A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

Example 2.13.

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \tag{2.33}$$

Transforming to polar coordinates, one can show that In order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and such nonlinear systems will be studied in the next chapter.

2.3 Matrix Formulation

We have investigated several linear systems in the plane and in the next chapter we will use some of these ideas to investigate nonlinear systems. We need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations later in this chapter.

We start with the usual homogeneous system in Equation (2.5). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix* A. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

[1]

[1] The *exponential of a matrix* is defined using the Maclaurin series expansion

We would like to investigate the solution of our system. Our investigations will lead to new techniques for solving linear systems using matrix methods.

We begin by recalling the solution to the specific problem (2.12). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.35}$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t \\ \frac{1}{3} c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \end{aligned} \tag{2.36}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v} e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and λ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (2.3) we insert this guess. Thus,

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \Rightarrow \\ \lambda \mathbf{v} e^{\lambda t} &= A \mathbf{v} e^{\lambda t}. \end{aligned} \tag{2.37}$$

For this to be true for all t, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{2.38}$$

This is an eigenvalue problem. A is a 2×2 matrix for our problem, but could easily be generalized to a system of n first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations..

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = \sum_{k=0}^{\infty} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.34}$$

In general, it is difficult computing e^A unless A is diagonal.

2.4 Eigenvalue Problems

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{2.39}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is called a *trivial solution*. Typically, we are given the matrix A and have to determine the *eigenvalues*, λ, and the associated *eigenvectors*, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (2.39), we obtain the homogeneous algebraic system

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d-\lambda)v_2 &= 0. \end{aligned} \tag{2.40}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation (2.8) for the general constant coefficient differential equation considered in the first chapter. Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

So, the process is to

a) Write the coefficient matrix;
b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,
c) Find the eigenvectors by solving the linear system $(A - \lambda I)\mathbf{v} = 0$ for each λ.

2.5 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{2.41}$$

The type of behavior depends upon the eigenvalues of matrix A. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions[2], then the general solution is given as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Then, setting $t = 0$, we get two linear equations for c_1 and c_2:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. On the next page we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

[2] Recall that linear independence means $c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

Classification of the Solutions for Two Linear First Order Differential Equations

1. **Case I: Two real, distinct roots.**
   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$
2. **Case II: One Repeated Root**
   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue λ, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.
   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.
3. **Case III: Two complex conjugate roots.**
   Solve the eigenvalue problem $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector $\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$. Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t) : \mathbf{y}_1(t) = Re(\mathbf{y}(t))$ and $\mathbf{y}_2(t) = Im(\mathbf{y}(t))$. Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

Let's consider Case III. Note that since the original system of equations does not have any i's, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

and

$$\frac{d}{dt}Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a+d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned}(te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right].\\ e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2.\\ e^{\lambda t}\left(\mathbf{v}_1 + \lambda\mathbf{v}_2\right) &= e^{\lambda t}A\mathbf{v}_2. \end{aligned} \tag{2.42}$$

Noting this is true for all t, we find that

$$\mathbf{v}_1 + \lambda\mathbf{v}_2 = A\mathbf{v}_2. \tag{2.43}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

2.6 Examples of the Matrix Method

Here we will give some examples for typical systems for the three cases mentioned in the last section.

Example 2.14. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

Eigenvalues: We first determine the eigenvalues.

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{2.44}$$

Therefore,

$$\begin{aligned} 0 &= (4-\lambda)(3-\lambda) - 6 \\ 0 &= \lambda^2 - 7\lambda + 6 \\ 0 &= (\lambda - 1)(\lambda - 6) \end{aligned} \tag{2.45}$$

The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.

Eigenvectors: Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda\mathbf{v}$ in each case.

Case $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.46}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.47}$$

This gives $3v_1 + 2v_2 = 0$. One possible solution yields an eigenvector of

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Case $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.48}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.49}$$

For this case we need to solve $-2v_1 + 2v_2 = 0$. This yields

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General Solution: We can now construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 \\ &= c_1 e^t \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^t + c_2 e^{6t} \\ -3c_1 e^t + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{2.50}$$

Example 2.15. $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues: Again, one solves the eigenvalue equation.

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{2.51}$$

Therefore,

$$\begin{aligned} 0 &= (3-\lambda)(-1-\lambda) + 5 \\ 0 &= \lambda^2 - 2\lambda + 2 \\ \lambda &= \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i. \end{aligned} \tag{2.52}$$

The eigenvalues are then $\lambda = 1+i, 1-i$. This is an example of Case III.

Eigenvectors: In order to find the general solution, we need only find the eigenvector associated with $1+i$.

$$\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.53}$$

We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{2.54}$$

Complex Solution: In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.

$$\begin{aligned} e^{\lambda t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} &= e^{(1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t (\cos t + i \sin t) \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix} \\ &= e^t \begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i \sin t \end{pmatrix} \\ &= e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

General Solution: Now we can construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\ &= e^t \begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1 \cos t + c_2 \sin t \end{pmatrix}. \end{aligned} \tag{2.55}$$

Note: This can be rewritten as

$$\mathbf{x}(t) = e^t \cos t \begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^t \sin t \begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.$$

Example 2.16. $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} \tag{2.56}$$

Therefore,

$$\begin{aligned} 0 &= (7-\lambda)(1-\lambda)+9 \\ 0 &= \lambda^2 - 8\lambda + 16 \\ 0 &= (\lambda-4)^2. \end{aligned} \tag{2.57}$$

There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.

Eigenvectors: In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.58}$$

Therefore, we have

$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Second Linearly Independent Solution:
Now we need to solve $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{2.59}$$

Expanding the matrix product, we obtain the system of equations

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{2.60}$$

The solution of this system is $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

General Solution: We construct the general solution as

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1). \\ &= c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}. \end{aligned} \tag{2.61}$$

2.6.1 Planar Systems - Summary

The reader should have noted by now that there is a connection between the behavior of the solutions obtained in Section 2.2 and the eigenvalues found from the coefficient matrices in the previous examples. Here we summarize some of these cases.

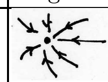
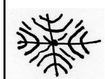
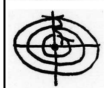
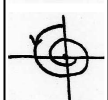
Type	Figure	Eigenvalues	Stability
Node		Real λ, same signs	$\lambda > 0$, stable
Saddle		Real λ opposite signs	Mostly Unstable
Center		λ pure imaginary	—
Focus/Spiral		Complex λ, $\mathrm{Re}(\lambda) \neq 0$	$\mathrm{Re}(\lambda > 0)$, stable
Degenerate Node		Repeated roots, $\lambda > 0$, stable	
Line of Equilibria		One zero eigenvalue	$\lambda > 0$, stable

Table 2.1. List of typical behaviors in planar systems.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 2.2 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

2.7 Theory of Homogeneous Constant Coefficient Systems

There is a general theory for solving homogeneous, constant coefficient systems of first order differential equations. We begin by once again recalling the specific problem (2.12). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.62}$$

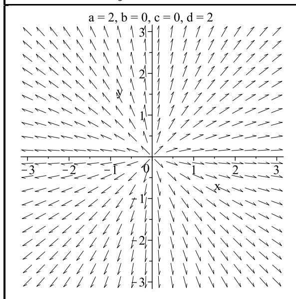
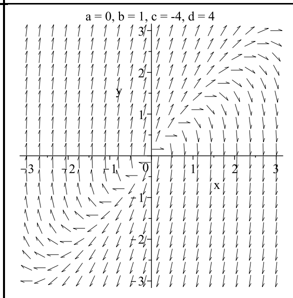
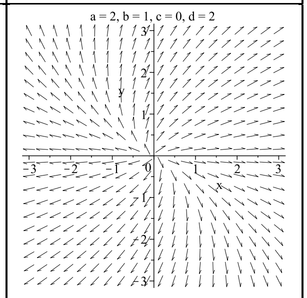
System 1	System 2	System 3
		
$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$

Table 2.2. Three examples of systems with a repeated root of $\lambda = 2$.

This time we rewrite the solution as

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{-4t} \\ \frac{1}{3} e^t & -\frac{1}{2} e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &\equiv \Phi(t)\mathbf{C}. \end{aligned} \tag{2.63}$$

Thus, we can write the general solution as a 2×2 matrix Φ times an arbitrary constant vector. The matrix Φ consists of two columns that are linearly independent solutions of the original system. This matrix is an example of what we will define as the *Fundamental Matrix* of solutions of the system. So, determining the Fundamental Matrix will allow us to find the general solution of the system upon multiplication by a constant matrix. In fact, we will see that it will also lead to a simple representation of the solution of the initial value problem for our system. We will outline the general theory.

Consider the homogeneous, constant coefficient system of first order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{aligned} \tag{2.64}$$

As we have seen, this can be written in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now, consider m vector solutions of this system: $\phi_1(t)$, $\phi_2(t), \ldots \phi_m(t)$. These solutions are said to be *linearly independent* on some domain if

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_m\phi_m(t) = 0$$

for all t in the domain implies that $c_1 = c_2 = \ldots = c_m = 0$.

Let $\phi_1(t)$, $\phi_2(t), \ldots \phi_n(t)$ be a set of n linearly independent set of solutions of our system, called a *fundamental set of solutions*. We construct a matrix from these solutions using these solutions as the column of that matrix. We define this matrix to be the *fundamental matrix solution*. This matrix takes the form

$$\Phi = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}.$$

What do we mean by a "matrix" solution? We have assumed that each ϕ_k is a solution of our system. Therefore, we have that $\phi_k' = A\phi_k$, for $k = 1, \ldots, n$. We say that Φ is a matrix solution because we can show that Φ also satisfies the matrix formulation of the system of differential equations. We can show this using the properties of matrices.

$$\begin{aligned} \frac{d}{dt}\Phi &= \begin{pmatrix} \phi_1' & \ldots & \phi_n' \end{pmatrix} \\ &= \begin{pmatrix} A\phi_1 & \ldots & A\phi_n \end{pmatrix} \\ &= A\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} \\ &= A\Phi. \end{aligned} \qquad (2.65)$$

Given a set of vector solutions of the system, when are they linearly independent? We consider a matrix solution $\Omega(t)$ of the system in which we have n vector solutions. Then, we define the *Wronskian* of $\Omega(t)$ to be

$$W = \det \Omega(t).$$

If $W(t) \neq 0$, then $\Omega(t)$ is a fundamental matrix solution.

Before continuing, we list the fundamental matrix solutions for the set of examples in the last section. (Refer to the solutions from those examples.) Furthermore, note that the fundamental matrix solutions are not unique as one can multiply any column by a nonzero constant and still have a fundamental matrix solution.

Example 2.14 $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix}.$$

We should note in this case that the Wronskian is found as

$$\begin{aligned} W &= \det \Phi(t) \\ &= \begin{vmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{vmatrix} \\ &= 5e^{7t} \neq 0. \end{aligned} \tag{2.66}$$

Example 2.15 $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^t(2\cos t - \sin t) & e^t(\cos t + 2\sin t) \\ e^t \cos t & e^t \sin t \end{pmatrix}.$$

Example 2.16 $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{4t}(1+t) \\ 3e^{4t} & e^{4t}(2+3t) \end{pmatrix}.$$

So far we have only determined the general solution. This is done by the following steps:

Procedure for Determining the General Solution

1. Solve the eigenvalue problem $(A - \lambda I)\mathbf{v} = 0$.
2. Construct vector solutions from $\mathbf{v}e^{\lambda t}$. The method depends if one has real or complex conjugate eigenvalues.
3. Form the fundamental solution matrix $\Phi(t)$ from the vector solution.
4. The general solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$ an arbitrary constant vector.

We are now ready to solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting with the general solution, we have that

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

As usual, we need to solve for the c_k's. Using matrix methods, this is now easy. Since the Wronskian is not zero, then we can invert Φ at any value of t. So, we have

$$\mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Putting $\mathbf{C}$ back into the general solution, we obtain the solution to the initial value problem:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0.$$

You can easily verify that this is a solution of the system and satisfies the initial condition at $t = t_0$.

The matrix combination $\Phi(t)\Phi^{-1}(t_0)$ is useful. So, we will define the resulting product to be the *principal matrix solution*, denoting it by

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0).$$

Thus, the solution of the initial value problem is $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$. Furthermore, we note that $\Psi(t)$ is a solution to the matrix initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = I,$$

where I is the $n \times n$ identity matrix.

Matrix Solution of the Homogeneous Problem

In summary, the matrix solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0,$$

where $\Phi(t)$ is the fundamental matrix solution and $\Psi(t)$ is the principal matrix solution.

Example 2.17. Let's consider the matrix initial value problem

$$\begin{aligned} x' &= 5x + 3y \\ y' &= -6x - 4y, \end{aligned} \tag{2.67}$$

satisfying $x(0) = 1$, $y(0) = 2$. Find the solution of this problem.

We first note that the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalue equation is easily found from

$$\begin{aligned} 0 &= -(5-\lambda)(4+\lambda)+18 \\ &= \lambda^2-\lambda-2 \\ &= (\lambda-2)(\lambda+1). \end{aligned} \tag{2.68}$$

So, the eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found to be

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now we construct the fundamental matrix solution. The columns are obtained using the eigenvectors and the exponentials, $e^{\lambda t}$:

$$\phi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \qquad \phi_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

So, the fundamental matrix solution is

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}.$$

The general solution to our problem is then

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \mathbf{C}$$

for $\mathbf{C}$ is an arbitrary constant vector.

In order to find the particular solution of the initial value problem, we need the principal matrix solution. We first evaluate $\Phi(0)$, then we invert it:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

The particular solution is then

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} + 4e^{2t} \\ 6e^{-t} - 4e^{2t} \end{pmatrix} \end{aligned} \tag{2.69}$$

Thus, $x(t) = -3e^{-t} + 4e^{2t}$ and $y(t) = 6e^{-t} - 4e^{2t}$.

2.8 Nonhomogeneous Systems

Before leaving the theory of systems of linear, constant coefficient systems, we will discuss nonhomogeneous systems. We would like to solve systems of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t). \tag{2.70}$$

We will assume that we have found the fundamental matrix solution of the homogeneous equation. Furthermore, we will assume that $A(t)$ and $\mathbf{f}(t)$ are continuous on some common domain.

As with second order equations, we can look for solutions that are a sum of the general solution to the homogeneous problem plus a particular solution of the nonhomogeneous problem. Namely, we can write the general solution as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \mathbf{x}_p(t),$$

where $\mathbf{C}$ is an arbitrary constant vector, $\Phi(t)$ is the fundamental matrix solution of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$\mathbf{x}_p' = A(t)\mathbf{x}_p + \mathbf{f}(t).$$

Such a representation is easily verified.

We need to find the particular solution, $\mathbf{x}_p(t)$. We can do this by applying *The Method of Variation of Parameters for Systems.* We consider a solution in the form of the solution of the homogeneous problem, but replace the constant vector by unknown parameter functions. Namely, we assume that

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{c}(t).$$

Differentiating, we have that

$$\mathbf{x}_p' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = A\Phi\mathbf{c} + \Phi\mathbf{c}',$$

or

$$\mathbf{x}_p' - A\mathbf{x}_p = \Phi\mathbf{c}'.$$

But the left side is $\mathbf{f}$. So, we have that,

$$\Phi\mathbf{c}' = \mathbf{f},$$

or, since Φ is invertible (why?),

$$\mathbf{c}' = \Phi^{-1}\mathbf{f}.$$

In principle, this can be integrated to give $\mathbf{c}$. Therefore, the particular solution can be written as

$$\mathbf{x}_p(t) = \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{2.71}$$

This is the *variation of parameters formula.*

The general solution of Equation (2.70) has been found as

$$\boxed{\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds.} \tag{2.72}$$

We can use the general solution to find the particular solution of an initial value problem consisting of Equation (2.70) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. This condition is satisfied for a solution of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.73}$$

provided

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

This can be solved for $\mathbf{C}$ as in the last section. Inserting the solution back into the general solution (2.73), we have

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.74}$$

This solution can be written a little neater in terms of the principal matrix solution, $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$:

$$\boxed{\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{f}(s)\,ds \qquad (2.75)}$$

Finally, one further simplification occurs when A is a constant matrix, which are the only types of problems we have solved in this chapter. In this case, we have that $\Psi^{-1}(t) = \Psi(-t)$. So, computing $\Psi^{-1}(t)$ is relatively easy.

Example 2.18. $x'' + x = 2\cos t,\ x(0) = 4,\ x'(0) = 0$. This example can be solved using the Method of Undetermined Coefficients. However, we will use the matrix method described in this section.

First, we write the problem in matrix form. The system can be written as

$$\begin{aligned} x' &= y \\ y' &= -x + 2\cos t. \end{aligned} \tag{2.76}$$

Thus, we have a nonhomogeneous system of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}.$$

Next we need the fundamental matrix of solutions of the homogeneous problem. We have that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm i$. An eigenvector associated with $\lambda = i$ is easily found as $\begin{pmatrix} 1 \\ i \end{pmatrix}$. This leads to a complex solution

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{pmatrix}.$$

From this solution we can construct the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So, the general solution to the homogeneous problem is

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}.$$

Next we seek a particular solution to the nonhomogeneous problem. From Equation (2.73) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have

$$\begin{aligned} \Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\ &= \begin{pmatrix} -2\sin s \cos s \\ 2\cos^2 s \end{pmatrix}. \end{aligned} \tag{2.77}$$

We now compute

$$\begin{aligned} \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\, ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_{t_0}^{t} \begin{pmatrix} -2\sin s \cos s \\ 2\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin^2 t \\ t + \frac{1}{2}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix}. \end{aligned} \tag{2.78}$$

therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} + \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix},$$

or

$$\mathbf{x} = \begin{pmatrix} 4\cos t + t \sin t \\ -3\sin t + t \cos t \end{pmatrix}.$$

2.9 Applications

In this section we will describe several applications leading to systems of differential equations. In keeping with common practice in areas like physics, we will denote differentiation with respect to time as