5

Fourier Series

5.1 Introduction

In this chapter we will look at trigonometric series. Previously, we saw that such series expansion occurred naturally in the solution of the heat equation and other boundary value problems. In the last chapter we saw that such functions could be viewed as a basis in an infinite dimensional vector space of functions. Given a function in that space, when will it have a representation as a trigonometric series? For what values of x will it converge? Finding such series is at the heart of Fourier, or spectral, analysis.

There are many applications using spectral analysis. At the root of these studies is the belief that many continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagorean study of the vibrations of strings, which lead to their view of a world of harmony. This idea was carried further by Johannes Kepler in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis, Brook Taylor and Jean le Rond d'Alembert.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, his solution led himself and others, like Leonhard Euler and Daniel Bernoulli, to investigate what "functions" could be the solutions of this equation. In fact, this lead to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x, 0) = h(x)$ has a discontinuous derivative!

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$. However, the initial conditions for such superpositions are

$$y(x,0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x,0) = \begin{cases} cx, & 0 \leq x \leq L/2 \\ c(L-x), & L/2 \leq x \leq L \end{cases}.$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's solution of the heat equation. The use of such Fourier expansions became an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. As seen in the last chapter, using the Method of Separation of Variables, allows higher dimensional problems to be reduced to several one dimensional boundary value problems. However, these studies lead to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums?
3. Do such infinite sums of trigonometric functions a actually converge to the functions they represents?

Sums over sinusoidal functions naturally occur in music and in studying sound waves. A pure note can be represented as

$$y(t) = A \sin(2\pi f t),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume, or intensity, of the sound. The larger the amplitude, the louder the sound. In Figure 5.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with

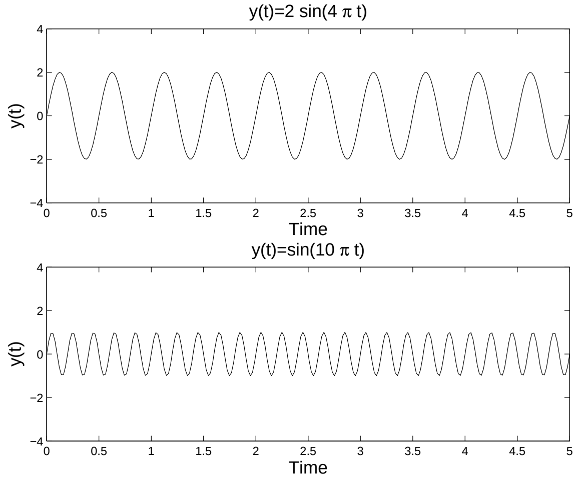

Fig. 5.1. Plots of $y(t) = \sin(2\pi f t)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

different amplitudes and frequencies. In Figure 5.2 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 5.2, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the *period.* We can define this more precisely.

Definition 5.1. *A function is said to be periodic with period T if $f(t+T) = f(t)$ for all t and the smallest such positive number T is called the period.*

For example, we consider the functions used in Figure 5.2. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 5.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

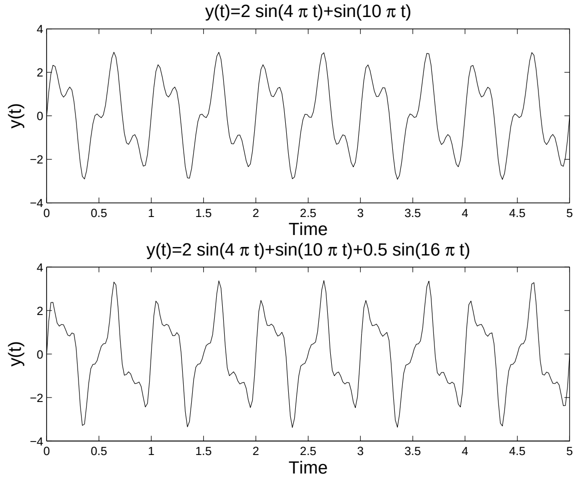

Fig. 5.2. Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and and $f = 8$ Hz.

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to the superpositions in Figure 5.2, we have that $y(t) = \sin(10\pi t)$ has a period of 0.2 Hz and $y(t) = \sin(16\pi t)$ has a period of 0.125 Hz. The two superpositions retain the largest period of the signals added, which is 0.5 Hz.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. First of all, we will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of sinusoidal functions.

Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like “Row, Row, Row your Boat” starting at slightly different times.

We can easily add shifted sine functions. In Figure 5.3 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/8$.

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up our signal, $f(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function. Consider the general shifted function

$$y(t) = A\sin(2\pi ft + \phi).$$

Note that $2\pi ft + \phi$ is called the *phase* of our sine function and ϕ is called the *phase shift*. We can use our trigonometric identity for the sine of the sum of two angles to obtain

$$y(t) = A\sin(2\pi ft + \phi) = A\sin(\phi)\cos(2\pi ft) + A\cos(\phi)\sin(2\pi ft).$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi ft) + b\sin(2\pi ft).$$

Thus, we see that our signal is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2} \quad \tan\phi = \frac{b}{a}.$$

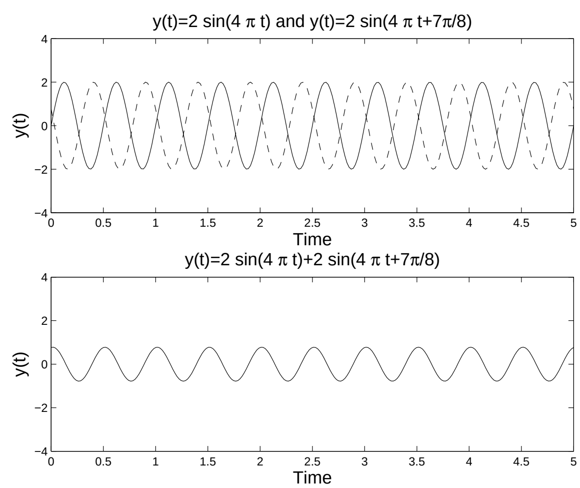

Fig. 5.3. Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

We are now in a position to state our goal in this chapter.

Goal

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called *Fourier Analysis.*

5.2 Fourier Trigonometric Series

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{5.1}$$

Notice that we have opted to drop reference to the frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

The series representation in Equation (5.1) is called a *Fourier trigonometric series.* We will simply refer to this as a *Fourier series* for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients.* The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that the infinite series is periodic. The largest period of the terms comes from the $n = 1$ terms. The periods of $\cos x$ and $\sin x$ are $T = 2\pi$. Thus, the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 5.4 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π.

Fourier Coefficients

Theorem 5.2. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$ when it exists, is given by (5.1) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.2}$$

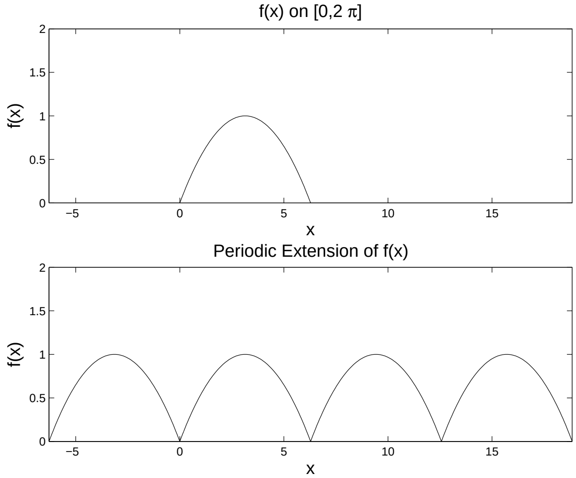

Fig. 5.4. Plot of the functions $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will look at the derivations of the a_n's. First we obtain a_0.

We begin by integrating the Fourier series term by term in Equation (5.1).

$$\int_0^{2\pi} f(x)\, dx = \int_0^{2\pi} \frac{a_0}{2}\, dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\, dx. \tag{5.3}$$

We assume that we can integrate the infinite sum term by term. Then we need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\, dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\, dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\, dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{5.4}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\, dx = \pi a_0.$$

This confirms the value for a_0.

Next, we need to find a_n. We will multiply the Fourier series (5.1) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\int_0^{2\pi} f(x)\cos mx\,dx = \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]\cos mx\,dx. \tag{5.5}$$

Integrating term by term, the right side becomes

$$\frac{a_0}{2}\int_0^{2\pi} \cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos mx\,dx + b_n \int_0^{2\pi} \sin nx \cos mx\,dx\right]. \tag{5.6}$$

We have already established that $\int_0^{2\pi} \cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals. For future reference, we list several useful identities, some of which we will prove along the way.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \tag{5.7}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{5.8}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{5.9}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{5.10}$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)) \tag{5.11}$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)) \tag{5.12}$$

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \tag{5.13}$$

We first want to evaluate $\int_0^{2\pi} \cos nx \cos mx\,dx$. We do this by using the product identity. We had done this in the last chapter, but will repeat the derivation for the reader's benefit. Recall the addition formulae for cosines:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Adding these equations gives

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$$

We can use this identity with $A = mx$ and $B = nx$ to complete the integration. We have

$$\begin{aligned}\int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{5.14}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

So, for $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx \, dx$. This can also be handled using a trigonometric identity. Recall that

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta.)$$

Inserting this into the integral, we find

$$\begin{aligned}\int_0^{2\pi} \cos^2 mx \, dx &= \frac{1}{2} \int_0^{2\pi} (1 + \cos^2 2mx) \, dx \\ &= \frac{1}{2} \left[x + \frac{1}{2m} \sin 2mx \right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{5.15}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{5.16}$$

This holds true for $m, n = 0, 1, \dots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition 5.3. *A set of (real) functions* $\{\phi_n(x)\}$ *is said to be orthogonal on* $[a, b]$ *if* $\int_a^b \phi_n(x)\phi_m(x) \, dx = 0$ *when* $n \neq m$. *Furthermore, if we also have that* $\int_a^b \phi_n^2(x) \, dx = 1$, *these functions are called orthonormal.*

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (5.15).

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x)\,dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. But that is another topic for later.

Returning to the evaluation of the integrals in equation (5.6), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\,dx$. This can also be evaluated using trigonometric identities. In this case, we need an identity involving products of sines and cosines. Such products occur in the addition formulae for sine functions:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A.$$

Adding these equations, we find that

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned}
\int_0^{2\pi} \sin nx \cos mx\,dx &= \frac{1}{2}\int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]\,dx \\
&= \frac{1}{2}\left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m}\right]_0^{2\pi} \\
&= (-1+1) + (-1+1) = 0. \qquad (5.17)
\end{aligned}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx\,dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx\,dx = \frac{1}{2}\left[\frac{-\cos 2mx}{2m}\right]_0^{2\pi} = 0.$$

Finally, we can finish our evaluation of (5.6). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx\,dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos mx\,dx.$$

Since this is true for all $m = 1, 2, \dots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 5.4. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \, dx = 0. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \cos nx \, dx = 0, \quad n \neq 2. \\
a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x \, dx = 3, \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \sin nx \, dx = 0, \forall n.
\end{aligned}
\tag{5.18}
$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$. Well, we should have know this before doing all of these integrals. So, if we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 5.5. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$
\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.
$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \dots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$.

Example 5.6. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals at this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

$$
a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \, dx
$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (-1)\,dx \\
&= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi + \pi) = 0. \qquad (5.19)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_{\pi}^{2\pi}\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\
&= 0. \qquad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\
&= \frac{2}{n\pi}(1 - \cos n\pi). \qquad (5.21)
\end{aligned}
$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (5.1), we note that $\cos n\pi = (-1)^n$. Therefore,

$$
1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \qquad (5.22)
$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$
f(x) \sim \frac{4}{\pi}\sum_{n=1,\ \text{odd}}^{\infty} \frac{1}{n}\sin nx,
$$

we could let $n = 2k - 1$ and write

$$
f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1},
$$

But does this series converge? Does it converge to $f(x)$? We will discuss this question later in the chapter.

5.3 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L. Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to our Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 5.5. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$.

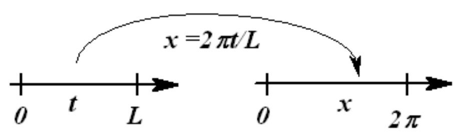

Fig. 5.5. A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

Recall the form of the Fourier representation for $f(x)$ in Equation (5.1):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{5.23}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L}\right]. \tag{5.24}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L}dt$. Thus, the resulting form for our coefficient is

$$a_n = \frac{2}{L}\int_0^{L} g(t)\cos\frac{2n\pi t}{L}\,dt. \tag{5.25}$$

Similarly, we find that

$$b_n = \frac{2}{L}\int_0^{L} g(t)\sin\frac{2n\pi t}{L}\,dt. \tag{5.26}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos\frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 5.1 we summarize some commonly used Fourier series representations.

We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

Example 5.7. Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \pi \end{aligned} \tag{5.33}$$

At this point we need to remind the reader about the integration of even and odd functions.

1. **Even Functions:** In this evaluation we made use of the fact that the integrand is an even function. Recall that $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 5.6(A). If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx. \tag{5.34}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

Table 5.1. Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{5.27}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.28}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{5.29}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.30}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{5.31}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.32}$$

$$\begin{aligned} \int_{-a}^{a} f(x)\, dx &= \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx \\ &= -\int_a^0 f(-y)\, dy + \int_0^a f(x)\, dx \\ &= \int_0^a f(y)\, dy + \int_0^a f(x)\, dx \\ &= 2 \int_0^a f(x)\, dx. \end{aligned} \tag{5.35}$$

This can be visually verified by looking at Figure 5.6(A).

2. **Odd Functions:** A similar computation could be done for odd functions. $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 5.6(B). If one integrates an odd function over a symmetric interval, then one has that
$$\int_{-a}^{a} f(x)\,dx = 0. \tag{5.36}$$

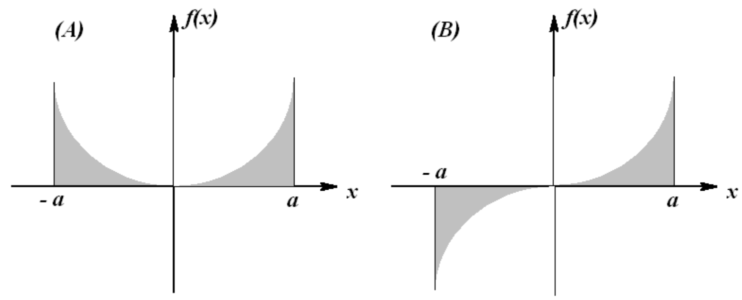

Fig. 5.6. Examples of the areas under (A) even and (B) odd functions on symmetric intervals, $[-a, a]$.

We now continue with our computation of the Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \tag{5.37}$$
Here we have made use of the fact that $|x|\cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts,
$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$
by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$. Continuing with the computation, we have
$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(1-(-1)^n). \end{aligned} \tag{5.38}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This lead to a factor $(1-(-1)^n)$. This factor can be simplified as

$$1-(-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} . \tag{5.39}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x| \sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \frac{\cos nx}{n^2}. \tag{5.40}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k-1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{5.41}$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 5.7 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 5.8, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 5.1. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{5.42}$$

As seen in Figure 5.9 we again obtain the periodic extension of our function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

Example 5.8. It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

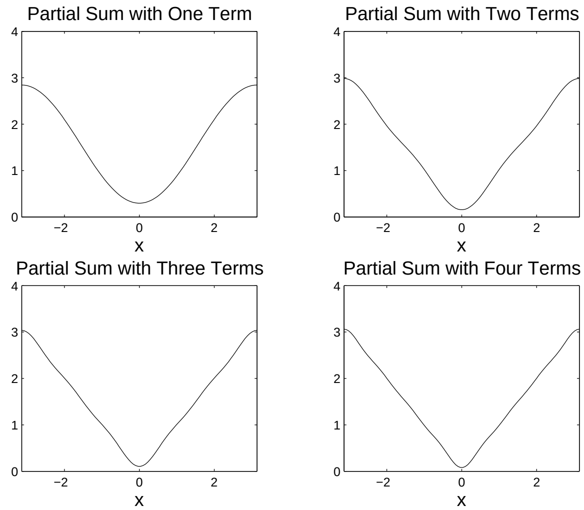

Fig. 5.7. Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

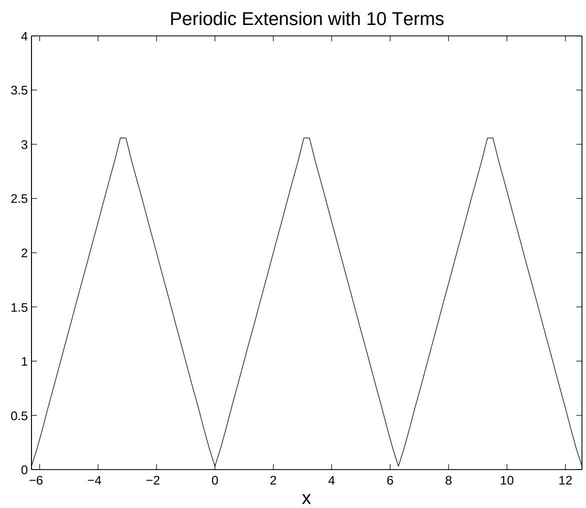

Fig. 5.8. Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right].$$

This gives a well known expression for π:

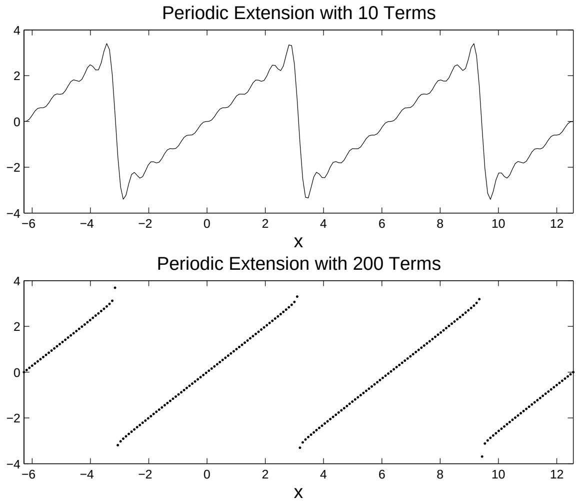

Fig. 5.9. Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right].$$

5.3.1 Fourier Series on $[a, b]$

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t - a}{b - a}.$$

Inserting this into the Fourier series (5.1) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t - a)}{b - a} + b_n \sin \frac{2n\pi(t - a)}{b - a}\right]. \tag{5.43}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right]. \end{aligned} \tag{5.44}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ B_n &\equiv a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a}, \end{aligned} \tag{5.45}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \tag{5.46}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \end{aligned} \tag{5.47}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \end{aligned} \tag{5.48}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$
\begin{aligned}
A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\
&= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\
&= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a}\, dt. \qquad (5.49)
\end{aligned}
$$

A similar computation gives

$$
B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a}\, dt. \qquad (5.50)
$$

Summarizing, we have shown that:

Theorem 5.9. *The Fourier series representation of $f(x)$ defined on $[a, b]$ when it exists, is given by*

$$
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \qquad (5.51)
$$

with Fourier coefficients

$$
\begin{aligned}
a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}\, dx. \quad n = 0, 1, 2, \ldots, \\
b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}\, dx. \quad n = 1, 2, \ldots. \qquad (5.52)
\end{aligned}
$$

5.4 Sine and Cosine Series

In the last two examples ($f(x) = |x|$ and $f(x) = x$ on $[-\pi, \pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 5.7-5.9 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 5.10 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

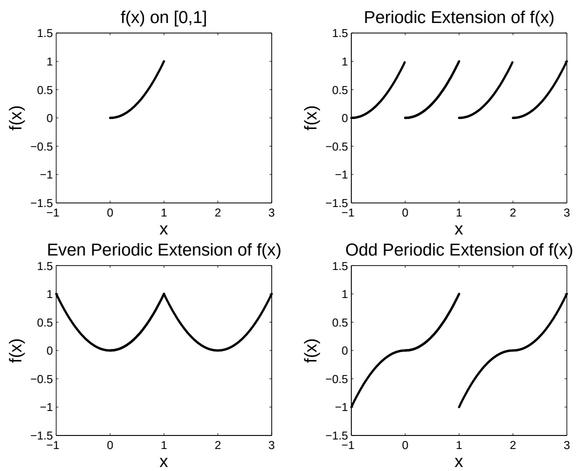

Fig. 5.10. This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0, 1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 5.2] We have already seen that the periodic extension of $f(x)$ is obtained through the Fourier series representation in Equation (5.53).

Table 5.2. Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{5.53}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.54}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{5.55}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{5.56}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{5.57}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{5.58}$$

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{5.59}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (5.51) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{5.60}$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots. \tag{5.61}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{5.62}$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots. \tag{5.63}$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{5.64}$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{5.65}$$

where

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \tag{5.66}$$

Example 5.10. In Figure 5.10 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 5.3.

Example 5.11. **Periodic Extension - Trigonometric Fourier Series**

Using the above code, we have that $a_0 = \frac{2}{3}$ $a_n = \frac{1}{n^2\pi^2}$ and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right].$$

Table 5.3. Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                              a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                     1
                            an := -------
                                    2   2
                                  n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                      1
                            bn := - -----
                                    n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=`Periodic Extension`,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

In Figure 5.11 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the appendix.

Example 5.12. **Even Periodic Extension - Cosine Series**

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

In Figure 5.12 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

Example 5.13. **Odd Periodic Extension - Sine Series**

Finally, we look at the sine series for this function. We find that $b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. Therefore,

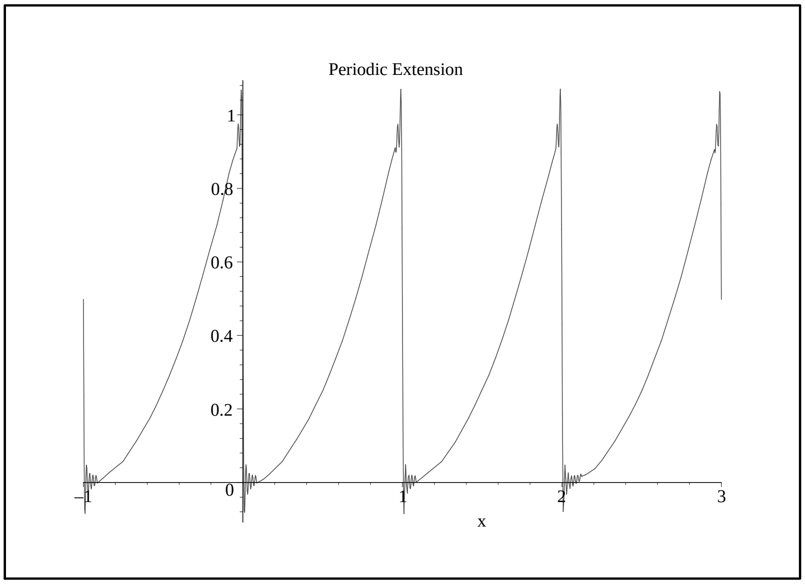

Fig. 5.11. The periodic extension of $f(x) = x^2$ on $[0, 1]$.

$$f(x) \sim -\frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} (n^2\pi^2(-1)^n - 2(-1)^n + 2) \sin n\pi x.$$

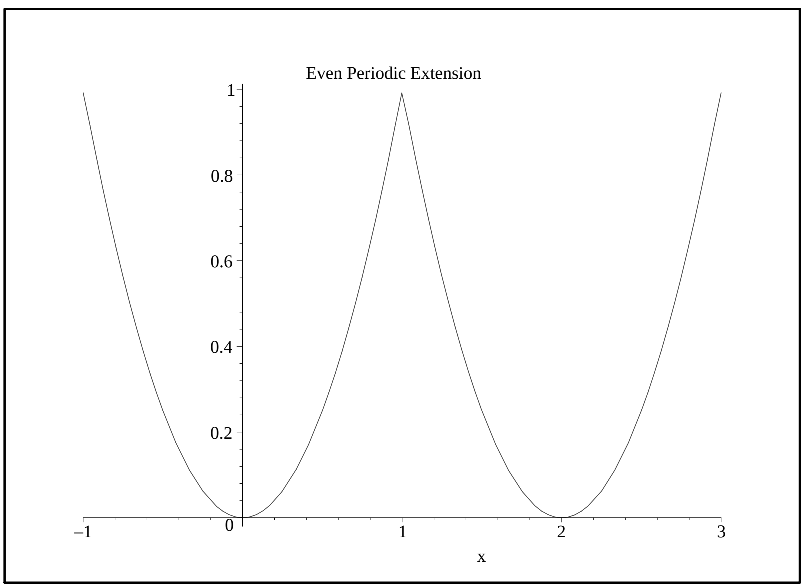

Fig. 5.12. The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

Once again we see discontinuities in the extension as seen in Figure 5.13. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 5.10.

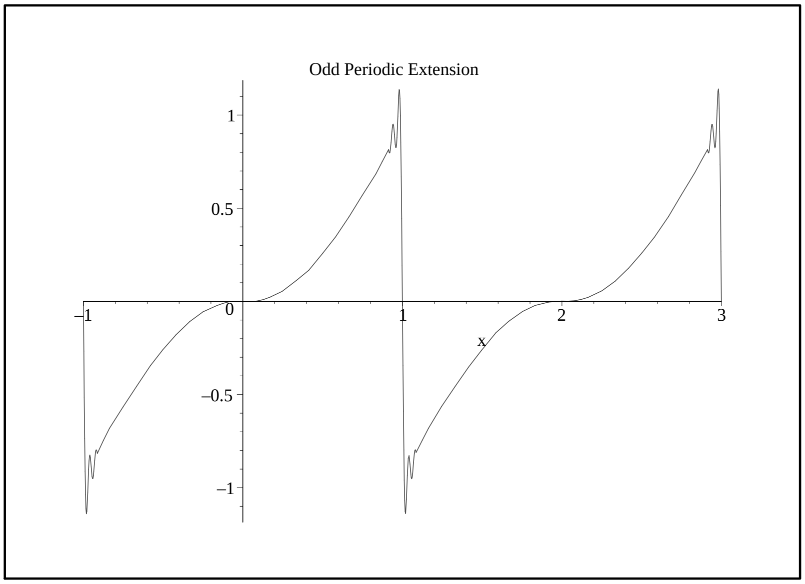

Fig. 5.13. The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.

5.5 Appendix: The Gibbs Phenomenon

We have seen that when there is a jump discontinuity in the periodic extension of our functions, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 5.9, 5.11 and 5.13. The Fourier series has a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the *Gibbs phenomenon* and the amount of overshoot can be computed.

In one of our first examples, Example 5.6, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

In Figure 5.14 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots near $x = 0, \pm\pi$. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 5.15. We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

One might expect that we only need to add more terms. In Figure 5.16 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 5.17 and 5.18 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

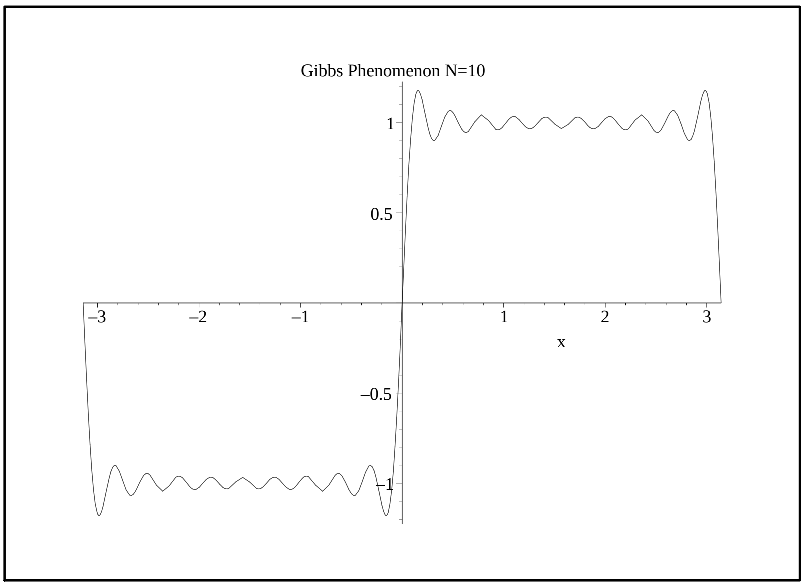

Fig. 5.14. The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

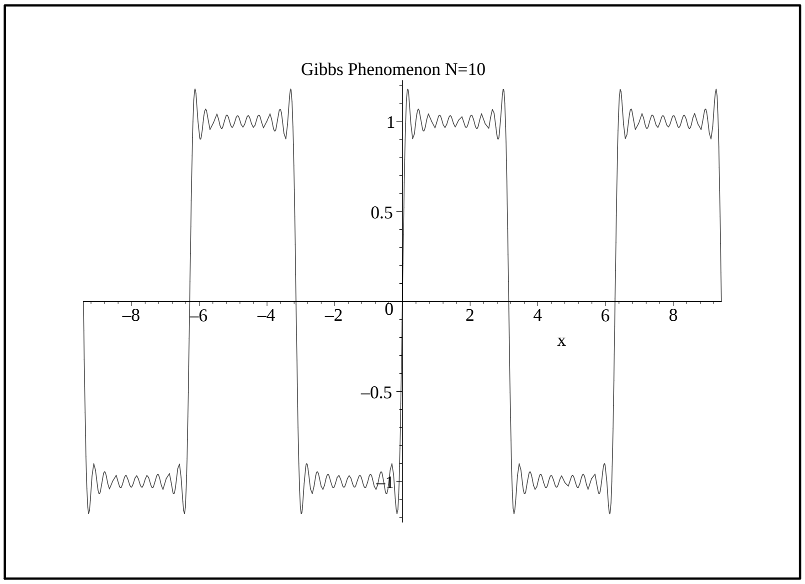

Fig. 5.15. The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

$$S_N(x) = a_0 + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

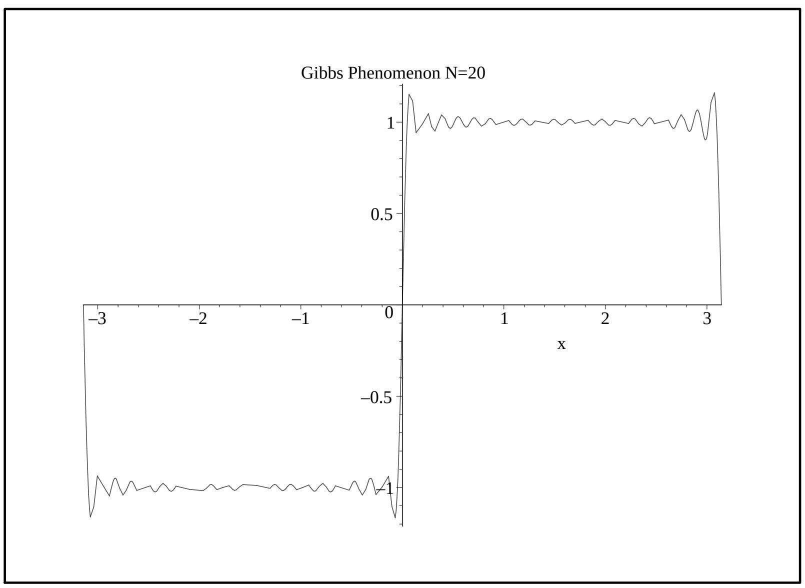

Fig. 5.16. The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

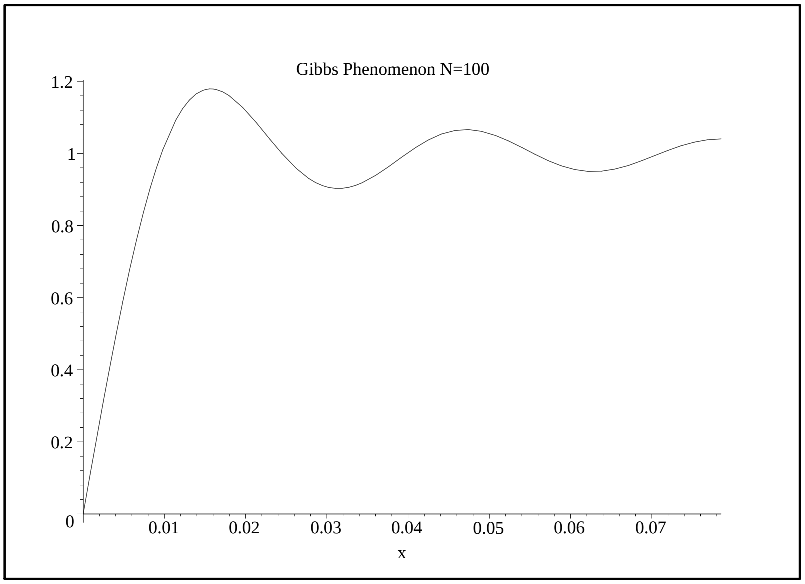

Fig. 5.17. The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

$$
\begin{aligned}
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L}\right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy\right)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy. \qquad (5.67)
\end{aligned}
$$

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L},$$

which is called the *N-th Dirichlet Kernel*. We now prove

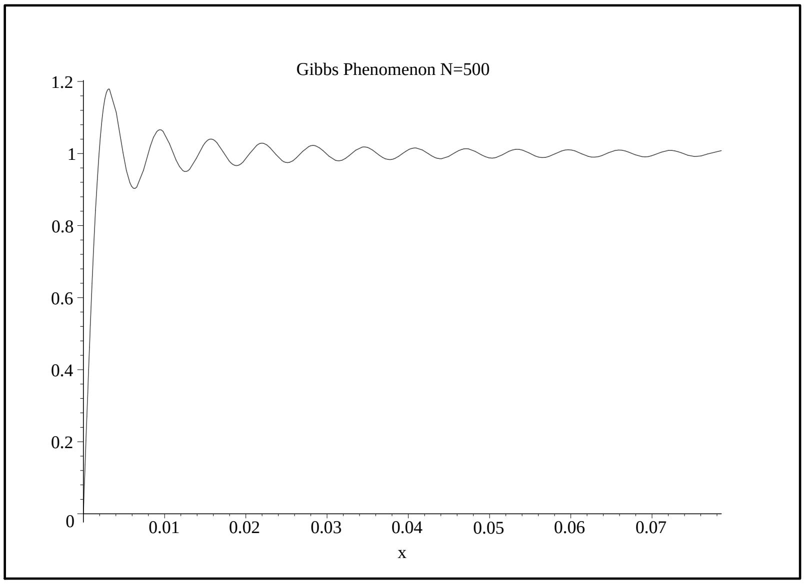

Fig. 5.18. The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

Proposition:

$$D_n(x) = \begin{cases} \frac{\sin((n+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0 \\ n + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0 \end{cases}.$$

Proof: Let $\theta = \frac{\pi x}{L}$ and multiply $D_n(x)$by $2\sin\frac{\theta}{2}$to obtain:

$$\begin{aligned} 2\sin\frac{\theta}{2}D_n(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2} + \cos\theta + \cdots + \cos n\theta\right] \\ &= \sin\frac{\theta}{2} + 2\cos\theta\sin\frac{\theta}{2} + 2\cos 2\theta\sin\frac{\theta}{2} + \cdots + 2\cos n\theta\sin\frac{\theta}{2} \\ &= \sin\frac{\theta}{2} + \left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right) + \left(\sin\frac{5\theta}{2} - \sin\frac{3\theta}{2}\right) + \cdots \\ &\quad + \left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\left(n-\frac{1}{2}\right)\theta\right] \\ &= \sin\left(n+\frac{1}{2}\right)\theta. \end{aligned} \tag{5.68}$$

Thus,

$$2\sin\frac{\theta}{2}D_n(x) = \sin\left(n+\frac{1}{2}\right)\theta,$$

or if $\sin\frac{\theta}{2} \neq 0$,

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta = \frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2} = 0$,then one needs to apply L'Hospital's Rule:

$$\begin{aligned} \lim_{\theta\to 2m\pi}\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\ &= \frac{\left(n+\frac{1}{2}\right)\cos\left(2mn\pi + m\pi\right)}{\cos m\pi} \\ &= n + \frac{1}{2}. \end{aligned} \tag{5.69}$$

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that

$$S_N(x) = \frac{1}{L} \int_{-L}^{L} D_N(y - x) f(y)\, dy. \tag{5.70}$$

Now, make the substitution $\xi = y - x$. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L} \int_{-L-x}^{L-x} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L} \int_{-L}^{L} D_N(\xi) f(\xi + x)\, d\xi. \end{aligned} \tag{5.71}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

Now split the integration and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L} \int_{-L}^{0} D_N(\xi) f(\xi + x)\, d\xi + \frac{1}{L} \int_{0}^{L} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L} \int_{0}^{L} \left[f(x - \xi) + f(\xi + x) \right] D_N(\xi)\, d\xi. \end{aligned} \tag{5.72}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on our earlier example. Namely,

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

For this case, we have

$$S_N(x) = \frac{1}{\pi} \int_{0}^{\pi} \left[f(x - \xi) + f(\xi + x) \right] D_N(\xi)\, d\xi \tag{5.73}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx.$$

Also, one can show that

$$f(x - \xi) + f(\xi + x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$
\begin{aligned}
S_N(x) &= \frac{2}{\pi}\int_0^x D_N(\xi)\,d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)\,d\xi \\
&= \frac{2}{\pi}\int_0^x D_N(z)\,dz + \frac{2}{\pi}\int_0^x D_N(\pi-z)\,dz.
\end{aligned} \tag{5.74}
$$

Here we made the substitution $z=\pi-\xi$ in the second integral. The Dirichlet kernel in the proposition for $L=\pi$ is given by

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N+\frac{1}{2}\approx N$, and for small x, we have $\sin\frac{x}{2}\approx\frac{x}{2}$. So, under these assumptions,

$$D_N(x)\approx\frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x)\to\frac{2}{\pi}\int_0^x\frac{\sin N\xi}{\xi}\,d\xi.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x)=\frac{2}{\pi}\frac{\sin Nx}{x}=0.$$

The extrema occur for $Nx=m\pi$, $m=\pm1,\pm2,\ldots$. One can show that there is a maximum at $x=\pi/N$ and a minimum for $x=2\pi/N$. The value for the overshoot can be computed as

$$
\begin{aligned}
S_N(\pi/N) &= \frac{2}{\pi}\int_0^{\pi/N}\frac{\sin N\xi}{\xi}\,d\xi \\
&= \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt \\
&= \frac{2}{\pi}\mathrm{Si}(\pi) \\
&= 1.178979744\ldots.
\end{aligned} \tag{5.75}
$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\mathrm{Si}(x)\equiv\int_0^x\frac{\sin t}{t}\,dt.$$

Problems

5.1. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some simple Maple code for computing partial sums is shown below.]

a. $f(x) = x$, $|x| < \pi$.
b. $f(x) = \frac{x^2}{4}$, $|x| < \pi$.
c. $f(x) = \pi - |x|$, $|x| < \pi$.
d. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$
e. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

A simple set of commands in Maple are shown below, where you fill in the Fourier coefficients that you have computed by hand and $f(x)$ so that you can compare your results. Of course, other modifications may be needed.

```
> restart:
> f:=x:
> F:=a0/2+sum(an*cos(n*x)+bn*sin(n*x),n=1..N):
> N:=10: plot({f,F},x=-Pi..Pi,color=black);
```

5.2. Consider the function $f(x) = 4\sin^3 2x$

a. Derive an identity relating $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ and express $f(x)$ in terms of simple sine functions.
b. Determine the Fourier coefficients of $f(x)$ in a Fourier series expansion on $[0, 2\pi]$ without computing any integrals!

5.3. Find the Fourier series of $f(x) = x$ on the given interval with the given period T. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$, $T = 2$.
b. $-2 < x < 2$, $T = 4$.

5.4. The result in problem 5.1b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 5.8.]

a.
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

5.5. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

a. $f(x) = x^2$, $0 < x < 1$.
b. $f(x) = x(2 - x)$, $0 < x < 2$.
c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$
d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$