8

Green's Functions

In this chapter we will investigate the solution of nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$L[u] = f,$$

where L is a differential operator. The solution is formally given by

$$u = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$u = \int G(x,\xi) f(\xi)\, d\xi.$$

The function $G(x,\xi)$ is referred to as the kernel of the integral operator and is called the *Green's function*.

The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function".

We will restrict our discussion to Green's functions for ordinary differential equations. Extensions to partial differential equations are typically one of the subjects of a PDE course. We will begin our investigations by examining solutions of nonhomogeneous second order linear differential equations using the Method of Variation of Parameters, which is typically seen in a first course on differential equations. We will identify the Green's function for both initial value and boundary value problems. We will then focus on boundary value Green's functions and their properties. Determination of Green's functions is also possible using Sturm-Liouville theory. This leads to series representation of Green's functions, which we will study in the last section of this chapter.

8.1 The Method of Variation of Parameters

We are interested in solving nonhomogeneous second order linear differential equations of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x). \tag{8.1}$$

The general solution of this nonhomogeneous second order linear differential equation is found as a sum of the general solution of the homogeneous equation,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \tag{8.2}$$

and a particular solution of the nonhomogeneous equation. Recall from Chapter 1 that there are several approaches to finding particular solutions of nonhomogeneous equations. Any guess would be sufficient. An intelligent guess, based upon the Method of Undetermined Coefficients, was reviewed previously in Chapter 1. However, a more methodical method, which is first seen in a first course in differential equations, is the Method of Variation of Parameters. Also, we explored the matrix version of this method in Section 2.8. We will review this method in this section and extend it to the solution of boundary value problems.

While it is sufficient to derive the method for the general differential equation above, we will instead consider solving equations that are in Sturm-Liouville, or self-adjoint, form. Therefore, we will apply the Method of Variation of Parameters to the equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \tag{8.3}$$

Note that $f(x)$ in this equation is not the same function as in the general equation posed at the beginning of this section.

We begin by assuming that we have determined two linearly independent solutions of the homogeneous equation. The general solution is then given by

$$y(x) = c_1y_1(x) + c_2y_2(x). \tag{8.4}$$

In order to determine a particular solution of the nonhomogeneous equation, we vary the *parameters* c_1 and c_2 in the solution of the homogeneous problem by making them functions of the independent variable. Thus, we seek a particular solution of the nonhomogeneous equation in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{8.5}$$

In order for this to be a solution, we need to show that it satisfies the differential equation. We first compute the derivatives of $y_p(x)$. The first derivative is

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x).$$

Without loss of generality, we will set the sum of the last two terms to zero. (One can show that the same results would be obtained if we did not. See Problem 8.2.) Then, we have

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{8.6}$$

Now, we take the second derivative of the remaining terms to obtain

$$y_p''(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x).$$

Expanding the derivative term in Equation (8.3),

$$p(x)y_p''(x) + p'(x)y_p'(x) + q(x)y_p(x) = f(x),$$

and inserting the expressions for y_p, $y_p'(x)$, and $y_p''(x)$, we have

$$\begin{aligned} f(x) = p(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ +p'(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] + q(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned}$$

Rearranging terms, we find

$$\begin{aligned} f(x) = c_1(x)\left[p(x)y_1''(x) + p'(x)y_1'(x) + q(x)y_1(x)\right] \\ +c_2(x)\left[p(x)y_2''(x) + p'(x)y_2'(x) + q(x)y_2(x)\right] \\ +p(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{8.7}$$

Since $y_1(x)$ and $y_2(x)$ are both solutions of the homogeneous equation. The first two bracketed expressions vanish. Dividing by $p(x)$, we have that

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{p(x)}. \tag{8.8}$$

Our goal is to determine $c_1(x)$ and $c_2(x)$. In this analysis, we have found that the derivatives of these functions satisfy a linear system of equations (in the c_i's):

Linear System for Variation of Parameters

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0. \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \end{aligned} \tag{8.9}$$

This system is easily solved to give

$$\begin{aligned} c_1'(x) &= -\frac{f(x)y_2(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]} \\ c_2'(x) &= \frac{f(x)y_1(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]}. \end{aligned} \tag{8.10}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem. Recall that

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Furthermore, we can show that the denominator, $p(x)W(x)$, is constant. Differentiating this expression and using the homogeneous form of the differential equation proves this assertion.

$$\begin{aligned} \frac{d}{dx}(p(x)W(x)) &= \frac{d}{dx}\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right] \\ &= y_1(x)\frac{d}{dx}(p(x)y_2'(x))) + p(x)y_2'(x)y_1'(x) \\ &\quad -y_2(x)\frac{d}{dx}(p(x)y_1'(x))) - p(x)y_1'(x)y_2'(x) \\ &= -y_1(x)q(x)y_2(x) + y_2(x)q(x)y_1(x) = 0. \end{aligned} \tag{8.11}$$

Therefore,

$$p(x)W(x) = \text{ constant.}$$

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(x) &= -\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi \\ c_2(x) &= \int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi, \end{aligned} \tag{8.12}$$

where x_0 and x_1 are arbitrary constants to be determined later.

Therefore, the particular solution of (8.3) can be written as

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi. \tag{8.13}$$

As a further note, we usually do not rewrite our initial value problems in self-adjoint form. Recall that for an equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \tag{8.14}$$

we obtained the self-adjoint form by multiplying the equation by

$$\frac{1}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}\, dx} = \frac{1}{a_2(x)}p(x).$$

This gives the standard form

$$(p(x)y'(x))' + q(x)y(x) = f(x),$$

where

$$f(x) = \frac{1}{a_2(x)} p(x) g(x).$$

With this in mind, Equation (8.13) becomes

$$y_p(x) = y_2(x) \int_{x_1}^{x} \frac{g(\xi) y_1(\xi)}{a_2(\xi) W(\xi)}\, d\xi - y_1(x) \int_{x_0}^{x} \frac{g(\xi) y_2(\xi)}{a_(\xi) W(\xi)}\, d\xi. \tag{8.15}$$

Example 8.1. Consider the nonhomogeneous differential equation

$$y'' - y' - 6y = 20e^{-2x}.$$

We seek a particular solution to this equation. First, we note two linearly independent solutions of this equation are

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}.$$

So, the particular solution takes the form

$$y_p(x) = c_1(x) e^{3x} + c_2(x) e^{-2x}.$$

We just need to determine the c_i's. Since this problem is not in self-adjoint form, we will use

$$\frac{f(x)}{p(x)} = \frac{g(x)}{a_2(x)} = 20e^{-2x}$$

as seen above. Then the linear system we have to solve is

$$\begin{aligned} c_1'(x) e^{3x} + c_2'(x) e^{-2x} &= 0. \\ 3c_1'(x) e^{3x} - 2c_2'(x) e^{-2x} &= 20e^{-2x}. \end{aligned} \tag{8.16}$$

Multiplying the first equation by 2 and adding the equations yields

$$5c_1'(x) e^{3x} = 20e^{-2x},$$

or

$$c_1'(x) = 4e^{-5x}.$$

Inserting this back into the first equation in the system, we have

$$4e^{-2x} + c_2'(x) e^{-2x} = 0,$$

leading to

$$c_2'(x) = -4.$$

These equations are easily integrated to give

$$c_1(x) = -\frac{4}{5} e^{-5x}, \quad c_2(x) = -4x.$$

Therefore, the particular solution has been found as

$$\begin{aligned} y_p(x) &= c_1(x)e^{3x} + c_2(x)e^{-2x} \\ &= -\frac{4}{5}e^{-5x}e^{3x} - 4xe^{-2x} \\ &= -\frac{4}{5}e^{-2x} - 4xe^{-2x}. \end{aligned} \tag{8.17}$$

Noting that the first term can be absorbed into the solution of the homogeneous problem. So, the particular solution can simply be written as

$$y_p(x) = -4xe^{-2x}.$$

This is the answer you would have found had you used the Modified Method of Undetermined Coefficients.

Example 8.2. Revisiting the last example, $y'' - y' - 6y = 20e^{-2x}$.

The formal solution in Equation (8.13) was not used in the last example. Instead, we proceeded from the Linear System for Variation of Parameters earlier in this section. This is the more natural approach towards finding the particular solution of the nonhomogeneous equation. Since we will be using Equation (8.13) to obtain solutions to initial value and boundary value problems, it might be useful to use it to solve this problem.

From the last example we have

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}.$$

We need to compute the Wronskian:

$$W(x) = W(y_1, y_2)(x) = \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -5e^{x}.$$

Also, we need $p(x)$, which is given by

$$p(x) = \exp\left(-\int dx\right) = e^{-x}.$$

So, we see that $p(x)W(x) = -5$. It is indeed constant, just as we had proven earlier.

Finally, we need $f(x)$. Here is where one needs to be careful as the original problem was not in self-adjoint form. We have from the original equation that $g(x) = 20e^{-2x}$ and $a_2(x) = 1$. So,

$$f(x) = \frac{p(x)}{a_2(x)}g(x) = 20e^{-3x}.$$

Now we are ready to construct the solution.

$$\begin{aligned}
y_p(x) &= y_2(x)\int_{x_1}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\
&= e^{-2x}\int_{x_1}^{x}\frac{20e^{-3\xi}e^{3\xi}}{-5}\,d\xi - e^{3x}\int_{x_0}^{x}\frac{20e^{-3\xi}e^{-2\xi}}{-5}\,d\xi \\
&= -4e^{-2x}\int_{x_1}^{x}d\xi + 4e^{3x}\int_{x_0}^{x}e^{-5x}\,d\xi \\
&= -4\xi e^{-2x}\Big|_{x_1}^{x} - \frac{4}{5}e^{3x}e^{-5\xi}\Big|_{x_0}^{x} \\
&= -4xe^{-2x} - \frac{4}{5}e^{-2x} + 4x_1e^{-2x} + \frac{4}{5}e^{-5x_0}e^{3x}. \qquad (8.18)
\end{aligned}$$

Note that the first two terms we had found in the last example. The remaining two terms are simply linear combinations of y_1 and y_2. Thus, we really have the solution to the homogeneous problem contained within the solution when we use the arbitrary constant limits in the integrals. In the next section we will make use of these constants when solving initial value and boundary value problems.

In the next section we will determine the unknown constants subject to either initial conditions or boundary conditions. This will allow us to combine the two integrals and then determine the appropriate Green's functions.

8.2 Initial and Boundary Value Green's Functions

We begin with the particular solution (8.13) of our nonhomogeneous differential equation (8.3). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_2(x)\int_{x_1}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \qquad (8.19)$$

As seen in the last section, an appropriate choice of x_0 and x_1 could be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(x) + c_2y_2(x)$. However, setting up the solution in this form will allow us to use x_0 and x_1 to determine particular solutions which satisfies certain homogeneous conditions.

We will now consider initial value and boundary value problems. Each type of problem will lead to a solution of the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \int_a^b G(x,\xi)f(\xi)\,d\xi, \qquad (8.20)$$

where the function $G(x,\xi)$ will be identified as the Green's function and the integration limits will be found on the integral. Having identified the Green's function, we will look at other methods in the last section for determining the Green's function.

8.2.1 Initial Value Green's Function

We begin by considering the solution of the initial value problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)=f(x).$$
$$y(0)=y_0,\quad y'(0)=v_0. \tag{8.21}$$

Of course, we could have studied the original form of our differential equation without writing it in self-adjoint form. However, this form is useful when studying boundary value problems. We will return to this point later.

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$\frac{d}{dx}\left(p(x)\frac{dy_h(x)}{dx}\right)+q(x)y_h(x)=0.$$
$$y_h(0)=y_0,\quad y_h'(0)=v_0. \tag{8.22}$$

We then assume that the particular solution satisfies the problem

$$\frac{d}{dx}\left(p(x)\frac{dy_p(x)}{dx}\right)+q(x)y_p(x)=f(x).$$
$$y_p(0)=0,\quad y_p'(0)=0. \tag{8.23}$$

Since the differential equation is linear, then we know that $y(x)=y_h(x)+y_p(x)$ is a solution of the nonhomogeneous equation. However, this solution satisfies the initial conditions:

$$y(0)=y_h(0)+y_p(0)=y_0+0=y_0,$$

$$y'(0)=y_h'(0)+y_p'(0)=v_0+0=v_0.$$

Therefore, we need only focus on solving for the particular solution that satisfies homogeneous initial conditions.

Recall Equation (8.13) from the last section,

$$y_p(x)=y_2(x)\int_{x_1}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi-y_1(x)\int_{x_0}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.24}$$

We now seek values for x_0 and x_1 which satisfies the homogeneous initial conditions, $y_p(0)=0$ and $y_p'(0)=0$.

First, we consider $y_p(0)=0$. We have

$$y_p(0) = y_2(0)\int_{x_1}^{0} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(0)\int_{x_0}^{0} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.25}$$

Here, $y_1(x)$ and $y_2(x)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then we have

$$y_p(0) = y_2(0)\int_{x_1}^{0} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.26}$$

We can force $y_p(0) = 0$ if we set $x_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(x) = y_2'(x)\int_{0}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1'(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi, \tag{8.27}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $x = 0$, we have

$$y_p'(0) = -y_1'(0)\int_{x_0}^{0} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.28}$$

Assuming that $y_1'(0) \neq 0$, we can set $x_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(x)\int_{0}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \\ &= \int_0^x \left[\frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}\right] f(\xi)\,d\xi. \end{aligned} \tag{8.29}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(x) = \int_0^x G(x,\xi)f(\xi)\,d\xi, \tag{8.30}$$

where the *initial value Green's function* is defined as

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}.$$

We summarize

Solution of Initial Value Problem (8.21)

The solution of the initial value problem (8.21) takes the form

$$y(x) = y_h(x) + \int_0^x G(x,\xi) f(\xi)\, d\xi, \tag{8.31}$$

where

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}$$

and the solution of the homogeneous problem satisfies the initial conditions,

$$y_h(0) = y_0, \quad y_h'(0) = v_0.$$

Example 8.3. Solve the forced oscillator problem

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

This problem was solved in Chapter 2 using the theory of nonhomogeneous systems. We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, to the homogeneous differential equation satisfying $y_1(0) = 0$ and $y_2'(0) = 0$. So, we pick $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $p(t) = 1$ in this problem, we have

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{p(\tau)W\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{8.32}$$

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form.

We can now determine the particular solution of the nonhomogeneous differential equation. We have

$$\begin{aligned} x_p(t) &= \int_0^t G(t,\tau) f(\tau)\, d\tau \\ &= \int_0^t (\sin t\cos\tau - \sin\tau\cos t)\,(2\cos\tau)\; d\tau \end{aligned}$$

$$
\begin{aligned}
&= 2\sin t \int_0^t \cos^2\tau d\tau - 2\cos t \int_0^t \sin\tau\cos\tau d\tau \\
&= 2\sin t \left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \\
&= t\sin t. \qquad (8.33)
\end{aligned}
$$

Therefore, the particular solution is $x(t) = 4\cos t + t\sin t$. This is the same solution we had found earlier in Chapter 2.

As noted in the last section, we usually are not given the differential equation in self-adjoint form. Generally, it takes the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \qquad (8.34)$$

The driving term becomes

$$f(x) = \frac{1}{a_2(x)}p(x)g(x).$$

Inserting this into the Green's function form of the particular solution, we obtain the following:

Solution Using the Green's Function

The solution of the initial value problem,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

takes the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \int_0^t G(x,\xi)g(\xi)\,d\xi, \qquad (8.35)$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a_2(\xi)W(\xi)} \qquad (8.36)$$

and $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0.$$

8.2.2 Boundary Value Green's Function

We now turn to boundary value problems. We will focus on the problem

$$
\begin{aligned}
\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,\\
y(a) = 0, \quad y(b) = 0. \qquad (8.37)
\end{aligned}
$$

However, the general theory works for other forms of homogeneous boundary conditions.

Once again, we seek x_0 and x_1 in the form

$$y(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi$$

so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Here we absorb $y_h(x)$ into the integrals with an appropriate choice of lower limits on the integrals.

We first pick solutions of the homogeneous differential equation such that $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. So, we have

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(a)\int_{x_0}^{a} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{8.38}$$

This expression is zero if $x_1 = a$.

At $x = b$ we find that

$$\begin{aligned} y(b) &= y_2(b)\int_{x_1}^{b} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= -y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{8.39}$$

This vanishes for $x_0 = b$.

So, we have found that

$$y(x) = y_2(x)\int_{a}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{b}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.40}$$

We are seeking a Green's function so that the solution can be written as one integral. We can move the functions of x under the integral. Also, since $a < x < b$, we can flip the limits in the second integral. This gives

$$y(x) = \int_{a}^{x} \frac{f(\xi)y_1(\xi)y_2(x)}{p(\xi)W(\xi)}\,d\xi + \int_{x}^{b} \frac{f(\xi)y_1(x)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.41}$$

This result can be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem takes the form

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi, \tag{8.42}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \frac{y_1(\xi)y_2(x)}{pW}, & a \le \xi \le x \\ \frac{y_1(x)y_2(\xi)}{pW} & x \le \xi \le b \end{cases}. \tag{8.43}$$

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at $x = a$ and $x = b$. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{pW} = 0,$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{pW} = 0.$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x) = \begin{cases} \frac{y_1(x)y_2(\xi)}{pW}, & a \le x \le \xi \\ \frac{y_1(\xi)y_2(x)}{pW} & \xi \le x \le b \end{cases}. \tag{8.44}$$

But a careful look at the original form shows that

$$G(x,\xi) = G(\xi,x).$$

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 8.4. Solve the boundary value problem $y'' = x^2, \quad y(0) = 0 = y(1)$ using the boundary value Green's function.

We first solve the homogeneous equation, $y'' = 0$. After two integrations, we have $y(x) = Ax + B$, for A and B constants to be determined.

We need one solution satisfying $y_1(0) = 0$ Thus, $0 = y_1(0) = B$. So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So, $0 = y_2(1) = A + B$. This can be solved for $B = -A$. Again, A is arbitrary and we will choose $A = -1$. Thus, $y_2(x) = 1 - x$.

For this problem $p(x) = 1$. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = x(-1) - 1(1-x) = -1.$$

Note that $p(x)W(x)$ is a constant, as it should be.

Now we construct the Green's function. We have

$$G(x,\xi) = \begin{cases} -\xi(1-x),\ 0 \le \xi \le x \\ -x(1-\xi),\ x \le \xi \le 1 \end{cases}. \tag{8.45}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\xi) = 0$, from the lower branch, and $G(1,\xi) = 0$, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution:

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^1 G(x,\xi)\xi^2\, d\xi \\ &= -\int_0^x \xi(1-x)\xi^2\, d\xi - \int_x^1 x(1-\xi)\xi^2\, d\xi \\ &= -(1-x)\int_0^x \xi^3\, d\xi - x\int_x^1 (\xi^2 - \xi^3)\, d\xi \\ &= -(1-x)\left[\frac{\xi^4}{4}\right]_0^x - x\left[\frac{\xi^3}{3} - \frac{\xi^4}{4}\right]_x^1 \\ &= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3 - 3x^4) \\ &= \frac{1}{12}(x^4 - x). \end{aligned} \tag{8.46}$$

8.3 Properties of Green's Functions

We have noted some properties of Green's functions in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. Here is a list of the properties based upon our previous solution.

Properties of the Green's Function

1. **Differential Equation:**
$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$
For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.
2. **Boundary Conditions:**
For $x = a$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x,\xi)$ will satisfy. A similar statement can be made for $x = b$.
3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi,x)$
We had shown this in the last section.
4. **Continuity of G at** $x = \xi$**:** $G(\xi^+,\xi) = G(\xi^-,\xi)$
Here we have defined

$$G(\xi^+,x) = \lim_{x\downarrow\xi} G(x,\xi), \quad x > \xi,$$

$$G(\xi^-,x) = \lim_{x\uparrow\xi} G(x,\xi), \quad x < \xi.$$

Setting $x = \xi$ in both branches, we have

$$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x = \xi$**:**

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\begin{aligned}\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} &= -\frac{1}{pW}y_1(\xi)y_2'(\xi) + \frac{1}{pW}y_1'(\xi)y_2(\xi) \\ &= -\frac{y_1'(\xi)y_2(\xi) - y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi))} \\ &= \frac{1}{p(\xi)}. \end{aligned} \tag{8.47}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function.

Example 8.5. Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$

$$y(0) = 0 = y(1),$$

with $\omega \neq 0$.

I. **Find solutions to the homogeneous equation.**
A general solution to the homogeneous equation is given as

$$y_h(x) = c_1 \sin \omega x + c_2 \cos \omega x.$$

Thus, for $x \neq \xi$,

$$G(x, \xi) = c_1(\xi) \sin \omega x + c_2(\xi) \cos \omega x.$$

II. **Boundary Conditions.**
First, we have $G(0, \xi) = 0$ for $0 \leq x \leq \xi$. So,

$$G(0, \xi) = c_2(\xi) \cos \omega x = 0.$$

So,

$$G(x, \xi) = c_1(\xi) \sin \omega x, \quad 0 \leq x \leq \xi.$$

Second, we have $G(1, \xi) = 0$ for $\xi \leq x \leq 1$. So,

$$G(1, \xi) = c_1(\xi) \sin \omega + c_2(\xi) \cos \omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi) \tan \omega.$$

This gives

$$G(x, \xi) = c_1(\xi) \sin \omega x - c_1(\xi) \tan \omega \cos \omega x.$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x, \xi) &= \frac{c_1(\xi)}{\cos \omega} [\sin \omega x \cos \omega - \sin \omega \cos \omega x] \\ &= -\frac{c_1(\xi)}{\cos \omega} \sin \omega(1 - x). \end{aligned} \tag{8.48}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x, \xi) = d_1(\xi) \sin \omega(1 - x), \quad \xi \leq x \leq 1.$$

We note that we could have started with $y_2(x) = \sin \omega(1-x)$ as one of our linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

III. **Symmetry or Reciprocity**
We now impose that $G(x,\xi) = G(\xi,x)$. To this point we have that

$$G(x,\xi) = \begin{cases} c_1(\xi)\sin\omega x, & 0 \le x \le \xi \\ d_1(\xi)\sin\omega(1-x), & \xi \le x \le 1 \end{cases}.$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi) = C\sin\omega(1-\xi)$ and $d_1(\xi) = C\sin\omega\xi$. Then,

$$G(x,\xi) = \begin{cases} C\sin\omega(1-\xi)\sin\omega x, \; 0 \le x \le \xi \\ C\sin\omega(1-x)\sin\omega\xi, \; \xi \le x \le 1 \end{cases}.$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where $C = \frac{1}{pW}$.

IV. **Continuity of $G(x,\xi)$**
We note that we already have continuity by virtue of the symmetry imposed in the last step.

V. **Jump Discontinuity in $\frac{\partial}{\partial x}G(x,\xi)$.**
We still need to determine C. We can do this using the jump discontinuity of the derivative:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}.$$

For our problem $p(x) = 1$. So, inserting our Green's function, we have

$$\begin{aligned} 1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\ &= \frac{\partial}{\partial x}\left[C\sin\omega(1-x)\sin\omega\xi\right]_{x=\xi} - \frac{\partial}{\partial x}\left[C\sin\omega(1-\xi)\sin\omega x\right]_{x=\xi} \\ &= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\xi \\ &= -\omega C\sin\omega(\xi+1-\xi) \\ &= -\omega C\sin\omega. \end{aligned} \tag{8.49}$$

Therefore,

$$C = -\frac{1}{\omega\sin\omega}.$$

Finally, we have our Green's function:

$$G(x,\xi) = \begin{cases} -\frac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, \; 0 \le x \le \xi \\ -\frac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, \; \xi \le x \le 1 \end{cases}. \tag{8.50}$$

It is instructive to compare this result to the Variation of Parameters result. We have the functions $y_1(x) = \sin\omega x$ and $y_2(x) = \sin\omega(1-x)$ as the solutions of the homogeneous equation satisfying $y_1(0) = 0$ and $y_2(1) = 0$. We need to compute pW:

$$\begin{aligned} p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= -\omega \sin\omega x \cos\omega(1-x) - \omega\cos\omega x \sin\omega(1-x) \\ &= -\omega \sin\omega \end{aligned} \tag{8.51}$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

8.3.1 The Dirac Delta Function

We will develop a more general theory of Green's functions for ordinary differential equations which encompasses some of the listed properties. The Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, \quad x \neq \xi. \tag{8.52}$$

When $x = \xi$, we saw that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x) = \begin{cases} 1, \ x > 0 \\ 0, \ x < 0 \end{cases}.$$

In the case of the step function, the derivative is zero everywhere except at the jump. At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this by introducing the Dirac delta function,

$$\delta(x) = \frac{d}{dx}H(x).$$

We will then show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{8.53}$$

The Dirac delta function, $\delta(x)$, is one example of what is known as a *generalized function*, or a *distribution.* Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Before defining the Dirac delta function and introducing some of its properties, we will look at some representations that lead to the definition. We will consider the limits of two sequences of functions.

First we define the sequence of functions

$$f_n(x) = \begin{cases} 0, \ |x| > \frac{1}{n} \\ \frac{n}{2}, \ |x| < \frac{1}{n} \end{cases}.$$

This is a sequence of functions as shown in Figure 8.1. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one since each box has height $\frac{n}{2}$ and width $\frac{2}{n}$. Thus, the limiting function is zero at most points but has area one. (At this point the reader who is new to this should be doing some head scratching!)

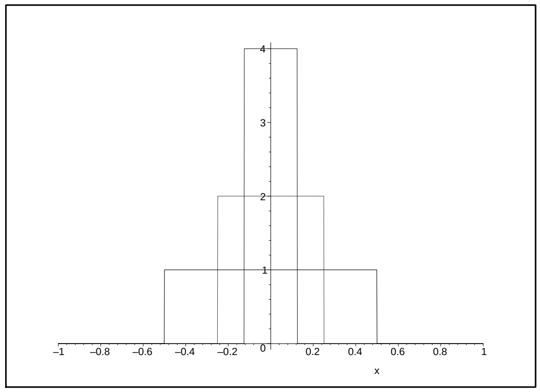

Fig. 8.1. A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a *generalized function*. It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

Another example is the sequence defined by

$$D_n(x) = \frac{2 \sin nx}{x}. \tag{8.54}$$

We can graph this function. We first rewrite this function as

$$D_n(x) = 2n \frac{\sin nx}{nx}.$$

Now it is easy to see that as $x \to 0$, $D_n(x) \to 2n$. For large x, The function tends to zero. A plot of this function is in Figure 8.2. For large n the peak grows and the values of $D_n(x)$ for $x \neq 0$ tend to zero as show in Figure 8.3.

We note that in the limit $n \to \infty$, $D_n(x) = 0$ for $x \neq 0$ and it is infinite at $x = 0$. However, using complex analysis one can show that the area is

$$\int_{-\infty}^{\infty} D_n(x)\, dx = 2\pi.$$

Thus, the area is constant for each n.

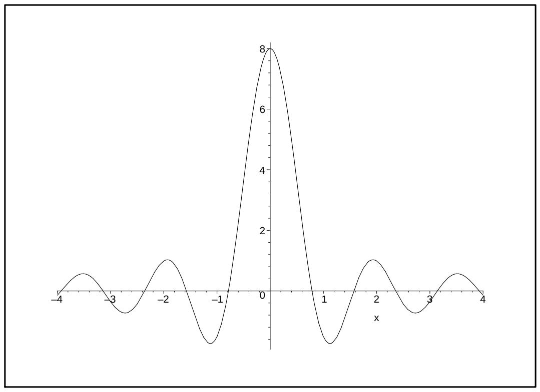

Fig. 8.2. A plot of the function $D_n(x)$ for $n = 4$.

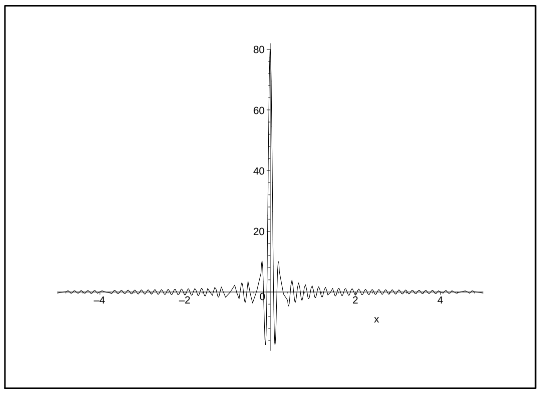

Fig. 8.3. A plot of the function $D_n(x)$ for $n = 40$.

There are two main properties that define a Dirac delta function. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)\,dx = 1, \quad 0 \in [a,b]$$

and

$$\int_a^b \delta(x)\,dx = 0, \quad 0 \notin [a,b].$$

Another common property is what is sometimes called the *sifting property.* Namely, integrating the product of a function and the delta function "sifts" out a specific value of the function. It is given by

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a).$$

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1} \delta(x). \tag{8.55}$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(ax)\, dx &= \lim_{L \to \infty} \int_{-L}^{L} \delta(ax)\, dx \\ &= \lim_{L \to \infty} \frac{1}{a} \int_{-aL}^{aL} \delta(y)\, dy. \end{aligned} \tag{8.56}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{\infty}^{-\infty} \delta(y)\, dy = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy. \tag{8.57}$$

Example 8.6. Evaluate $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx$. This is a straight forward integration:

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{11}{4}.$$

A more general scaling of the argument takes the form $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|}\delta(x - x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|}\delta(x - x_1).$$

Example 8.7. Evaluate $\int_{-\infty}^{\infty} \delta(3x - 2)x^2\, dx$.

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x) = 3x-2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x - 2)x^2\, dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(x - \frac{2}{3})x^2\, dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y = 3x - 2$. Then, $dy = 3dx$ and $x = (y + 2)/3$. This gives

$$\int_{-\infty}^{\infty} \delta(3x - 2)x^2\, dx = \frac{1}{3}\int_{-\infty}^{\infty} \delta(y)\left(\frac{y+2}{3}\right)^2 dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for x_j, $j = 1, 2, \ldots, n$, (i.e.; when one has n simple zeros), then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|}\delta(x - x_j).$$

Example 8.8. Evaluate $\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx$.

In this case the argument of the delta function has two simple roots. Namely, $f(x) = x^2 - \pi^2 = 0$ when $x = \pm\pi$. Furthermore, $f'(x) = 2x$. Therefore, $|f'(\pm\pi)| = 2\pi$. This gives

$$\delta(x^2 - \pi^2) = \frac{1}{2\pi}[\delta(x - \pi) + \delta(x + \pi)].$$

Inserting this expression into the integral and noting that $x = -\pi$ is not in the integration interval, we have

$$\begin{aligned}\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx &= \frac{1}{2\pi}\int_0^{2\pi} \cos x\, [\delta(x - \pi) + \delta(x + \pi)]\, dx \\ &= \frac{1}{2\pi}\cos\pi = -\frac{1}{2\pi}. \qquad (8.58)\end{aligned}$$

Finally, we previously noted there is a relationship between the Heaviside, or step, function and the Dirac delta function. We defined the Heaviside function as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Then, it is easy to see that $H'(x) = \delta(x)$.

8.3.2 Green's Function Differential Equation

As noted, the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=\delta(x-\xi) \tag{8.59}$$

and satisfies homogeneous conditions. We have used the Green's function to solve the nonhomogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)=f(x). \tag{8.60}$$

These equations can be written in the more compact forms

$$\begin{aligned}\mathcal{L}[y]&=f(x)\\ \mathcal{L}[G]&=\delta(x-\xi).\end{aligned} \tag{8.61}$$

Multiplying the first equation by $G(x,\xi)$, the second equation by $y(x)$, and then subtracting, we have

$$G\mathcal{L}[y]-y\mathcal{L}[G]=f(x)G(x,\xi)-\delta(x-\xi)y(x).$$

Now, integrate both sides from $x=a$ to $x=b$. The left side becomes

$$\int_a^b\left[f(x)G(x,\xi)-\delta(x-\xi)y(x)\right]\,dx=\int_a^b f(x)G(x,\xi)\,dx-y(\xi)$$

and, using Green's Identity, the right side is

$$\int_a^b\left(G\mathcal{L}[y]-y\mathcal{L}[G]\right)\,dx=\left[p(x)\left(G(x,\xi)y'(x)-y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Combining these results and rearranging, we obtain

$$y(\xi)=\int_a^b f(x)G(x,\xi)\,dx-\left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi)-G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \tag{8.62}$$

Next, one uses the boundary conditions in the problem in order to determine which conditions the Green's function needs to satisfy. For example, if we have the boundary condition $y(a)=0$ and $y(b)=0$, then the boundary terms yield

$$\begin{aligned}y(\xi)&=\int_a^b f(x)G(x,\xi)\,dx-\left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi)-G(b,\xi)y'(b)\right)\right]\\ &\quad+\left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi)-G(a,\xi)y'(a)\right)\right]\\ &=\int_a^b f(x)G(x,\xi)\,dx+p(b)G(b,\xi)y'(b)-p(a)G(a,\xi)y'(a).\end{aligned} \tag{8.63}$$

The right hand side will only vanish if $G(x,\xi)$ also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi) = \int_a^b f(x)G(x,\xi)\,dx.$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi)G(\xi,x)\,d\xi.$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, $G(x,\xi)$ is symmetric in its arguments. So, we can simply switch the arguments getting the desired result.

We can now see that the theory works for other boundary conditions. If we had $y'(a)=0$, then the $y(a)\frac{\partial G}{\partial x}(a,\xi)$ term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}(a,\xi)=0$. So, this confirms that other boundary value problems can be posed besides the one elaborated upon in the chapter so far.

We can even adapt this theory to nonhomogeneous boundary conditions. We first rewrite Equation (8.62) as

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi - \left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b}. \tag{8.64}$$

Let's consider the boundary conditions $y(a)=\alpha$ and $y'(b)=beta$. We also assume that $G(x,\xi)$ satisfies homogeneous boundary conditions,

$$G(a,\xi)=0, \quad \frac{\partial G}{\partial \xi}(b,\xi)=0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned}\left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b} &= \left[p(b)\left(y(b)\frac{\partial G}{\partial \xi}(x,b) - G(x,b)y'(b)\right)\right] \\ &\quad - \left[p(a)\left(y(a)\frac{\partial G}{\partial \xi}(x,a) - G(x,a)y'(a)\right)\right] \\ &= -\beta p(b)G(x,b) - \alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \end{aligned} \tag{8.65}$$

Therefore, we have the solution

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi + \beta p(b)G(x,b) + \alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \tag{8.66}$$

This solution satisfies the nonhomogeneous boundary conditions. Let's see how it works.

Example 8.9. Modify Example 8.4 to solve the boundary value problem $y'' = x^2, \quad y(0) = 1,\, y(1) = 2$ using the boundary value Green's function that we found:

$$G(x, \xi) = \begin{cases} -\xi(1-x),\, 0 \le \xi \le x \\ -x(1-\xi),\, x \le \xi \le 1 \end{cases}. \tag{8.67}$$

We insert the Green's function into the solution and use the given conditions to obtain

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)\xi^2\, d\xi - \left[y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi) \right]_{\xi=0}^{\xi=1} \\ &= \int_0^x (x-1)\xi^3\, d\xi + \int_x^1 x(\xi-1)\xi^2\, d\xi + y(0)\frac{\partial G}{\partial \xi}(x,0) - y(1)\frac{\partial G}{\partial \xi}(x,1) \\ &= \frac{(x-1)x^4}{4} + \frac{x(1-x^4)}{4} - \frac{x(1-x^3)}{3} + (x-1) - 2x \\ &= \frac{x^4}{12} + \frac{35}{12}x - 1. \end{aligned} \tag{8.68}$$

Of course, this problem can be solved more directly by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (8.59), can lead to the solution of boundary value problems. The Dirac delta function also aids in our interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the nonhomogeneous function is $\delta(x-\xi)$. Note that if we multiply the delta function by $f(\xi)$ and integrate we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x).$$

We can view the delta function as a unit impulse at $x = \xi$ which can be used to build $f(x)$ as a sum of impulses of different strengths, $f(\xi)$. Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x} \right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{8.69}$$

Now, integrate both sides from $\xi - \epsilon$ to $\xi + \epsilon$ and take the limit as $\epsilon \to 0$. Then,

$$\lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} \left[\frac{\partial}{\partial x} \left(p(x) \frac{\partial G(x,\xi)}{\partial x} \right) + q(x) G(x,\xi) \right] dx = \lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx$$
$$= 1. \quad (8.70)$$

Since the $q(x)$ term is continuous, the limit of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon \to 0} \left[p(x) \frac{\partial G(x,\xi)}{\partial x} \right]_{\xi-\epsilon}^{\xi+\epsilon} = 1. \quad (8.71)$$

This is the jump condition that we have been using!

8.4 Series Representations of Green's Functions

There are times that it might not be so simple to find the Green's function in the simple closed form that we have seen so far. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. We will finish our discussion of Green's functions for ordinary differential equations by showing how one obtains such series representations. (Note that we are really just repeating the steps towards developing eigenfunction expansion which we had seen in Chapter 6.)

We will make use of the complete set of eigenfunctions of the differential operator, $\mathcal{L}$, satisfying the homogeneous boundary conditions:

$$\mathcal{L}[\phi_n] = -\lambda_n \sigma \phi_n, \quad n = 1, 2, \ldots$$

We want to find the particular solution y satisfying $\mathcal{L}[y] = f$ and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Inserting this into the differential equation, we obtain

$$\mathcal{L}[y] = \sum_{n=1}^{\infty} a_n \mathcal{L}[\phi_n] = -\sum_{n=1}^{\infty} \lambda_n a_n \sigma \phi_n = f.$$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \sigma \phi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n.$$

We have seen how to compute these coefficients earlier in the text. We multiply both sides by $\phi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \phi_n(x)\phi_k(x)\sigma(x)\,dx = N_k\delta_{nk},$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = -\frac{(f,\phi_k)}{N_k\lambda_k},$$

where $(f,\phi_k) \equiv \int_a^b f(x)\phi_k(x)\,dx$.

As before, we can rearrange the solution to obtain the Green's function. Namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f,\phi_n)}{-N_n\lambda_n}\phi_n(x) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-N_n\lambda_n}}_{G(x,\xi)} f(\xi)\,d\xi$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{8.72}$$

Example 8.10. **Eigenfunction Expansion Example**

We will conclude this discussion with an example. Consider the boundary value problem

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

The Green's function for this problem can be constructed fairly quickly for this problem once the eigenvalue problem is solved. We will solve this problem three different ways in order to summarize the methods we have used in the text.

The eigenvalue problem is

$$\phi''(x) + 4\phi(x) = -\lambda\phi(x),$$

where $\phi(0) = 0$ and $\phi(1) = 0$. The general solution is obtained by rewriting the equation as

$$\phi''(x) + k^2\phi(x) = 0,$$

where

$$k^2 = 4 + \lambda.$$

Solutions satisfying the boundary condition at $x = 0$ are of the form

$$\phi(x) = A\sin kx.$$

Forcing $\phi(1) = 0$ gives

$$0 = A \sin k \Rightarrow k = n\pi, \quad k = 1, 2, 3 \ldots.$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\phi_n = \sin n\pi x, \quad n = 1, 2, \ldots.$$

We need the normalization constant, N_n. We have that

$$N_n = \|\phi_n\|^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}.$$

We can now construct the Green's function for this problem using Equation (8.72).

$$G(x, \xi) = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)}. \tag{8.73}$$

We can use this Green's function to determine the solution of the boundary value problem. Thus, we have

$$\begin{aligned} y(x) &= \int_0^1 G(x, \xi) f(\xi)\, d\xi \\ &= \int_0^1 \left(2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)} \right) \xi^2\, d\xi \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \int_0^1 \xi^2 \sin n\pi\xi\, d\xi \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \left[\frac{(2 - n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right] \end{aligned} \tag{8.74}$$

We can compare this solution to the one one would obtain if we did not employ Green's functions directly. The eigenfunction expansion method for solving boundary value problems, which we saw earlier proceeds as follows. We assume that our solution is in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Inserting this into the differential equation $\mathcal{L}[y] = x^2$ gives

$$
\begin{aligned}
x^2 &= \mathcal{L}\left[\sum_{n=1}^{\infty} c_n \sin n\pi x\right] \\
&= \sum_{n=1}^{\infty} c_n \left[\frac{d^2}{dx^2}\sin n\pi x + 4\sin n\pi x\right] \\
&= \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x \qquad (8.75)
\end{aligned}
$$

We need the Fourier sine series expansion of x^2 on $[0,1]$ in order to determine the c_n's. Thus, we need

$$
\begin{aligned}
b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x \\
&= 2\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right], \quad n = 1, 2, \ldots. \qquad (8.76)
\end{aligned}
$$

Thus,

$$
x^2 = 2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]\sin n\pi x.
$$

Inserting this in Equation (8.75), we find

$$
2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]\sin n\pi x = \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2]\sin n\pi x.
$$

Due to the linear independence of the eigenfunctions, we can solve for the unknown coefficients to obtain

$$
c_n = 2\frac{(2-n^2\pi^2)(-1)^n - 2}{(4-n^2\pi^2)n^3\pi^3}.
$$

Therefore, the solution using the eigenfunction expansion method is

$$
\begin{aligned}
y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\
&= 2\sum_{n=1}^{\infty}\frac{\sin n\pi x}{(4-n^2\pi^2)}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]. \qquad (8.77)
\end{aligned}
$$

We note that this is the same solution as we had obtained using the Green's function obtained in series form.

One remaining question is the following: Is there a closed form for the Green's function and the solution to this problem? The answer is yes! We note that the differential operator is a special case of the example done is section 8.2.2. Namely, we pick $\omega = 2$. The Green's function was already found in that section. For this special case, we have

$$G(x,\xi) = \begin{cases} -\frac{\sin 2(1-\xi)\sin 2x}{2\sin 2}, & 0 \le x \le \xi \\ -\frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}, & \xi \le x \le 1 \end{cases}. \tag{8.78}$$

What about the solution to the boundary value problem? This solution is given by

$$\begin{aligned}
y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\
&= -\int_0^x \frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}\xi^2\, d\xi + \int_x^1 \frac{\sin 2(\xi-1)\sin 2x}{2\sin 2}\xi^2\, d\xi \\
&= -\frac{1}{4\sin 2}\left[-x^2\sin 2 - \sin 2\cos^2 x + \sin 2 + \cos 2 \sin x \cos x + \sin x \cos x\right]. \\
&= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + (1-\cos^2 x)\sin 2 + \sin x \cos x(1+\cos 2)\right]. \\
&= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + 2\sin^2 x \sin 1 \cos 1 + 2\sin x\cos x\cos^2 1)\right]. \\
&= -\frac{1}{8\sin 1\cos 1}\left[-x^2\sin 2 + 2\sin x\cos 1(\sin x \sin 1 + \cos x \cos 1)\right]. \\
&= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}.
\end{aligned} \tag{8.79}$$

In Figure 8.4 we show a plot of this solution along with the first five terms of the series solution. The series solution converges quickly.

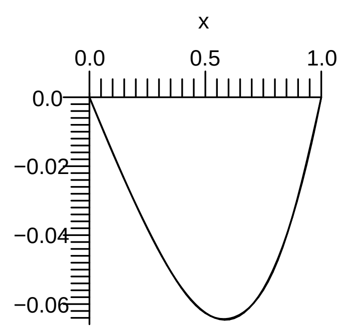

Fig. 8.4. Plots of the exact solution to Example 8.10 with the first five terms of the series solution.

As one last check, we solve the boundary value problem directly, as we had done in Chapter 4. Again, the problem is

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

The problem has the general solution

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + y_p(x),$$

where y_p is a particular solution of the nonhomogeneous differential equation. Using the Method of Undetermined Coefficients, we assume a solution of the form

$$y_p(x) = Ax^2 + Bx + C.$$

Inserting this in the nonhomogeneous equation, we have

$$2A + 4(Ax^2 + Bx + C) = x^2,$$

Thus, $B = 0$, $4A = 1$ and $2A + 4C = 0$. The solution of this system is

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}.$$

So, the general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8}.$$

We now determine the arbitrary constants using the boundary conditions. We have

$$\begin{aligned} 0 &= y(0) \\ &= c_1 - \frac{1}{8} \\ 0 &= y(1) \\ &= c_1 \cos 2 + c_2 \sin 2 + \frac{1}{8} \end{aligned} \tag{8.80}$$

Thus, $c_1 = \frac{1}{8}$ and

$$c_2 = -\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}.$$

Inserting these constants in the solution we find the same solution as before.

$$\begin{aligned} y(x) &= \frac{1}{8}\cos 2x - \left[\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}\right]\sin 2x + \frac{x^2}{4} - \frac{1}{8} \\ &= \frac{\cos 2x \sin 2 - \sin 2x \cos 2 - \sin 2x}{8 \sin 2} + \frac{x^2}{4} - \frac{1}{8} \\ &= \frac{(1 - 2\sin^2 x)\sin 1 \cos 1 - \sin x \cos x (2\cos^2 1 - 1) - \sin x \cos x - \sin 1 \cos 1}{8 \sin 1 \cos 1} + \frac{x^2}{4} \\ &= -\frac{\sin^2 x \sin 1 + \sin x \cos x \cos 1}{4 \sin 1} + \frac{x^2}{4} \\ &= \frac{x^2}{4} - \frac{\sin x \cos(1 - x)}{4 \sin 1}. \end{aligned} \tag{8.81}$$

Problems

8.1. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.
b. $y'' - 4y' + 4y = 6xe^{2x}$.

8.2. Instead of assuming that $c_1' y_1 + c_2' y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1' y_1 + c_2' y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

8.3. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$
b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$
c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$
d. $x^2 y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

8.4. Consider the problem $y'' = \sin x$, $y'(0) = 0$, $y(\pi) = 0$.

a. Solve by direct integration.
b. Determine the Green's function.
c. Solve the boundary value problem using the Green's function.
d. Change the boundary conditions to $y'(0) = 5$, $y(\pi) = -3$.
   i. Solve by direct integration.
   ii. Solve using the Green's function.

8.5. Consider the problem:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0), \quad \frac{\partial G}{\partial x}(0, x_0) = 0, \quad G(\pi, x_0) = 0.$$

a. Solve by direct integration.
b. Compare this result to the Green's function in part b of the last problem.
c. Verify that G is symmetric in its arguments.

8.6. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1 + n^2 x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.
b. Show that the area under each function is one.

8.7. Verify that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, defined by $f_n(x) = \frac{n}{2} e^{-n|x|}$, approaches a delta function.

8.8. Evaluate the following integrals:

a. $\int_0^{\pi} \sin x\delta\left(x - \frac{\pi}{2}\right)\, dx$.
b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\, dx$.
c. $\int_0^{\pi} x^2\delta\left(x + \frac{\pi}{2}\right)\, dx$.
d. $\int_0^{\infty} e^{-2x}\delta(x^2 - 5x + 6)\, dx$. [See Problem 8.10.]
e. $\int_{-\infty}^{\infty} (x^2 - 2x + 3)\delta(x^2 - 9)\, dx$. [See Problem 8.10.]

8.9. Find a Fourier series representation of the Dirac delta function, $\delta(x)$, on $[-L, L]$.

8.10. For the case that a function has multiple simple roots, $f(x_i) = 0$, $f'(x_i) \neq 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2 - 5x + 6)(3x^2 - 7x + 2)\, dx$.

8.11. Consider the boundary value problem: $y'' - y = x$, $x \in (0, 1)$, with boundary conditions $y(0) = y(1) = 0$.

a. Find a closed form solution without using Green's functions.
b. Determine the closed form Green's function using the properties of Green's functions. Use this Green's function to obtain a solution of the boundary value problem.
c. Determine a series representation of the Green's function. Use this Green's function to obtain a solution of the boundary value problem.
d. Confirm that all of the solutions obtained give the same results.