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A Second Course in Ordinary
Differential Equations:
Dynamical Systems and
Boundary Value Problems

Monograph

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