4

Boundary Value Problems

4.1 Introduction

Until this point we have solved initial value problems. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{4.1}$$

In the next chapters we will study boundary value problems and various tools for solving such problems. In this chapter we will motivate our interest in boundary value problems by looking into solving the one-dimensional heat equation, which is a partial differential equation. for the rest of the section, we will use this solution to show that in the background of our solution of boundary value problems is a structure based upon linear algebra and analysis leading to the study of inner product spaces. Though technically, we should be lead to Hilbert spaces, which are complete inner product spaces.

For an initial value problem one has to solve a differential equation subject to conditions on the unknown function or its derivatives at more than one value of the independent variable. As an example, we have a slight modification of the above problem: Find the solution $x = x(t)$ for $0 \leq t \leq 1$ that satisfies the problem

$$x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0. \tag{4.2}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge.

The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary value conditions in our discussions and exercises.

Let's solve the above boundary value problem. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so we have that the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is easily found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to our problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as many of the initial values problems we have seen. We will see in the next section that boundary value problems for ordinary differential equations often appear in the solution of partial differential equations.

4.2 Partial Differential Equations

In this section we will introduce some generic partial differential equations and see how the discussion of such equations leads naturally to the study of boundary value problems for ordinary differential equations. However, we will not derive the particular equations, leaving that to courses in differential equations, mathematical physics, etc.

For ordinary differential equations, the unknown functions are functions of a single variable, e.g., $y = y(x)$. Partial differential equations are equations involving an unknown function of several variables, such as $u = u(x, y)$, $u = u(x, y)$, $u = u(x, y, z, t)$, and its (partial) derivatives. Therefore, the derivatives are partial derivatives. We will use the standard notations $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

There are a few standard equations that one encounters. These can be studied in one to three dimensions and are all linear differential equations. A list is provided in Table 4.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems for ordinary differential equations.

Name	2 Vars	3 D
Heat Equation	$u_t = k u_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x, t)u$	$iu_t = \nabla^2 u + F(x, y, z, t)u$

Table 4.1. List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation.

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x, 0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end for all time. So, the problem one would need to solve is given as

1D Heat Equation

$$\begin{array}{lll} \text{PDE} & u_t = k u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.3}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed. Think of a violin string or a guitar string. Then the string is plucked, giving the string an initial profile. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. The initial-boundary value problem for this problem is given as

1D Wave Equation

$$\begin{array}{lll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.4}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string and the tension placed on the string.

4.2.1 Solving the Heat Equation

We would like to see how the solution of such problems involving partial differential equations will lead naturally to studying boundary value problems for ordinary differential equations. We will see this as we attempt the solution of the heat equation problem 4.3. We will employ a method typically used in studying linear partial differential equations, called *the method of separation of variables.*

We assume that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

Dividing both sides by k and $u = XT$, we then get

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ :

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}} .$$

This leads to two equations:

$$T' = k\lambda T, \tag{4.5}$$

$$X'' = \lambda X. \tag{4.6}$$

These are ordinary differential equations. The general solutions to these equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{4.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{\sqrt{-\lambda}x}. \tag{4.8}$$

We need to be a little careful at this point. The aim is to force our product solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on u lead to conditions on X.

The first condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0$$

for all t. The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{4.9}$$

We are seeking nonzero solutions, as $X \equiv 0$ is an obvious and uninteresting solution. We call such solutions *trivial solutions*.

There are three cases to consider, depending on the sign of λ.

I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{\sqrt{-\lambda}x}. \tag{4.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then, $X(x) = c_1(e^{\sqrt{\lambda}x} - e^{\sqrt{-\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x$. Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda} L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$. Thus, the only solution in this case is $X(x) = 0$. This leads to a trivial solution, $u(x,t) = 0$.

II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$. At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L == n\pi, \quad n = 1, 2, \ldots .$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (4.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right), \quad n = 1, 2, 3, \ldots .$$

Product solutions of the heat equation (4.3) satisfying the boundary conditions are therefore

$$\boxed{u_n(x,t) = b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots, \qquad (4.11)}$$

where b_n is an arbitrary constant. However, these do not necessarily satisfy the initial condition $u(x,0) = f(x)$. What we do get is

$$u_n(x,0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if our initial condition is in one of these forms, we can pick out the right n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, we can write a linear combination of our product solutions and obtain the *general solution* satisfying the given boundary conditions as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{4.12}$$

The only thing to impose is the initial condition:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we are given $f(x)$, can we find the constants b_n? If we can, then we will have the solution to the full initial-boundary value problem. This will be the subject of the next chapter. However, first we will look at the general form of our boundary value problem and relate what we have done to the theory of infinite dimensional vector spaces.

4.3 Connections to Linear Algebra

We have already seen in earlier chapters that ideas from linear algebra crop up in our studies of differential equations. Namely, we solved eigenvalue problems associated with our systems of differential equations in order to determine the local behavior of dynamical systems near fixed points. In our study of boundary value problems we will find more connections with the theory of vector spaces. However, we will find that our problems lie in the realm of infinite dimensional vector spaces. In this section we will begin to see these connections.

4.3.1 Eigenfunction Expansions for PDEs

In the last section we sought solutions of the heat equation. Let's formally write the heat equation in the form

$$\frac{1}{k} u_t = L[u], \tag{4.13}$$

where

$$L = \frac{\partial^2}{\partial x^2}.$$

L is another example of a linear differential operator. [See Section 1.1.2.] It is a differential operator because it involves derivative operators. We sometimes define $D_x = \frac{\partial}{\partial x}$, so that $L = D_x^2$. It is linear, because for functions $f(x)$ and $g(x)$ and constants α, β we have

$$L[\alpha f + \beta g] = \alpha L[f] + \beta L[g]$$

When solving the heat equation, using the method of separation of variables, we found an infinite number of product solutions $u_n(x,t) = T_n(t)X_n(x)$. We did this by solving the boundary value problem

$$L[X] = \lambda X, \quad X(0) = 0 = X(L). \tag{4.14}$$

Here we see that an operator acts on an unknown function and spits out an unknown constant times that unknown. Where have we done this before? This is the same form as $A\mathbf{v} = \lambda\mathbf{v}$. So, we see that Equation (4.14) is really an eigenvalue problem for the operator L and given boundary conditions. When we solved the heat equation in the last section, we found the *eigenvalues*

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and the *eigenfunctions*

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

We used these to construct the general solution that is essentially a linear combination over the eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x).$$

Note that these eigenfunctions live in an infinite dimensional function space.

We would like to generalize this method to problems in which L comes from an assortment of linear differential operators. So, we consider the more general partial differential equation

$$u_t = L[u], \quad a \le x \le b, \quad t > 0,$$

satisfying the boundary conditions

$$B[u](a,t) = 0, \qquad B[u](b,t) = 0, \quad t > 0,$$

and initial condition

$$u(x,0) = f(x), \quad a \le x \le b.$$

The form of the allowed boundary conditions $B[u]$ will be taken up later. Also, we will later see specific examples and properties of linear differential operators that will allow for this procedure to work.

We assume product solutions of the form $u_n(x,t) = b_n(t)\phi_n(x)$, where the ϕ_n's are the eigenfunctions of the operator L,

$$L\phi_n = \lambda_n \phi_n, \quad n = 1, 2, \ldots, \tag{4.15}$$

satisfying the boundary conditions

$$B[\phi_n](a) = 0, \qquad B[\phi_n](b) = 0. \tag{4.16}$$

Inserting the general solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

into the partial differential equation, we have

$$\begin{aligned} u_t &= L[u], \\ \frac{\partial}{\partial t} \sum_{n=1}^{\infty} b_n(t)\phi_n(x) &= L\left[\sum_{n=1}^{\infty} b_n(t)\phi_n(x)\right] \end{aligned} \tag{4.17}$$

On the left we differentiate term by term[1] and on the right side we use the linearity of L:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt}\phi_n(x) = \sum_{n=1}^{\infty} b_n(t) L[\phi_n(x)] \tag{4.18}$$

Now, we make use of the result of applying L to the eigenfunction ϕ_n:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt}\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\lambda_n \phi_n(x). \tag{4.19}$$

Comparing both sides, or using the linear independence of the eigenfunctions, we see that

$$\frac{db_n(t)}{dt} = \lambda_n b_n(t),$$

whose solution is

$$b_n(t) = b_n(0)e^{\lambda_n t}.$$

So, the general solution becomes

[1] Infinite series cannot always be differentiated, so one must be careful. When we ignore such details for the time being, we say that we *formally* differentiate the series and formally apply the differential operator to the series. Such operations need to be justified later.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0) e^{\lambda_n t} \phi_n(x).$$

This solution satisfies, at least formally, the partial differential equation and satisfies the boundary conditions.

Finally, we need to determine the $b_n(0)$'s, which are so far arbitrary. We use the initial condition $u(x,0) = f(x)$ to find that

$$f(x) = \sum_{n=1}^{\infty} b_n(0) \phi_n(x).$$

So, given $f(x)$, we are left with the problem of extracting the coefficients $b_n(0)$ in an expansion of f in the eigenfunctions ϕ_n. We will see that this is related to Fourier series expansions, which we will take up in the next chapter.

4.3.2 Eigenfunction Expansions for Nonhomogeneous ODEs

Partial differential equations are not the only applications of the method of eigenfunction expansions, as seen in the last section. We can apply these method to nonhomogeneous two point boundary value problems for ordinary differential equations assuming that we can solve the associated eigenvalue problem.

Let's begin with the nonhomogeneous boundary value problem:

$$\begin{aligned} L[u] &= f(x), \quad a \leq x \leq b \\ B[u](a) &= 0, \quad B[u](b) = 0. \end{aligned} \tag{4.20}$$

We first solve the eigenvalue problem,

$$\begin{aligned} L[\phi] &= \lambda \phi, \quad a \leq x \leq b \\ B[\phi](a) &= 0, \quad B[\phi](b) = 0, \end{aligned} \tag{4.21}$$

and obtain a family of eigenfunctions, $\{\phi_n(x)\}_{n=1}^{\infty}$. Then we assume that $u(x)$ can be represented as a linear combination of these eigenfunctions:

$$u(x) = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Inserting this into the differential equation, we have

$$\begin{aligned} f(x) &= L[u] \\ &= L\left[\sum_{n=1}^{\infty} b_n \phi_n(x)\right] \\ &= \sum_{n=1}^{\infty} b_n L\left[\phi_n(x)\right] \end{aligned}$$

$$= \sum_{n=1}^{\infty} \lambda_n b_n \phi_n(x)$$

$$\equiv \sum_{n=1}^{\infty} c_n \phi_n(x). \tag{4.22}$$

Therefore, we have to find the expansion coefficients $c_n = \lambda_n b_n$ of the given $f(x)$ in a series expansion over the eigenfunctions. This is similar to what we had found for the heat equation problem and its generalization in the last section.

There are a lot of questions and details that have been glossed over in our formal derivations. Can we always find such eigenfunctions for a given operator? Do the infinite series expansions converge? Can we differentiate our expansions terms by term? Can one find expansions that converge to given functions like $f(x)$ above? We will begin to explore these questions in the case that the eigenfunctions are simple trigonometric functions like the $\phi_n(x) = \sin \frac{n\pi x}{L}$ in the solution of the heat equation.

4.3.3 Linear Vector Spaces

Much of the discussion and terminology that we will use comes from the theory of vector spaces. Until now you may only have dealt with finite dimensional vector spaces in your classes. Even then, you might only be comfortable with two and three dimensions. We will review a little of what we know about finite dimensional spaces so that we can deal with the more general function spaces, which is where our eigenfunctions live.

The notion of a vector space is a generalization of our three dimensional vector spaces. In three dimensions, we have things called vectors, which are arrows of a specific length and pointing in a given direction. To each vector, we can associate a point in a three dimensional Cartesian system. We just attach the tail of the vector $\mathbf{v}$ to the origin and the head lands at (x, y, z). We then use unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the coordinate axes to write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Having defined vectors, we then learned how to add vectors and multiply vectors by numbers, or scalars. Under these operations, we expected to get back new vectors. Then we learned that there were two types of multiplication of vectors. We could multiply then to get a scalar or a vector. This lead to the dot and cross products, respectively. The dot product was useful for determining the length of a vector, the angle between two vectors, or if the vectors were orthogonal.

These notions were later generalized to spaces of more than three dimensions in your linear algebra class. The properties outlined roughly above need to be preserved. So, we have to start with a space of vectors and the operations between them. We also need a set of scalars, which generally come from some *field*. However, in our applications the field will either be the set of real numbers or the set of complex numbers.

Definition 4.1. *A vector space V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions: For any $u, v, w \in V$ and $a, b \in F$*

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. *There exists a* 0 *such that* $0 + v = v$.
4. *There exists a* $-v$ *such that* $v + (-v) = 0$.
5. $a(bv) = (ab)v$.
6. $(a + b)v = av + bv$.
7. $a(u + v) = au + bv$.
8. $1(v) = v$.

Now, for an n-dimensional vector space, we have the idea that any vector in the space can be represented as the sum over n linearly independent vectors. Recall that a linearly independent set of vectors $\{\mathbf{v}_j\}_{j=1}^n$ satisfies

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = \mathbf{0} \quad \Leftrightarrow \quad c_j = 0.$$

This leads to the idea of a basis set. The standard basis in an n-dimensional vector space is a generalization of the standard basis in three dimensions ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$). We define

$$\mathbf{e}_k = (0, \dots, 0, \underbrace{1}_{k\text{th space}}, 0, \dots, 0), \quad k = 1, \dots, n. \tag{4.23}$$

Then, we can expand any $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k, \tag{4.24}$$

where the v_k's are called the components of the vector in this basis and one can write $\mathbf{v}$ as an n-tuple $(v_1, v_2, \dots, v_n)$.

The only other thing we will need at this point is to generalize the dot product, or scalar product. Recall that there are two forms for the dot product in three dimensions. First, one has that

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta, \tag{4.25}$$

where u and v denote the length of the vectors. The other form, is the component form:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k. \tag{4.26}$$

Of course, this form is easier to generalize. So, we define the *scalar product* between to n-dimensional vectors as

$$< \mathbf{u}, \mathbf{v} >= \sum_{k=1}^{n} u_k v_k. \tag{4.27}$$

Actually, there are a number of notations that are used in other texts. One can write the scalar product as $(\mathbf{u}, \mathbf{v})$ or even use the Dirac notation $< \mathbf{u}|\mathbf{v} >$ for applications in quantum mechanics.

While it does not always make sense to talk about angles between general vectors in higher dimensional vector spaces, there is one concept that is useful. It is that of orthogonality, which in three dimensions another way of say vectors are perpendicular to each other. So, we also say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if and only if $< \mathbf{u}, \mathbf{v} >= 0$. If $\{\mathbf{a}_k\}_{k=1}^n$, is a set of basis vectors such that

$$< \mathbf{a}_j, \mathbf{a}_k >= 0, \quad k \neq j,$$

then it is called an *orthogonal basis*. If in addition each basis vector is a unit vector, then one has an *orthonormal basis*

Let $\{\mathbf{a}_k\}_{k=1}^n$, be a set of basis vectors for vector space V. We know that any vector $\mathbf{v}$ can be represented in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. If we know the basis and vector, can we find the components? The answer is, yes. We can use the scalar product of $\mathbf{v}$ with each basis element $\mathbf{a}_j$. So, we have for $j = 1, \ldots, n$

$$\begin{aligned} < \mathbf{a}_j, \mathbf{v} > &= < \mathbf{a}_j, \sum_{k=1}^{n} v_k \mathbf{a}_k > \\ &= \sum_{k=1}^{n} v_k < \mathbf{a}_j, \mathbf{a}_k > . \end{aligned} \tag{4.28}$$

Since we know the basis elements, we can easily compute the numbers

$$A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$$

and

$$b_j \equiv < \mathbf{a}_j, \mathbf{v} > .$$

Therefore, the system (4.28) for the v_k's is a linear algebraic system, which takes the form $A\mathbf{v} = \mathbf{b}$. However, if the basis is orthogonal, then the matrix A is diagonal and the system is easily solvable. We have that

$$< \mathbf{a}_j, \mathbf{v} >= v_j < \mathbf{a}_j, \mathbf{a}_j >, \tag{4.29}$$

or

$$v_j = \frac{< \mathbf{a}_j, \mathbf{v} >}{< \mathbf{a}_j, \mathbf{a}_j >}. \tag{4.30}$$

In fact, if the basis is orthonormal, A is the identity matrix and the solution is simpler:

$$v_j = < \mathbf{a}_j, \mathbf{v} > . \tag{4.31}$$

We spent some time looking at this simple case of extracting the components of a vector in a finite dimensional space. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are the key ingredients that we will need in the infinite dimensional case. Recall that when we solved the heat equation, we had a function (vector) that we wanted to expand in a set of eigenfunctions (basis) and we needed to find the expansion coefficients (components). As you can see, we need to extend the concepts for finite dimensional spaces to their analogs in infinite dimensional spaces. Linear algebra will provide some of the backdrop for what is to follow: The study of many boundary value problems amounts to the solution of eigenvalue problems over infinite dimensional vector spaces (complete inner product spaces, the space of square integrable functions, or Hilbert spaces).

We will consider the space of functions of a certain type. They could be the space of continuous functions on [0,1], or the space of differentiably continuous functions, or the set of functions integrable from a to b. Later, we will specify the types of functions needed. We will further need to be able to add functions and multiply them by scalars. So, we can easily obtain a vector space of functions.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. For a real vector space, we define

Definition 4.2. *An inner product* $<,>$ *on a real vector space* V *is a mapping from* $V \times V$ *into* R *such that for* $u, v, w \in V$ *and* $\alpha \in R$ *one has*

1. $< u + v, w > = < u, w > + < v, w > .$
2. $< \alpha v, w > = \alpha < v, w > .$
3. $< v, w > = < w, v > .$
4. $< v, v > \geq 0$ *and* $< v, v > = 0$ *iff* $v = 0$.

A real vector space equipped with the above inner product leads to a real inner product space. A more general definition with the third item replaced with $< v, w > = \overline{< w, v >}$ is needed for complex inner product spaces.

For the time being, we are dealing just with real valued functions. We need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$. Then, we define the *inner product*, if the integral exists, as

$$< f, g > = \int_a^b f(x)g(x)\, dx. \tag{4.32}$$

So far, we have functions spaces equipped with an inner product. Can we find a basis for the space? For an n-dimensional space we need n basis vectors.

For an infinite dimensional space, how many will we need? How do we know when we have enough? We will think about those things later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^{\infty}$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

How do we find the c_n's? Does this remind you of the problem we had earlier?

Formally, we take the inner product of f with each ϕ_j, to find

$$\begin{aligned} < \phi_j, f > &= < \phi_j, \sum_{n=1}^{\infty} c_n \phi_n > \\ &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > . \end{aligned} \tag{4.33}$$

If our basis is an *orthogonal basis*, then we have

$$< \phi_j, \phi_n >= N_j \delta_{jn}, \tag{4.34}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, \ i \neq j \\ 1, \ i = j. \end{cases} \tag{4.35}$$

Thus, we have

$$\begin{aligned} < \phi_j, f > &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{jn} \\ &= c_1 N_j \delta_{j1} + c_2 N_j \delta_{j2} + \ldots + c_j N_j \delta_{jj} + \ldots \\ &= c_j N_j. \end{aligned} \tag{4.36}$$

So, the expansion coefficient is

$$c_j = \frac{< \phi_j, f >}{N_j} = \frac{< \phi_j, f >}{< \phi_j, \phi_j >}.$$

We summarize this important result:

Generalized Basis Expansion

Let $f(x)$ be represented by an expansion over a basis of orthogonal functions, $\{\phi_n(x)\}_{n=1}^{\infty}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Then, the expansion coefficients are formally determined as

$$c_n = \frac{< \phi_n, f >}{< \phi_n, \phi_n >}.$$

In our preparation for later sections, let's determine if the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. We need to show that $< \phi_n, \phi_m >= 0$ for $n \neq m$. Thus, we have for $n \neq m$

$$\begin{aligned} < \phi_n, \phi_m > &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} = 0. \end{aligned} \quad (4.37)$$

Here we have made use of a trigonometric identity for the product of two sines. We recall how this identity is derived. Recall the addition formulae for cosines:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Adding, or subtracting, these equations gives

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B),$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B).$$

So, we have determined that the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[= \pi, \pi]$. Just as with vectors in three dimensions, we can normalize our basis functions to arrive at an *orthonormal basis*, $< \phi_n, \phi_m >= \delta_{nm}$, $m, n = 1, 2, \ldots$. This is simply done by dividing by the *length* of the vector. Recall that the length of a vector was obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ In the same way, we define the *norm* of our functions by

$$\|f\| = \sqrt{< f, f >}.$$

Note, there are many types of norms, but this will be sufficient for us.

For the above basis of sine functions, we want to first compute the norm of each function. Then we would like to find a new basis from this one such that each basis eigenfunction has unit length and is therefore an orthonormal basis. We first compute

$$\begin{aligned}\|\phi_n\|^2 &= \int_{-\pi}^{\pi} \sin^2 nx\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [1-\cos 2nx]\,dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi. \end{aligned} \tag{4.38}$$

We have found for our example that

$$< \phi_n, \phi_m >= \pi\delta_{nm} \tag{4.39}$$

and that $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}}\phi_n(x)$, we have *normalized* the ϕ_n's and have obtained an orthonormal basis of functions on $[-\pi, \pi]$.

Expansions of functions in trigonometric bases occur often and originally resulted from the study of partial differential equations. They have been named Fourier series and will be the topic of the next chapter.

Problems

4.1. Solve the following problem:

$$x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$$

4.2. Find product solutions, $u(x,t) = b(t)\phi(x)$, to the heat equation satisfying the boundary conditions $u_x(0,t) = 0$ and $u(L,t) = 0$. Use these solutions to find a general solution of the heat equation satisfying these boundary conditions.

4.3. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.[2]

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$
b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$
c. $x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$
d. $(x^2 y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

[2] In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

4.4. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set.

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \dots, \quad 0 \le x \le \pi.$
b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \dots, \quad 0 \le x \le 2.$
c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \dots, \quad x \in [-L, L].$

4.5. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4 y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.