7

Special Functions

In this chapter we will look at some additional functions which arise often in physical applications and are eigenfunctions for some Sturm-Liouville boundary value problem. We begin with a collection of special functions, called the classical orthogonal polynomials. These include such polynomial functions as the Legendre polynomials, the Hermite polynomials, the Tchebychef and the Gegenbauer polynomials. Also, Bessel functions occur quite often. We will spend more time exploring the Legendre and Bessel functions. These functions are typically found as solutions of differential equations using power series methods in a first course in differential equations.

7.1 Classical Orthogonal Polynomials

We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstrass Approximation Theorem this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. We are familiar with being able to expand functions over this basis, since the expansions are just power series representations of the functions,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and (a, b)=$(-1, 1)$. For example,

$$< 1, x^2 >= \int_{-1}^{1} x^0 x^2 \, dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask if it is possible to obtain an orthogonal basis involving these powers of x. Of course, finite combinations of these basis element are just polynomials!

OK, we will ask."Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the **Gram-Schmidt Orthogonalization Process.** We will recall this process for finite dimensional vectors and then generalize to function spaces.

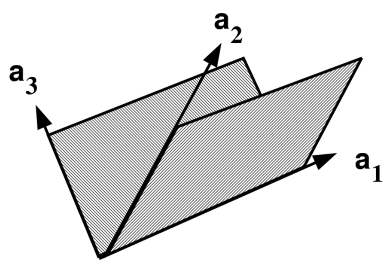

Fig. 7.1. The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$ considered in the text.

Let's assume that we have three vectors that span $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 7.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

Of course, we might want to normalize our new basis vectors, so we would denote such a normalized vector with a "hat":

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 7.2 we see the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. Note that $\mathbf{a}_2$ can be written as a sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{7.1}$$

We recall the projection of one vector onto another from our vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.2}$$

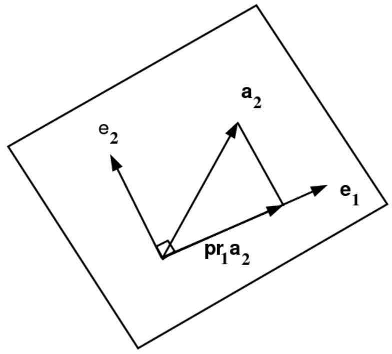

Fig. 7.2. A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

Note that this is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (7.1)-(7.2), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.3}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned}\mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0.\end{aligned} \tag{7.4}$$

Now, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the new vector. This is shown in Figure 7.3. Then we have,

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{7.5}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

We can easily generalize the procedure to the N-dimensional case.

Gram-Schmidt Orthogonalization in N-Dimensions

Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and for $n > 1$,

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j. \tag{7.6}$$

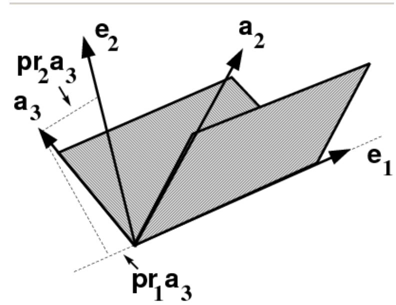

Fig. 7.3. A plot of the vectors and their projections for determining $\mathbf{e}_3$.

Now, we can generalize this idea to (real) function spaces.

Gram-Schmidt Orthogonalization for Function Spaces

Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{< f_n, \phi_j >}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \tag{7.7}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\, dx. \tag{7.8}$$

Note the similarity between the orthogonal basis in (7.7) and the expression for the finite dimensional case in Equation (7.6).

Example 7.1. Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = \frac{1}{2}.$$

We could use this result to fix the normalization of our new basis, but we will hold off on doing that for now.

Now, we compute the second basis element:

$$\begin{aligned}\phi_1(x) &= f_1(x) - \frac{< f_1, \phi_0 >}{\|\phi_0\|^2}\phi_0(x) \\ &= x - \frac{< x, 1 >}{\|1\|^2} 1 = x,\end{aligned} \tag{7.9}$$

since $< x, 1 >$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned}\phi_2(x) &= f_2(x) - \frac{< f_2, \phi_0 >}{\|\phi_0\|^2}\phi_0(x) - \frac{< f_2, \phi_1 >}{\|\phi_1\|^2}\phi_1(x) \\ &= x^2 - \frac{< x^2, 1 >}{\|1\|^2} 1 - \frac{< x^2, x >}{\|x\|^2} x \\ &= x^2 - \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} dx} \\ &= x^2 - \frac{1}{3}.\end{aligned} \tag{7.10}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x) = \phi_1(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see that $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\, dx = \frac{2}{5}.$$

The set of Legendre polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Other orthogonal polynomials in this group are shown in Table 7.1.

For reference, we also note the differential equations satisfied by these functions.

7.2 Legendre Polynomials

In the last section we saw the Legendre polynomials in the context of orthogonal bases for a set of square integrable functions in $L^2(-1, 1)$. In your first course in differential equations, you saw these polynomials as one of the solutions of the differential equation

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty,\infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0,\infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

Table 7.1. Common classical orthogonal polynomials with the interval and weight function used to define them.

Polynomial	Differential Equation
Hermite	$y''-2xy'+2ny=0$
Laguerre	$xy''+(\alpha+1-x)y'+ny=0$
Legendre	$(1-x^2)y''-2xy'+n(n+1)y=0$
Gegenbauer	$(1-x^2)y''-(2n+3)xy'+\lambda y=0$
Tchebychef of the 1st kind	$(1-x^2)y''-xy'+n^2y=0$
Jacobi	$(1-x^2)y''+(\nu-\mu+(\mu+\nu+2)x)y'+n(n+1+\mu+\nu)y=0$

Table 7.2. Differential equations satisfied by some of the common classical orthogonal polynomials.

$$(1-x^2)y''-2xy'+n(n+1)y=0, \quad n\in N_0. \tag{7.11}$$

Recall that these were obtained by using power series expansion methods. In this section we will explore a few of the properties of these functions.

For completeness, we recall the solution of Equation (7.11) using the power series method. We assume that the solution takes the form

$$y(x)=\sum_{k=0}^\infty a_kx^k.$$

The goal is to determine the coefficients, a_k. Inserting this series into Equation (7.11), we have

$$(1-x^2)\sum_{k=0}^\infty k(k-1)a_kx^{k-2}-\sum_{k=0}^\infty 2a_kkx^k+\sum_{k=0}^\infty n(n+1)a_kx^k=0,$$

or

$$\sum_{k=2}^\infty k(k-1)a_kx^{k-2}-\sum_{k=2}^\infty k(k-1)a_kx^k+\sum_{k=0}^\infty\left[-2k+n(n+1)\right]a_kx^k=0.$$

We can combine some of these terms:

$$\sum_{k=2}^\infty k(k-1)a_kx^{k-2}+\sum_{k=0}^\infty\left[-k(k-1)-2k+n(n+1)\right]a_kx^k=0.$$

Further simplification yields

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty} [n(n+1) - k(k+1)]\, a_k x^k = 0.$$

We need to collect like powers of x. This can be done by reindexing each sum. In the first sum, we let $m = k - 2$, or $k = m + 2$. In the second sum we independently let $k = m$. Then all powers of x are of the form x^m. This gives

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=0}^{\infty} [n(n+1) - m(m+1)]\, a_m x^m = 0.$$

Combining these sums, we have

$$\sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} + (n(n+1) - m(m+1))a_m]\, x^m = 0.$$

This has to hold for all x. So, the coefficients of x^m must vanish:

$$(m+2)(m+1)a_{m+2} + (n(n+1) - m(m+1))a_m.$$

Solving for a_{m+2}, we obtain the recursion relation

$$a_{m+2} = \frac{n(n+1) - m(m+1)}{(m+2)(m+1)} a_m, \quad m \geq 0.$$

Thus, a_{m+2} is proportional to a_m. We can iterate and show that each coefficient is either proportional to a_0 or a_1. However, for n an integer, sooner, or later, $m = n$ and the series truncates. $a_m = 0$ for $m > n$. Thus, we obtain polynomial solutions. These polynomial solutions are the Legendre polynomials, which we designate as $y(x) = P_n(x)$. Furthermore, for n an even integer, $P_n(x)$ is an even function and for n an odd integer, $P_n(x)$ is an odd function.

Actually, this is a trimmed down version of the method. We would need to find a second linearly independent solution. We will not discuss these solutions and leave that for the interested reader to investigate.

7.2.1 The Rodrigues Formula

The first property that the Legendre polynomials have is the *Rodrigues formula*:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n \in N_0. \tag{7.12}$$

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

As an example, we determine $P_2(x)$ from Rodrigues formula:

$$\begin{aligned}P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned} \tag{7.13}$$

Note that we get the same result as we found in the last section using orthogonalization.

One can systematically generate the Legendre polynomials in tabular form as shown in Table 7.2.1. In Figure 7.4 we show a few Legendre polynomials.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

Table 7.3. Tabular computation of the Legendre polynomials using the Rodrigues formula.

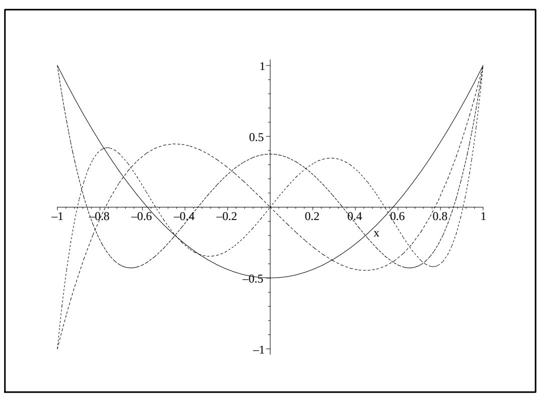

Fig. 7.4. Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

7.2.2 Three Term Recursion Formula

The classical orthogonal polynomials also satisfy *three term recursion formulae.* In the case of the Legendre polynomials, we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 1, 2, \ldots. \tag{7.14}$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{7.15}$$

We will prove this recursion formula in two ways. First we use the orthogonality properties of Legendre polynomials and the following lemma.

Lemma 7.2. *The leading coefficient of* x^n *in* $P_n(x)$ *is* $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.

Proof. We can prove this using Rodrigues formula. first, we focus on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\ldots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\ldots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned}\frac{1}{2^n n!}[2n(2n-1)\ldots(n+1)] &= \frac{1}{2^n n!}[2n(2n-1)\ldots(n+1)]\frac{n(n-1)\ldots 1}{n(n-1)\ldots 1}\\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \end{aligned}\tag{7.16}$$

In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$(2n-1)\frac{1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

It is easy to see that the leading order terms in $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \ldots + c_{n-2}P_{n-2}(x). \quad (7.17)$$

Multiplying Equation (7.17) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1, and using orthogonality, we obtain

$$0 = c_m\|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)\, dx = 0$ for $k \leq n-1$. Thus, $\int_{-1}^{1} xP_{n-1}(x)P_m(x)\, dx = 0$ for $m \leq n-3$.]

Thus, all of these c_m's are zero, leaving Equation (7.17) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$.

7.2.3 The Generating Function

A second proof of the three term recursion formula can be obtained from the *generating function* of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| < 1, |t| < 1. \quad (7.18)$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions. For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 7.5. The tidal potential Φ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

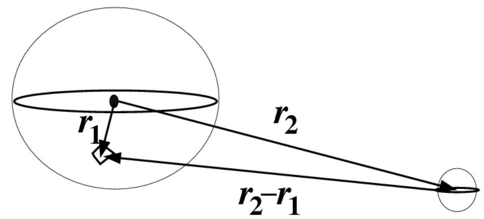

Fig. 7.5. The position vectors used to describe the tidal force on the Earth due to the moon.

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the force is the gradient of the potential, $\mathbf{F} = \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 7.3. Evaluate $P_n(0)$. $P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (7.18), we have

$$g(0,t) = \frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} P_n(0)t^n. \tag{7.19}$$

We can use the binomial expansion to find our final answer. [See the last section of this chapter for a review.] Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 7.10) that

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{7.20}$$

where $n!!$ is the *double factorial*,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, \text{odd}, \\ n(n-2)\ldots(4)2, & n > 0, \text{even}, \\ 1 & n = 0, -1 \end{cases}.$$

Example 7.4. Evaluate $P_n(-1)$. This is a simpler problem. In this case we have

$$g(-1, t) = \frac{1}{\sqrt{1 + 2t + t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots.$$

Therefore, $P_n(-1) = (-1)^n$.

We can also use the generating function to find recursion relations. To prove the three term recursion (7.14) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (7.18) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x - t}{(1 - 2xt + t^2)^{3/2}} = \frac{x - t}{1 - 2xt + t^2} g(x, t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1},$$

we have

$$(x - t) g(x, t) = (1 - 2xt + t^2) \sum_{n=0}^{\infty} n P_n(x) t^{n-1}.$$

Inserting the series expression for $g(x, t)$ and distributing the sum on the right side, we obtain

$$(x - t) \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} - \sum_{n=0}^{\infty} 2nx P_n(x) t^n + \sum_{n=0}^{\infty} n P_n(x) t^{n+1}.$$

Rearranging leads to three separate sums:

$$\sum_{n=0}^{\infty} n P_n(x) t^{n-1} - \sum_{n=0}^{\infty} (2n+1) x P_n(x) t^n + \sum_{n=0}^{\infty} (n+1) P_n(x) t^{n+1} = 0. \tag{7.21}$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n - 1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} n P_n(x) t^{n-1} = \sum_{k=-1}^{\infty} (k+1) P_{k+1}(x) t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty} (k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. However, we will leave the k's in the first term and now reindex the next sums in Equation (7.21). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n + 1$. Therefore, Equation (7.21) becomes

$$\sum_{k=-1}^{\infty} (k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty} kP_{k-1}(x)t^k = 0. \quad (7.22)$$

We can now combine all of the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give

$$P_1(x) - xP_0(x) = 0. \quad (7.23)$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty} \left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right] t^k = 0. \quad (7.24)$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

There are other recursion relations. For example,

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \quad (7.25)$$

This can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 7.4.

Another use of the generating function is to obtain the normalization constant. Namely, $\|P_n\|^2$. Squaring the generating function, we have

$$\frac{1}{1-2xt+t^2} = \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \quad (7.26)$$

Integrating from -1 to 1 and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx. \qquad (7.27)\end{aligned}$$

However, one can show that

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t=0$, we obtain

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (7.27), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}. \qquad (7.28)$$

7.2.4 Eigenfunction Expansions

Finally, we can expand other functions in this orthogonal basis. This is just a generalized Fourier series. A Fourier-Legendre series expansion for $f(x)$ on $[-1,1]$ takes the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \qquad (7.29)$$

As before, we can determine the coefficients by multiplying both sides by $P_m(x)$ and integrating. Orthogonality gives the usual form for the generalized Fourier coefficients. In this case, we have

$$c_n = \frac{< f, P_n >}{\|P_n\|^2},$$

where

$$< f, P_n > = \int_{-1}^{1} f(x)P_n(x)\,dx.$$

We have just found $\|P_n\|^2 = \frac{2}{2n+1}$. Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx. \qquad (7.30)$$

Example 7.5. Expand $f(x) = x^3$ in a Fourier-Legendre series.

We simply need to compute

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} x^3 P_n(x)\, dx. \tag{7.31}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m < n.$$

This is simply proven using Rodrigues formula. Inserting Equation (7.12), we have

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n\, dx.$$

Since $m < n$, we can integrate by parts m-times to show the result, using $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. As a result, we will have for this example that $c_n = 0$ for $n > 3$.

We could just compute $\int_{-1}^{1} x^3 P_m(x)\, dx$ for $m = 0, 1, 2, \ldots$ outright. But, noting that x^3 is an odd function, we easily confirm that $c_0 = 0$ and $c_2 = 0$. This leaves us with only two coefficients to compute. These are

$$c_1 = \frac{3}{2} \int_{-1}^{1} x^4\, dx = \frac{3}{5}$$

and

$$c_3 = \frac{7}{2} \int_{-1}^{1} x^3 \left[\frac{1}{2}(5x^3 - 3x)\right] dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x).$$

Of course, this is simple to check using Table 7.2.1:

$$\frac{3}{5} P_1(x) + \frac{2}{5} P_3(x) = \frac{3}{5} x + \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] = x^3.$$

Well, maybe we could have guessed this without doing any integration. Let's see,

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2} c_2 (5x^3 - 3x) \\ &= (c_1 - \frac{3}{2} c_2) x + \frac{5}{2} c_2 x^3. \end{aligned} \tag{7.32}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2} c_2 = \frac{3}{5}$.

Example 7.6. Expand the Heaviside function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, \ x > 0, \\ 0, \ x < 0. \end{cases} \tag{7.33}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \\ &= \frac{2n+1}{2} \int_{0}^{1} P_n(x)\, dx, \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{7.34}$$

For $n = 0$, we have

$$c_0 = \frac{1}{2} \int_0^1 dx = \frac{1}{2}.$$

For $n > 1$, we make use of the identity (7.25) to find

$$c_n = \frac{1}{2} \int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\, dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

Thus, the Fourier-Bessel series for the Heaviside function is

$$f(x) \sim \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} [P_{n-1}(0) - P_{n+1}(0)] P_n(x).$$

We need to evaluate $P_{n-1}(0) - P_{n+1}(0)$. Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. Letting $n = 2k - 1$, we have

$$f(x) \sim \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} [P_{2k-2}(0) - P_{2k}(0)] P_{2k-1}(x).$$

We can use Equation (7.20),

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!},$$

to compute the coefficients:

$$\begin{aligned} f(x) &\sim \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} [P_{2k-2}(0) - P_{2k}(0)] P_{2k-1}(x) \\ &= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \left[(-1)^{k-1} \frac{(2k-3)!!}{(2k-2)!!} - (-1)^k \frac{(2k-1)!!}{(2k)!!} \right] P_{2k-1}(x) \\ &= \frac{1}{2} - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{(2k-3)!!}{(2k-2)!!} \left[1 + \frac{2k-1}{2k} \right] P_{2k-1}(x) \\ &= \frac{1}{2} - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{(2k-3)!!}{(2k-2)!!} \frac{4k-1}{2k} P_{2k-1}(x). \end{aligned} \tag{7.35}$$

The sum of the first 21 terms are shown in Figure 7.6. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$.

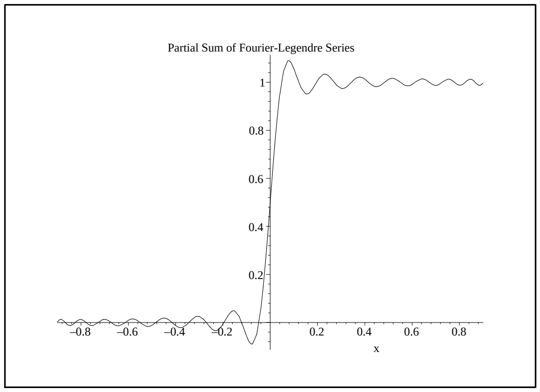

Fig. 7.6. Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

7.3 Gamma Function

Another function that often occurs in the study of special functions is the Gamma function. We will need the Gamma function in the next section on Bessel functions.

For $x > 0$ we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0. \tag{7.36}$$

The Gamma function is a generalization of the factorial function. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 7.7.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x < 0, \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 7.7. We now prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

which can be performed using a standard trick. Consider the integral

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

Then,

$$I^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \int_{-\infty}^\infty e^{-y^2}\, dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2}\, rdrd\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

We have seen that the factorial function can be written in terms of Gamma functions. One can write the even and odd double factorials as

$$(2n)!! = 2^n n!, \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}.$$

In particular, one can write

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

7.4 Bessel Functions

Another important differential equation that arises in many physics applications is

$$x^2 y'' + xy' + (x^2 - p^2)y = 0. \tag{7.37}$$

This equation is readily put into self-adjoint form as

$$(xy')' + (x - \frac{p^2}{x})y = 0. \tag{7.38}$$

This equation was solved in the first course on differential equations using power series methods, namely by using the Frobenius Method. One assumes a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s},$$

and one seeks allowed values of the constant s and a recursion relation for the coefficients, a_n. One finds that $s = \pm p$ and

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2}, \quad n \geq 2.$$

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{7.39}$$

In Figure 7.7 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

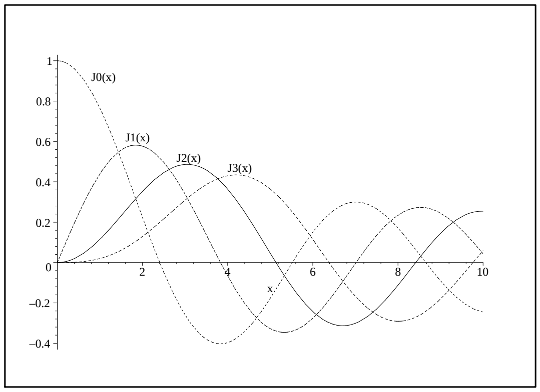

Fig. 7.7. Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{7.40}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

In Figure 7.8 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications these functions do not satisfy the boundary condition that one desires a bounded solution at $x = 0$. For example, one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the wave equation using separation of variables in cylindrical coordinates. The r equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind.

Derivative Identities

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{7.41}$$

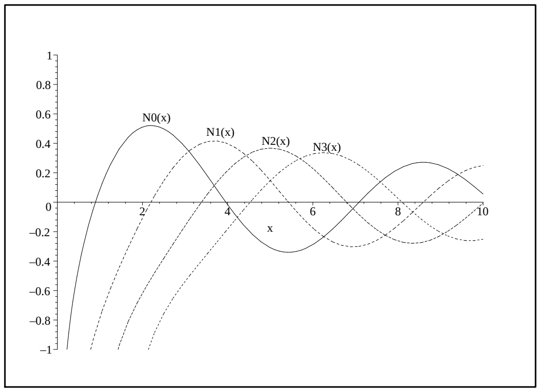

Fig. 7.8. Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{7.42}$$

Recursion Formulae

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x). \tag{7.43}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{7.44}$$

Orthogonality

$$\int_0^a xJ_p(j_{pn}\frac{x}{a})J_p(j_{pm}\frac{x}{a})\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2\delta_{n,m} \tag{7.45}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots are provided in Table 7.4.

n	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
1	2.405	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.147	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813
9	27.494	29.047	30.571	32.050	33.512	34.983

Table 7.4. The zeros of Bessel Functions

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{7.46}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta - n\theta)\, d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{7.47}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of eigenfunctions of a Sturm-Liouville problem, we can expand square integrable functions in this basis. In fact, the eigenvalue problem is given in the form

$$x^2 y'' + xy' + (\lambda x^2 - p^2)y = 0. \tag{7.48}$$

The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation.
Furthermore, one can solve the differential equation on a finite domain, $[0, a]$, with the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. One can show that $J_p(j_{pn}\frac{x}{a})$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0, a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{7.49}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2 \left[J_{p+1}(j_{pn})\right]^2} \int_0^a x f(x) J_p(j_{pn}\frac{x}{a})\, dx. \tag{7.50}$$

Example 7.8. Expand $f(x) = 1$ for $0 \leq x \leq 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (7.50):

$$c_n = \frac{2}{\left[J_1(j_{0n})\right]^2} \int_0^1 x J_0(j_{0n}x)\, dx. \tag{7.51}$$

From Equation (7.41) we have

$$\begin{aligned}\int_0^1 xJ_0(j_{0n}x)\,dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} yJ_0(y)\,dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[yJ_1(y)\right]\,dy \\ &= \frac{1}{j_{0n}^2}\left[yJ_1(y)\right]_0^{j_{0n}} \\ &= \frac{1}{j_{0n}}J_1(j_{0n}). \end{aligned} \tag{7.52}$$

As a result, we have found that the desired Fourier-Bessel expansion is

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \tag{7.53}$$

In Figure 7.9 we show the partial sum for the first fifty terms of this series. We see that there is slow convergence due to the Gibbs' phenomenon. Note: For reference, the partial sums of the Fourier-Bessel series was computed in Maple using the following code:

```
2*sum(BesselJ(0,BesselJZeros(0,n)*x)
/(BesselJZeros(0,n)*BesselJ(1,BesselJZeros(0,n))),n=1..50)
```

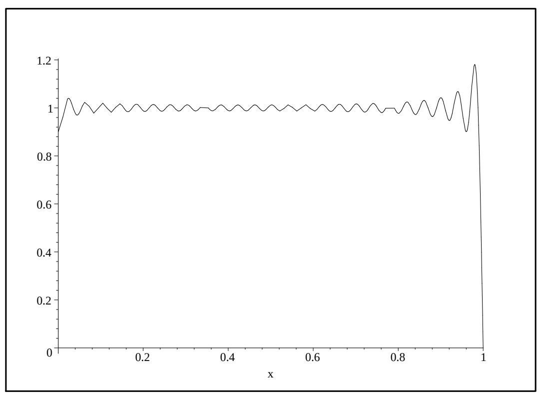

Fig. 7.9. Plot of the first 50 terms of the Fourier-Bessel series in Equation (7.53) for $f(x) = 1$ on $0 < x < 1$.

7.5 Hypergeometric Functions

Hypergeometric functions are probably the most useful, but least understood, class of functions. They typically do not make it into the undergraduate curriculum and seldom in graduate curriculum. Most functions that you know can be expressed using hypergeometric functions. There are many approaches to these functions and the literature can fill books. [1]

In 1812 Gauss published a study of the *hypergeometric series*

$$y(x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(1+\alpha)(1+\beta)}{2!\gamma(1+\gamma)}x^2 + \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{3!\gamma(1+\gamma)(2+\gamma)}x^3 + \ldots . \tag{7.54}$$

Here α, β, γ, and x are real numbers. If one sets $\alpha = 1$ and $\beta = \gamma$, this series reduces to the familiar geometric series

$$y(x) = 1 + x + x^2 + x^3 + \ldots .$$

The hypergeometric series is actually a solution of the differential equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]\, y' - \alpha\beta y = 0. \tag{7.55}$$

This equation was first introduced by Euler and latter studied extensively by Gauss, Kummer and Riemann. It is sometimes called Gauss' equation. Note that there is a symmetry in that α and β may be interchanged without changing the equation. The points $x = 0$ and $x = 1$ are regular singular points. Series solutions may be sought using the Frobenius method. It can be confirmed that the above hypergeometric series results.

A more compact form for the hypergeometric series may be obtained by introducing new notation. One typically introduces the *Pochhammer symbol,* $(\alpha)_n$, satisfying (i) $(\alpha)_0 = 1$ if $\alpha \neq 0$. and (ii) $(\alpha)_k = \alpha(1+\alpha)\ldots(k-1+\alpha)$, for $k = 1, 2, \ldots$.

Consider $(1)_n$. For $n = 0$, $(1)_0 = 1$. For $n > 0$,

$$(1)_n = 1(1+1)(2+1)\ldots[(n-1)+1].$$

This reduces to $(1)_n = n!$. In fact, one can show that

$$(k)_n = \frac{(n+k-1)!}{(k-1)!}$$

for k and n positive integers. In fact, one can extend this result to noninteger values for k by introducing the gamma function:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

We can now write the hypergeometric series in standard notation as

[1] See for example *Special Functions* by G. E. Andrews, R. Askey, and R. Roy, 1999, Cambridge University Press.

$$_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} x^n.$$

Using this one can show that the general solution of Gauss' equation is

$$y(x) = A_2F_1(\alpha, \beta; \gamma; x) + B_2 x_2^{1-\gamma} F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x).$$

By carefully letting β approach ∞, one obtains what is called the *confluent hypergeometric function*. This in effect changes the nature of the differential equation. Gauss' equation has three regular singular points at $x = 0, 1, \infty$. One can transform Gauss' equation by letting $x = u/\beta$. This changes the regular singular points to $u = 0, \beta, \infty$. Letting $\beta \to \infty$, two of the singular points merge.

The new confluent hypergeometric function is then given as

$$_1F_1(\alpha; \gamma; u) = \lim_{\beta \to \infty} {}_2F_1\left(\alpha, \beta; \gamma; \frac{u}{\beta}\right).$$

This function satisfies the differential equation

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

The purpose of this section is only to introduce the hypergeometric function. Many other special functions are related to the hypergeometric function after making some variable transformations. For example, the Legendre polynomials are given by

$$P_n(x) ={}_2 F_1(-n, n+1; 1; \frac{1-x}{2}).$$

In fact, one can also show that

$$\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right).$$

The Bessel function $J_p(x)$ can be written in terms of confluent geometric functions as

$$J_p(x) = \frac{1}{\Gamma(p+1)} \left(\frac{z}{2}\right)^p e^{-iz} {}_1F_1\left(\frac{1}{2}+p, 1+2p; 2iz\right).$$

These are just a few connections of the powerful hypergeometric functions to some of the elementary functions that you know.

7.6 Appendix: The Binomial Expansion

In this section we had to recall the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\cdots
\end{aligned} \tag{7.56}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{7.57}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. Next the second term and next to last term has a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have

$$\begin{array}{lccccccc}
n=2: & & 1 & & 2 & & 1 & \\
 & & & \searrow \; \swarrow & & \searrow \; \swarrow & & \\
n=3: & 1 & & 3 & & 3 & & 1
\end{array} \tag{7.58}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccccccc}
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array} \tag{7.59}$$

Of course, it would take a while to compute each row up to the desired n. We need a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r = k-1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form.

$$C_r^n = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange the products of r b's with $n-r$ a's. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation. The original problem that concerned Pascal was in gambling.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r. \tag{7.60}$$

What if $a \gg b$? Can we use this to get an approximation to $(a+b)^n$? If we neglect b then $(a+b)^n \simeq a^n$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big O notation. In order to do this we first divide out a as

$$(a+b)^n = a^n(1+\frac{b}{a})^n.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen above, we can use the binomial expansion to write

$$(1+\frac{b}{a})^n = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{b}{a}\right)^r. \tag{7.61}$$

Thus, we have a finite sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$(1+\frac{b}{a})^n = 1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

note that we have used the observation that the second coefficient in the nth row of Pascal's triangle is n.

Summarizing, this then gives

$$
\begin{aligned}
(a+b)^n &= a^n(1+\frac{b}{a})^n \\
&= a^n(1+n\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2)\right) \\
&= a^n + na^n\frac{b}{a} + a^nO\left(\left(\frac{b}{a}\right)^2\right). \qquad (7.62)
\end{aligned}
$$

Therefore, we can approximate $(a+b)^n \simeq a^n + nba^{n-1}$, with an error on the order of ba^{n-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^n \simeq a^n$, but it is not as good because the error in this case is of the order ba^{n-1}.

We have seen that

$$
\frac{1}{1-x} = 1 + x + x^2 + \ldots .
$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (7.60).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$
(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r. \qquad (7.63)
$$

However, we quickly run into problems with this form. Consider the coefficient for $r=1$ in an expansion of $(1+x)^{-1}$. This is given by

$$
\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.
$$

But what is $(-1)!$? By definition, it is

$$
(-1)! = (-1)(-2)(-3)\cdots .
$$

This product does not seem to exist! But with a little care, we note that

$$
\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.
$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$
\begin{aligned}
\binom{p}{r} &= \frac{p!}{(p-r)!r!} \\
&= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\
&= \frac{p(p-1)\cdots(p-r+1)}{r!}. \qquad (7.64)
\end{aligned}
$$

With this in mind we now state the theorem:

General Binomial Expansion The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (7.60). For p real, we have that

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r$$
$$= \sum_{r=0}^{\infty} \frac{\Gamma(p+1)}{r!\Gamma(p-r+1)} x^r. \quad (7.65)$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \quad (7.66)$$

Problems

7.1. Consider the set of vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.
b. What do you get if you do reverse the order of these vectors?

7.2. Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty,\infty)$ Weight Function: e^{-x^2}.
b. Interval: $(0,\infty)$ Weight Function: e^{-x}.

7.3. Find $P_4(x)$ using

a. The Rodrigues Formula in Equation (7.12).
b. The three term recursion formula in Equation (7.14).

7.4. Use the generating function for Legendre polynomials to derive the recursion formula $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$. Namely, consider $\frac{\partial g(x,t)}{\partial x}$ using Equation (7.18) to derive a three term derivative formula. Then use three term recursion formula (7.14) to obtain the above result.

7.5. Use the recursion relation (7.14) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \le m$.

7.6. Expand the following in a Fourier-Legendre series for $x \in (-1,1)$.

a. $f(x) = x^2$.
b. $f(x) = 5x^4 + 2x^3 - x + 3$.
c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

7.7. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

7.8. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^\infty x^{2/3} e^{-x}\,dx$.
b. $\int_0^\infty x^5 e^{-x^2}\,dx$
c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx$

7.9. The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $< H_n, H_m >= \int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x)\,dx = \sqrt{\pi}2^n n!\delta_{n,m}$.
ii. $H_n'(x) = 2nH_{n-1}(x)$.
iii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.
iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2xH_n' + 2nH_n = 0$. [Use properties ii. and iii.]
b. $\int_{-\infty}^\infty xe^{-x^2} H_n(x) H_m(x)\,dx = \sqrt{\pi}2^{n-1}n!\left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]
c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 1$.]

7.10. In Maple one can type **simplify(LegendreP(2*n-2,0)-LegendreP(2*n,0));** to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 7.6 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.
b. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x) = (x-1)\Gamma(x-1)$ and iteration.
c. Verify the result from Maple that $P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.
d. Can either expression for $P_{2n-2}(0) - P_{2n}(0)$ be simplified further?

7.11. A solution Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0$, , can be found using the guess $y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)} a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.$$

7.12. Use the infinite series in the last problem to derive the derivative identities (7.41) and (7.42):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.
b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

7.13. Bessel functions $J_p(\lambda x)$ are solutions of $x^2y'' + xy' + (\lambda^2 x^2 - p^2)y = 0$. Assume that $x \in (0, 1)$ and that $J_p(\lambda) = 0$ and $J_p(0)$ is finite.

a. Put this differential equation into Sturm-Liouville form.
b. Prove that solutions corresponding to different eigenvalues are orthogonal by first writing the corresponding Green's identity using these Bessel functions.
c. Prove that

$$\int_0^1 x J_p(\lambda x) J_p(\mu x)\, dx = \frac{1}{2} J_{p+1}^2(\lambda) = \frac{1}{2} J_p'^2(\lambda).$$

Note that λ is a zero of $J_p(x)$.

7.14. We can rewrite our Bessel function in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 7.10b for $\Gamma\left(k + \frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 7.11 using the gamma function.
b. Extend the series to $J_{-\nu(x)}$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.
c. Use these results to obtain closed form expressions for $J_{1/2}(x)$ and $J_{-1/2}(x)$. Use the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

7.15. In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4\sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j's$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 7.4 and Figure 7.7. [Note: Be careful determining α_1.]

b. Show that $\|J_0(\alpha_1 x)\|^2 = \frac{c^2}{2}$. Recall,

$$\|J_0(\alpha_j x)\|^2 = \int_0^c x J_0^2(\alpha_j x)\, dx.$$

c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2}\left[J_0(\alpha_j c)\right]^2$, $j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 7.13 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + xy' + \alpha_j^2 x^2 y = 0.$$

i. Show that the Sturm-Liouville form of this differential equation is $(xy')' = -\alpha_j^2 xy$.

ii. Multiply the equation in part i. by $y(x)$ and integrate from $x = 0$ to $x = c$ to obtain

$$\begin{aligned} \int_0^c (xy')' y\, dx &= -\alpha_j^2 \int_0^c xy^2\, dx \\ &= -\alpha_j^2 \int_0^c x J_0^2(\alpha_j x)\, dx. \end{aligned} \tag{7.67}$$

iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.

1. $J_0'(x) = -J_1(x)$ from Equation (7.42).
2. Equation (7.45).
3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (7.43).

iv. Now you should have enough information to complete this part.

d. Use the results from parts b and c to derive the expansion coefficients for

$$x^2 = \sum_{j=1}^{\infty} c_j J_0(\alpha_j x)$$

in order to obtain the desired expansion.

7.16. Use the derivative identities of Bessel functions,(7.41)-(7.42), and integration by parts to show that

$$\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x).$$