \cos\theta). \tag{3.56}$$

We note that a little rearranging shows that we can relate this to Equation (3.18):

$$\frac{1}{2}(\theta')^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2 = c.$$

We can use Equation (3.56) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total energy is pure potential energy. Letting θ_0 denote the angle at the highest position, we have that

$$E = mgL(1 - \cos\theta_0) = mL^2\omega^2(1 - \cos\theta_0).$$

Here we have used the relation $g = L\omega^2$.

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \omega^2(1 - \cos\theta_0). \tag{3.57}$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta),$$

we can rewrite Equation (3.57) as

$$\frac{1}{2}\dot{\theta}^2 = 2\omega^2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]. \tag{3.58}$$

Solving for θ', we have

$$\frac{d\theta}{dt} = 2\omega\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}. \tag{3.59}$$

One can now apply separation of variables and obtain an integral similar to the solution we had obtained previously. Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, then we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0}\frac{d\phi}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{3.60}$$

This result is not much different than our previous result, but we can now easily transform the integral into an elliptic integral. We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then Equation (3.60) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.61}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is an elliptic integral of the first kind. In particular, the elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

In some contexts, this is known as the incomplete elliptic integral of the first kind and $K(k) = F(\frac{\pi}{2},k)$ is called the complete integral of the first kind.

There are tables of values for elliptic integrals. Historically, that is how one found values of elliptic integrals. However, we now have access to computer algebra systems which can be used to compute values of such integrals. For small angles, we have that k is small. So, we can develop a series expansion for the period, T, for small k. This is done by first expanding

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6).$$

Substituting this in the integrand and integrating term by term, one finds that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]. \tag{3.62}$$

This expression gives further corrections to the linear result, which only provides the first term. In Figure 3.37 we show the relative errors incurred when keeping the k^2 and k^4 terms versus not keeping them. The reader is asked to explore this further in Problem 3.8.

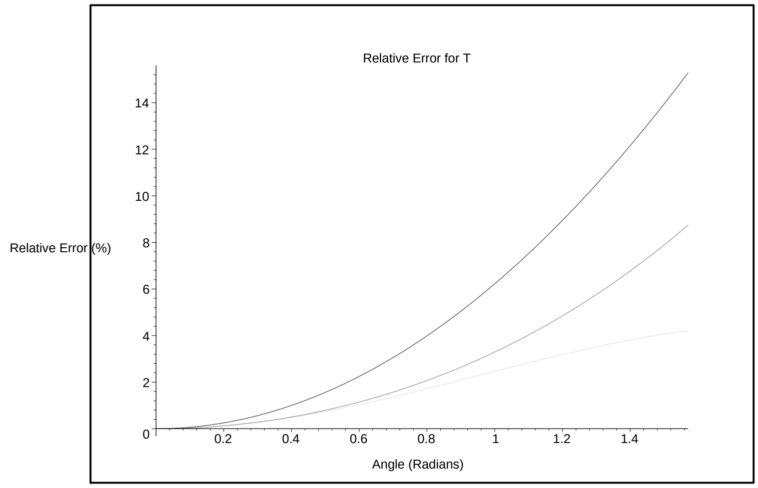

Fig. 3.37. The relative error in percent when approximating the exact period of a nonlinear pendulum with one, two, or three terms in Equation (3.62).

Problems

3.1. Find the equilibrium solutions and determine their stability for the following systems. For each case draw representative solutions and phase lines.

a. $y' = y^2 - 6y - 16$.
b. $y' = \cos y$.
c. $y' = y(y-2)(y+3)$.
d. $y' = y^2(y+1)(y-4)$.

3.2. For $y' = y - y^2$, find the general solution corresponding to $y(0) = y_0$. Provide specific solutions for the following initial conditions and sketch them: a. $y(0) = 0.25$, b. $y(0) = 1.5$, and c. $y(0) = -0.5$.

3.3. For each problem determine equilibrium points, bifurcation points and construct a bifurcation diagram. Discuss the different behaviors in each system.

a. $y' = y - \mu y^2$
b. $y' = y(\mu - y)(\mu - 2y)$
c. $x' = \mu - x^3$
d. $x' = x - \frac{\mu x}{1+x^2}$

3.4. Consider the family of differential equations $x' = x^3 + \delta x^2 - \mu x$.

a. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta = 0$.
b. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta > 0$.

Hint: Pick a few values of δ and μ in order to get a feel for how this system behaves.

3.5. Consider the system

$$x' = -y + x\left[\mu - x^2 - y^2\right],$$
$$y' = x + y\left[\mu - x^2 - y^2\right].$$

Rewrite this system in polar form. Look at the behavior of the r equation and construct a bifurcation diagram in μr space. What might this diagram look like in the three dimensional μxy space? (Think about the symmetry in this problem.) This leads to what is called a *Hopf bifurcation.*

3.6. Find the fixed points of the following systems. Linearize the system about each fixed point and determine the nature and stability in the neighborhood of each fixed point, when possible. Verify your findings by plotting phase portraits using a computer.

a.

$$x' = x(100 - x - 2y),$$
$$y' = y(150 - x - 6y).$$

b.

$$x' = x + x^3,$$
$$y' = y + y^3.$$

c.

$$x' = x - x^2 + xy,$$
$$y' = 2y - xy - 6y^2.$$

d.

$$x' = -2xy,$$
$$y' = -x + y + xy - y^3.$$

3.7. Plot phase portraits for the Lienard system

$$x' = y - \mu(x^3 - x)$$
$$y' = -x.$$

for a small and a not so small value of μ. Describe what happens as one varies μ.

3.8. Consider the period of a nonlinear pendulum. Let the length be $L = 1.0$ m and $g = 9.8$ m/s^2. Sketch T vs the initial angle θ_0 and compare the linear and nonlinear values for the period. For what angles can you use the linear approximation confidently?

3.9. Another population model is one in which species compete for resources, such as a limited food supply. Such a model is given by

$$x' = ax - bx^2 - cxy,$$
$$y' = dy - ey^2 - fxy.$$

In this case, assume that all constants are positive.

a Describe the effects/purpose of each terms.
b Find the fixed points of the model.
c Linearize the system about each fixed point and determine the stability.
d From the above, describe the types of solution behavior you might expect, in terms of the model.

3.10. Consider a model of a food chain of three species. Assume that each population on its own can be modeled by logistic growth. Let the species be labeled by $x(t)$, $y(t)$, and $z(t)$. Assume that population x is at the bottom of the chain. That population will be depleted by population y. Population y is sustained by x's, but eaten by z's. A simple, but scaled, model for this system can be given by the system

$$x' = x(1 - x) - xy$$
$$y' = y(1 - y) + xy - yz$$
$$z' = z(1 - z) + yz.$$

a. Find the equilibrium points of the system.
b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.
c. Find the eigenvalues and eigenvectors.
d. Describe the solution behavior near each equilibrium point.
f. Which of these equilibria are important in the study of the population model and describe the interactions of the species in the neighborhood of these point(s).

3.11. Show that the system $x' = x - y - x^3$, $y' = x + y - y^3$, has a unique limit cycle by picking an appropriate $\psi(x, y)$ in Dulac's Criteria.