$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{pmatrix}.$$

From this solution we can construct the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So, the general solution to the homogeneous problem is

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}.$$

Next we seek a particular solution to the nonhomogeneous problem. From Equation (2.73) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have

$$\begin{aligned} \Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\ &= \begin{pmatrix} -2 \sin s \cos s \\ 2\cos^2 s \end{pmatrix}. \end{aligned} \tag{2.77}$$

We now compute

$$\begin{aligned} \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\, ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_{t_0}^{t} \begin{pmatrix} -2 \sin s \cos s \\ 2\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin^2 t \\ t + \frac{1}{2}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix}. \end{aligned} \tag{2.78}$$

therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} + \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix},$$

or

$$\mathbf{x} = \begin{pmatrix} 4\cos t + t \sin t \\ -3 \sin t + t \cos t \end{pmatrix}.$$

2.9 Applications

In this section we will describe several applications leading to systems of differential equations. In keeping with common practice in areas like physics, we will denote differentiation with respect to time as

$$\dot{x} = \frac{dx}{dt}.$$

We will look mostly at linear models and later modify some of these models to include nonlinear terms.

2.9.1 Spring-Mass Systems

There are many problems in physics that result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. In particular, the net force is proportional to the acceleration with a proportionality constant called the mass, m. This is summarized as

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$, Newton's Second Law is mathematically a system of second order differential equations for three dimensional problems, or one second order differential equation for one dimensional problems. If there are several masses, then we would naturally end up with systems no matter how many dimensions are involved.

A standard problem encountered in a first course in differential equations is that of a single block on a spring as shown in Figure 2.18. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the *spring constant*. Here x is the elongation of the spring, or the displacement of the block from equilibrium. When x is positive, the spring force is negative and when x is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface.

A similar model can be provided for vertically oriented springs. Place the block on a vertically hanging spring. It comes to equilibrium, stretching the spring by ℓ_0. Newton's Second Law gives

$$-mg + k\ell_0 = 0.$$

Now, pulling the mass further by x_0, and releasing it, the mass begins to oscillate. Letting x be the displacement from the new equilibrium, Newton's Second Law now gives $m\ddot{x} = -mg + k(\ell_0 - x) = -kx$.

In both examples (a horizontally or vetically oscillating mass) Newton's Second Law of motion reults in the differential equation

$$m\ddot{x} + kx = 0. \tag{2.79}$$

This is the equation for simple harmonic motion which we have already encountered in Chapter 1.

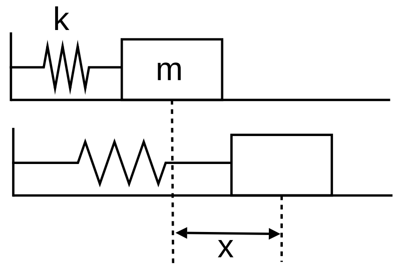

Fig. 2.18. Spring-Mass system.

This second order equation can be written as a system of two first order equations.

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x. \end{aligned} \tag{2.80}$$

The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix},$$

where $\omega^2 = \frac{k}{m}$. The eigenvalues of this system are $\lambda = \pm i\omega$ and the solutions are simple sines and cosines,

$$\begin{aligned} x(t) &= c_1 \cos \omega t + c_2 \sin \omega t, \\ y(t) &= \omega(-c_1 \sin \omega t + c_2 \cos \omega t). \end{aligned} \tag{2.81}$$

We further note that ω is called the angular frequency of oscillation and is given in rad/s. The frequency of oscillation is

$$f = \frac{\omega}{2\pi}.$$

It typically has units of s^{-1}, cps, or Hz. The multiplicative inverse has units of time and is called the period,

$$T = \frac{1}{f}.$$

Thus, the period of oscillation for a mass m on a spring with spring constant k is given by

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{2.82}$$

Of course, we did not need to convert the last problem into a system. In fact, we had seen this equation in Chapter 1. However, when one considers

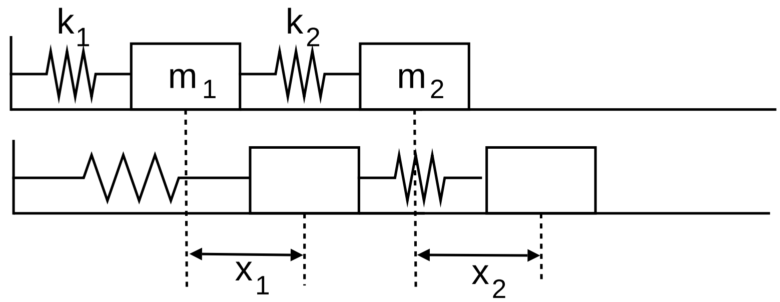

Fig. 2.19. Spring-Mass system for two masses and two springs.

more complicated spring-mass systems, systems of differential equations occur naturally. Consider two blocks attached with two springs as shown in Figure 2.19. In this case we apply Newton's second law for each block.

First, consider the forces acting on the first block. The first spring is stretched by x_1. This gives a force of $F_1 = -k_1 x_1$. The second spring may also exert a force on the block depending if it is stretched, or not. If both blocks are displaced by the same amount, then the spring is not displaced. So, the amount by which the spring is displaced depends on the relative displacements of the two masses. This results in a second force of $F_2 = k_2(x_2 - x_1)$.

There is only one spring connected to mass two. Again the force depends on the relative displacement of the masses. It is just oppositely directed to the force which mass one feels from this spring.

Combining these forces and using Newton's Second Law for both masses, we have the system of second order differential equations

$$\begin{aligned} m_1 \ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.83}$$

One can rewrite this system of two second order equations as a system of four first order equations. This is done by introducing two new variables $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. Note that these physically are the velocities of the two blocks.

The resulting system of first order equations is given as

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1) \end{aligned} \tag{2.84}$$

We can write our new system in matrix form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.85}$$

2.9.2 Electrical Circuits

Another problem often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.20. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, stores magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Since there is only one loop, we will only need Kirchoff's Loop Rule. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V_R = IR$.
2. Capacitor: $V_C = \frac{q}{C}$.
3. Inductor: $V_L = L\frac{dI}{dt}$.

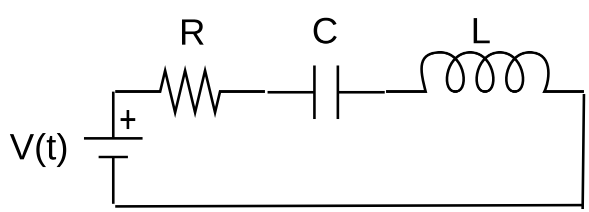

Fig. 2.20. Series LRC Circuit.

Adding these potential drops and setting the sum equal to the voltage supplied by the voltage source, $V(t)$, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Furthermore, we recall that the current is defined as $I = \frac{dq}{dt}$. where q is the charge in the circuit. Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t). \tag{2.86}$$

This is a second order differential equation for $q(t)$. One can set up a system of equations and proceed to solve them. However, this is a constant coefficient differential equation and can also be solved using the methods in Chapter 1.

In the next examples we will look at special cases that arise for the series LRC circuit equation. These include RC circuits, solvable by first order methods and LC circuits, leading to oscillatory behavior.

Example 2.19. **RC Circuits**

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$).

For charging a capacitor, we have the initial value problem

$$R\frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \tag{2.87}$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite this equation and solve it as a separable equation, since V_0 is a constant. We will do the former only as another example of finding the integrating factor.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \tag{2.88}$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt}\left(qe^{t/RC}\right) = \frac{V_0}{R}e^{t/RC}. \tag{2.89}$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R}\int e^{t/RC}\,dt = CV_0e^{t/RC} + K. \tag{2.90}$$

Note that we introduced the integration constant, K. Now divide out the exponential to get the general solution:

$$q = CV_0 + Ke^{-t/RC}. \tag{2.91}$$

(If we had forgotten the K, we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get our particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = CV_0 + K.$$

So, $K = -CV_0$. Inserting this into our solution, we have

$$q(t) = CV_0(1 - e^{-t/RC}). \tag{2.92}$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = CV_0$. This is what we expect, because the current is no longer flowing over R and this just gives the relation between the potential difference across the capacitor plates when a charge of q_0 is established on the plates.

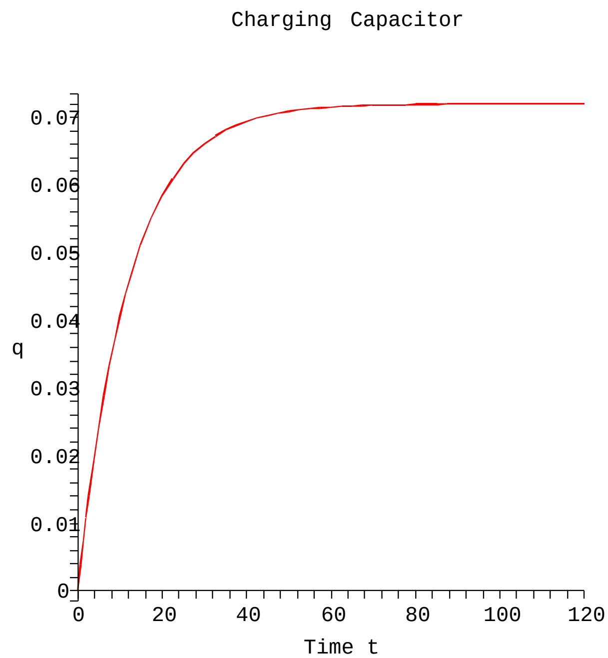

Fig. 2.21. The charge as a function of time for a charging capacitor with $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V.

Let's put in some values for the parameters. We let $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V. A plot of the solution is given in Figure 2.21. We see that the charge builds up to the value of $CV_0 = 72$ mC. If we use a smaller resistance, $R = 200\ \Omega$, we see in Figure 2.22 that the capacitor charges to the same value, but much faster.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays. If we set $t = \tau$, we find that

$$q(\tau) = CV_0(1 - e^{-1}) = (1 - 0.3678794412\ldots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12$s. For the second set, $\tau = 1.2$s.

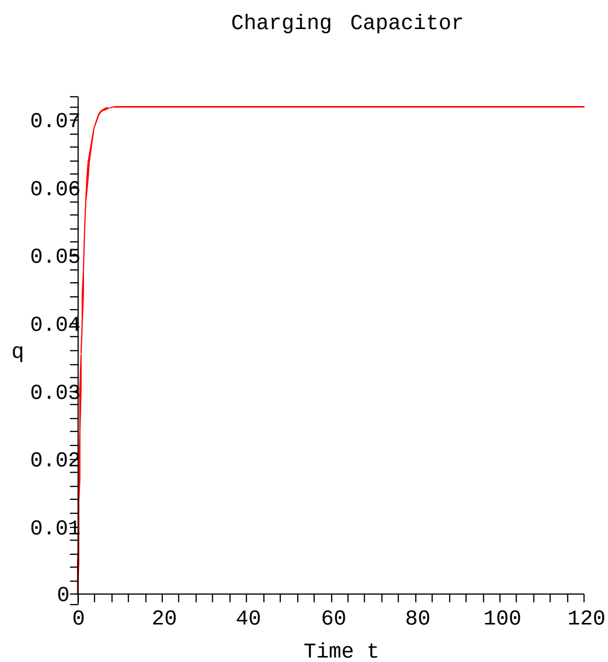

Fig. 2.22. The charge as a function of time for a charging capacitor with $R = 200$ Ω, $C = 6.00$ mF, and $V_0 = 12$ V.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit, the charge will then move off the plates, discharging the capacitor. The relevant form of our initial value problem becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \tag{2.93}$$

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \tag{2.94}$$

This is a simple exponential decay problem, which you can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.23 we show the discharging of our two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412\ldots)q_0 \approx 0.37 q_0.$$

So, at this time the capacitor only has about a third of its original value.

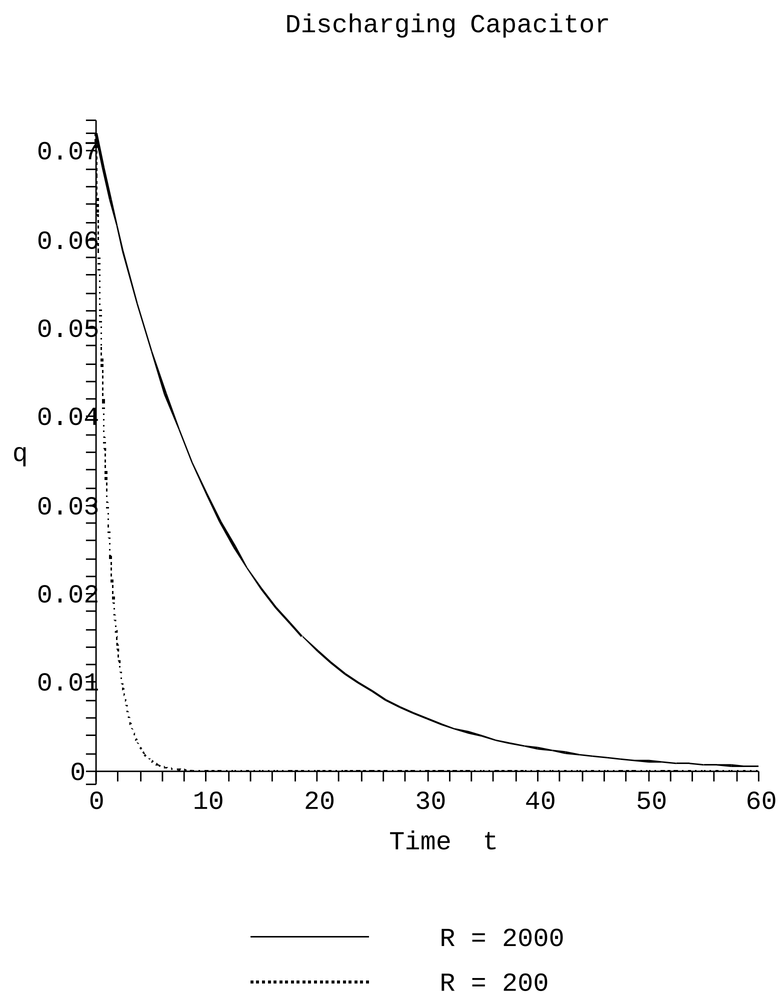

Fig. 2.23. The charge as a function of time for a discharging capacitor with $R = 2.00$ kΩ or $R = 200\ \Omega$, and $C = 6.00$ mF, and $q_0 = 72$ mC.

Example 2.20. **LC Circuits**

Another simple result comes from studying LC circuits. We will now connect a charged capacitor to an inductor. In this case, we consider the initial value problem

$$L\ddot{q} + \frac{1}{C}q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \tag{2.95}$$

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC}q = 0. \tag{2.96}$$

This equation is a second order, constant coefficient equation. It is of the same form as the one we saw earlier for simple harmonic motion of a mass on a spring. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm\frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.96) is of the form

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \omega = (LC)^{-1/2}. \tag{2.97}$$

Inserting the initial conditions yields

$$q(t) = q_0 \cos(\omega t). \tag{2.98}$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}. \tag{2.99}$$

This is called the tuning frequency because of its role in tuning circuits.

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. an example of another circuit setup is shown in Figure 2.24. This is not a problem that can be covered in the first year physics course.

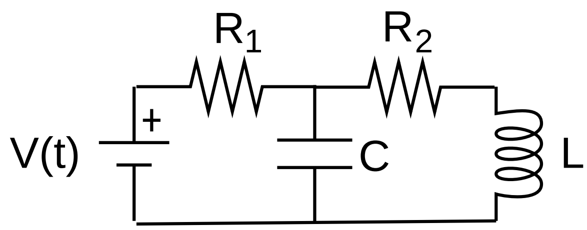

Fig. 2.24. A circuit with two loops containing several different circuit elements.

There are two loops, indicated in Figure 2.25 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled I_i, $i = 1, 2, 3$. Corresponding to each current is a changing charge, q_i such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1, 2, 3.$$

For loop one we have

$$I_1 R_1 + \frac{q_2}{C} = V(t). \tag{2.100}$$

For loop two

$$I_3 R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{2.101}$$

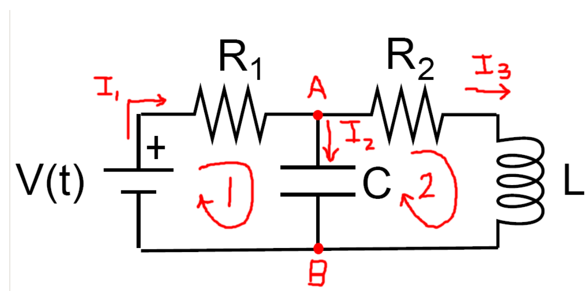

Fig. 2.25. The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

We have three unknown functions for the charge. Once we know the charge functions, differentiation will yield the currents. However, we only have two equations. We need a third equation. This is found from Kirchoff's Point (Junction) Rule. Consider the points A and B in Figure 2.25. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{2.102}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{2.103}$$

Equations (2.100), (2.101), and (2.103) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$\begin{aligned} R_1\dot{q}_1 + \frac{q_2}{C} &= V(t) \\ R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{2.104}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions for the other terms. $\dot{q}_3$ is gotten from the definition of q_4 and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned} \dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1C} \\ \dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1C} - q_4 \\ \dot{q}_3 &= q_4 \\ \dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L}q_4. \end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations. In the last section we learned how to solve such systems.

2.9.3 Love Affairs

The next application is one that has been studied by several authors as a cute system involving relationships. One considers what happens to the affections that two people have for each other over time. Let R denote the affection that Romeo has for Juliet and J be the affection that Juliet has for Romeo. positive values indicate love and negative values indicate dislike.

One possible model is given by

$$\frac{dR}{dt} = bJ$$
$$\frac{dJ}{dt} = cR \tag{2.105}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ. \tag{2.106}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

2.9.4 Predator Prey Models

Another common model studied is that of competing species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\frac{dR}{dt} = aR - cF,$$
$$\frac{dF}{dt} = -bF + dR, \tag{2.107}$$

where all of the constants are positive numbers. Studying this coupled system would lead to as study of the dynamics of these populations. We will discuss other (nonlinear) systems in the next chapter.

2.9.5 Mixture Problems

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out.}$$

This can be generalized to the case of two interconnected tanks. We provide some examples.

Example 2.21. **Single Tank Problem**

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 2.26.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.

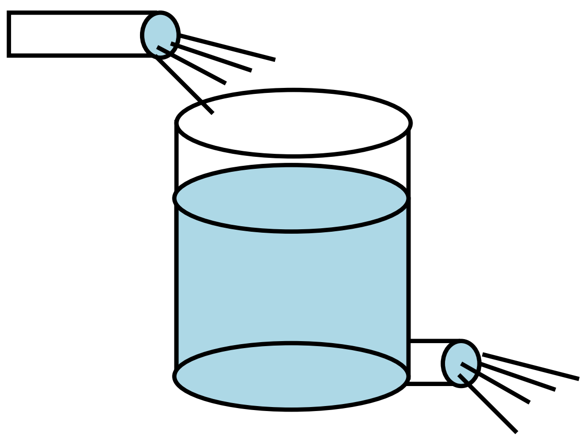

Fig. 2.26. A typical mixing problem.

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = 10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x \text{ pounds}}{50 \text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = \frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is easily solved using the methods for first order equations.

Example 2.22. **Double Tank Problem**

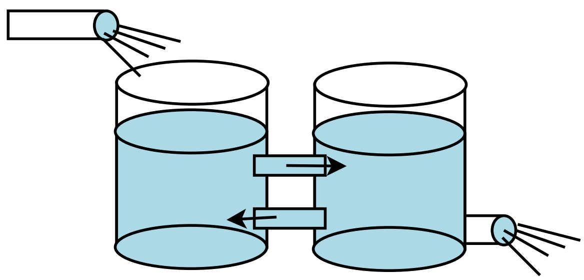

Fig. 2.27. The two tank problem.

One has two tanks connected together, labelled tank X and tank Y, as shown in Figure 2.27. Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Now pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?

In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100}\\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}.\end{aligned} \tag{2.108}$$

This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve.

2.9.6 Chemical Kinetics

There are many problems that come from studying chemical reactions. The simplest reaction is when a chemical A turns into chemical B. This happens at a certain rate, $k > 0$. This can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of A, $[A]$, and B, $[B]$, are given by

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{2.109}$$

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

In this case we can add to the above equation the rates of change of concentrations $[B]$ and $[C]$. The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.110}$$

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$\boxed{A \overset{k_3}{\underset{k_1}{\longleftrightarrow}} B \xrightarrow[k_2]{} C.}$$

The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A] + k_3[B], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.111}$$

More complicated chemical reactions will be discussed at a later time.

2.9.7 Epidemics

Another interesting area of application of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let break the population into three classes. First, $S(t)$ are the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others recover. Let's assume that initially there in one infected person and the rest, say N, are obviously healthy. Can we predict how many deaths have occurred by time t?

Let's try and model this problem using the compartmental analysis we had seen in the mixing problems. The total rate of change of any population would be due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that the rate of infection is proportional to the number of healthy people,aS. Also, we assume that the number who recover is proportional to the number of infected, rI. Thus, the rate of change of the healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infectives is due to healthy people getting infected and the infectives who either recover or die. Using the corresponding terms in the other equations, we can write

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \tag{2.112}$$

The eigenvalue equation for this system is

$$\lambda\left[\lambda^2 + (a+r+d)\lambda + ad\right] = 0.$$

The reader can find the solutions of this system and determine if this is a realistic model.

2.10 Appendix: Diagonalization and Linear Systems

As we have seen, the matrix formulation for linear systems can be powerful, especially for n differential equations involving n unknown functions. Our ability to proceed towards solutions depended upon the solution of eigenvalue problems. However, in the case of repeated eigenvalues we saw some additional complications. This all depends deeply on the background linear algebra. Namely, we relied on being able to diagonalize the given coefficient matrix. In this section we will discuss the limitations of diagonalization and introduce the Jordan canonical form.

We begin with the notion of similarity. Matrix A is *similar* to matrix B if and only if there exists a nonsingular matrix P such that

$$B = P^{-1}AP. \tag{2.113}$$

Recall that a nonsingular matrix has a nonzero determinant and is invertible.

We note that the similarity relation is an equivalence relation. Namely, it satisfies the following

1. A is similar to itself.
2. If A is similar to B, then B is similar to A.
3. If A is similar to B and B is similar to C, the A is similar to C.

Also, if A is similar to B, then they have the same eigenvalues. This follows from a simple computation of the eigenvalue equation. Namely,

$$\begin{aligned} 0 &= \det(B - \lambda I) \\ &= \det(P^{-1}AP - \lambda P^{-1}IP) \\ &= \det(P)^{-1}\det(A - \lambda I)\det(P) \\ &= \det(A - \lambda I). \end{aligned} \tag{2.114}$$

Therefore, $\det(A - \lambda I) = 0$ and λ is an eigenvalue of both A and B.

An $n \times n$ matrix A is *diagonalizable* if and only if A is similar to a diagonal matrix D; i.e., there exists a nonsingular matrix P such that

$$D = P^{-1}AP. \tag{2.115}$$

One of the most important theorems in linear algebra is the Spectral Theorem. This theorem tells us when a matrix can be diagonalized. In fact, it goes beyond matrices to the diagonalization of linear operators. We learn in linear algebra that linear operators can be represented by matrices once we pick a particular representation basis. Diagonalization is simplest for finite dimensional vector spaces and requires some generalization for infinite dimensional vectors spaces. Examples of operators to which the spectral theorem applies are self-adjoint operators (more generally normal operators on Hilbert spaces). We will explore some of these ideas later in the course. The spectral theorem provides a canonical decomposition, called the spectral decomposition, or eigendecomposition, of the underlying vector space on which it acts.

The next theorem tells us how to diagonalize a matrix:

Theorem 2.23. *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors. If so, then*

$$D = P^{-1}AP.$$

If $\{v_1, \ldots, v_n\}$ are the eigenvectors of A and $\{\lambda_1, \ldots, \lambda_n\}$ are the corresponding eigenvalues, then v_j is the jth column of P and $D_{jj} = \lambda_j$.

A simpler determination results by noting

Theorem 2.24. *Let A be an $n \times n$ matrix with n real and distinct eigenvalues. Then A is diagonalizable.*

Therefore, we need only look at the eigenvalues and determine diagonalizability. In fact, one also has from linear algebra the following result.

Theorem 2.25. *Let A be an $n \times n$ real symmetric matrix. Then A is diagonalizable.*

Recall that a symmetric matrix is one whose transpose is the same as the matrix, or $A_{ij} = A_{ji}$.

Example 2.26. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

This is a real symmetric matrix. The characteristic polynomial is found to be

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)(\lambda + 1) = 0.$$

As before, we can determine the corresponding eigenvectors (for $\lambda = -1, 3, 5$, respectively) as

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We can use these to construct the diagonalizing matrix P. Namely, we have

$$P^{-1}AP = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \quad (2.116)$$

Now diagonalization is an important idea in solving linear systems of first order equations, as we have seen for simple systems. If our system is originally diagonal, that means our equations are completely uncoupled. Let our system take the form

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y}, \tag{2.117}$$

where D is diagonal with entries λ_i, $i = 1, \ldots, n$. The system of equations, $y_i' = \lambda_i y_i$, has solutions

$$y_i(t) = c_c e^{\lambda_i t}.$$

Thus, it is easy to solve a diagonal system.

Let A be similar to this diagonal matrix. Then

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y}. \tag{2.118}$$

This can be rewritten as

$$\frac{dP\mathbf{y}}{dt} = AP\mathbf{y}.$$

Defining $\mathbf{x} = P\mathbf{y}$, we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{2.119}$$

This simple derivation shows that if A is diagonalizable, then a transformation of the original system in $\mathbf{x}$ to new *coordinates*, or a new basis, results in a simpler system in $\mathbf{y}$.

However, it is not always possible to diagonalize a given square matrix. This is because some matrices do not have enough linearly independent vectors, or we have repeated eigenvalues. However, we have the following theorem:

Theorem 2.27. *Every $n \times n$ matrix A is similar to a matrix of the form*

$$J = diag[J_1, J_2, \ldots, J_n],$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} \tag{2.120}$$

We will not go into the details of how one finds this **Jordan Canonical Form** or proving the theorem. In practice you can use a computer algebra system to determine this and the similarity matrix. However, we would still need to know how to use it to solve our system of differential equations.

Example 2.28. Let's consider a simple system with the 3×3 Jordan block

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The corresponding system of coupled first order differential equations takes the form

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + x_2, \\ \frac{dx_2}{dt} &= 2x_2 + x_3, \\ \frac{dx_3}{dt} &= 2x_3. \end{aligned} \tag{2.121}$$

The last equation is simple to solve, giving $x_3(t) = c_3 e^{2t}$. Inserting into the second equation, you have a

$$\frac{dx_2}{dt} = 2x_2 + c_3 e^{2t}.$$

Using the integrating factor, e^{-2t}, one can solve this equation to get $x_2(t) = (c_2 + c_3 t)e^{2t}$. Similarly, one can solve the first equation to obtain $x_1(t) = (c_1 + c_2 t + \frac{1}{2} c_3 t^2)e^{2t}$.

This should remind you of a problem we had solved earlier leading to the generalized eigenvalue problem in (2.43). This suggests that there is a more general theory when there are multiple eigenvalues and relating to Jordan canonical forms.

Let's write the solution we just obtained in vector form. We have

$$\mathbf{x}(t) = \left[c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \frac{1}{2}t^2 \\ t \\ 1 \end{pmatrix} \right] e^{2t}. \tag{2.122}$$

It looks like this solution is a linear combination of three linearly independent solutions,

$$\begin{aligned} \mathbf{x} &= \mathbf{v}_1 e^{2\lambda t} \\ \mathbf{x} &= (t\mathbf{v}_1 + \mathbf{v}_2)e^{\lambda t} \\ \mathbf{x} &= (\frac{1}{2}t^2 \mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3)e^{\lambda t}, \end{aligned} \tag{2.123}$$

where $\lambda = 2$ and the vectors satisfy the equations

$$\begin{aligned} (A - \lambda I)\mathbf{v}_1 &= 0, \\ (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1, \\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_2, \end{aligned} \tag{2.124}$$

and

$$\begin{aligned}(A - \lambda I)\mathbf{v}_1 &= 0,\\ (A - \lambda I)^2\mathbf{v}_2 &= 0,\\ (A - \lambda I)^3\mathbf{v}_3 &= 0.\end{aligned} \tag{2.125}$$

It is easy to generalize this result to build linearly independent solutions corresponding to multiple roots (eigenvalues) of the characteristic equation.

Problems

2.1. Consider the system

$$\begin{aligned}x' &= -4x - y\\ y' &= x - 2y.\end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.
b. Solve the differential equation for $x(t)$.
c. Using this solution, find $y(t)$.
d. Verify your solutions for $x(t)$ and $y(t)$.
e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

2.2. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned}x' &= 3x\\ y' &= -2y.\end{aligned}$$

b.

$$\begin{aligned}x' &= -y\\ y' &= -5x.\end{aligned}$$

c.

$$\begin{aligned}x' &= 2y\\ y' &= -3x.\end{aligned}$$

d.

$$\begin{aligned}x' &= x - y\\ y' &= y.\end{aligned}$$

e.

$$x' = 2x + 3y$$
$$y' = -3x + 2y.$$

2.3. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

2.4. In Equation (2.34) the exponential of a matrix was defined.

a. Let
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$
Compute e^A.

b. Give a definition of $\cos A$ and compute $\cos\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in simplest form.

c. Prove $e^{PAP^{-1}} = Pe^AP^{-1}$.

2.5. Consider the general system

$$x' = ax + by$$
$$y' = cx + dy.$$

Can one determine the family of trajectories for the general case? Recall, this means we have to solve the first order equation

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

[Actually, this equation is *homogeneous of degree 0*.] It can be written in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$. For such equations, one can make the substitution $z = \frac{y}{x}$, or $y(x) = xz(x)$, and obtain a separable equation for z.

a. Using the general system, show that $z = z(x)$ satisfies and equation of the form
$$x\frac{dz}{dx} = F(z) - z.$$
Identify the function $F(z)$.

b. Use the equation for $z(x)$ in part a to find the family of trajectories of the system
$$x' = x - y$$
$$y' = x + y.$$
First determine the appropriate $F(z)$ and then solve the resulting separable equation as a relation between z and x. Then write the solution of the original equation in terms of x and y.

c. Use polar coordinates to describe the family of solutions obtained. You can rewrite the solution in polar coordinates and/or solve the system rewritten in polar coordinates.

2.6. Find the eigenvalue(s) and eigenvector(s) for the following:

a. $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$

b. $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$

c. $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

d. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

2.7. Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 2.2, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

a.

$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

d.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

e.

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

f.

$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

2.8. For each of the following matrices consider the system $\mathbf{x}' = A\mathbf{x}$ and

a. Find the fundamental solution matrix.
b. Find the principal solution matrix.

a.

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

b.

$$A = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}.$$

c.

$$A = \begin{pmatrix} 4 & -13 \\ 2 & -6 \end{pmatrix}.$$

d.

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}.$$

2.9. For the following problems

1) Rewrite the problem in matrix form.
2) Find the fundamental matrix solution.
3) Determine the general solution of the nonhomogeneous system.
4) Find the principal matrix solution.
5) Determine the particular solution of the initial value problem.

a. $y'' + y = 2\sin 3x, \quad y(0) = 2, \quad y'(0) = 0.$
b. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

2.10. Prove Equation (2.75),

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{f}(s)\,ds,$$

starting with Equation (2.73).

2.11. Add a third spring connected to mass two in the coupled system shown in Figure 2.19 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.
b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

2.12. Consider the series circuit in Figure 2.20 with $L = 1.00$ H, $R = 1.00\times10^2$ Ω, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.
b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.
c. Plot your solutions and describe how the system behaves over time.

2.13. You live in a cabin in the mountains and you would like to provide yourself with water from a water tank that is 25 feet above the level of the pipe going into the cabin. [See Figure 2.28.] The tank is filled from an aquifer 125 ft below the surface and being pumped at a maximum rate of 7 gallons per minute. As this flow rate is not sufficient to meet your daily needs, you would like to store water in the tank and have gravity supply the needed pressure. So, you design a cylindrical tank that is 35 ft high and has a 10 ft diameter. The water then flows through pipe at the bottom of the tank. You are interested in the height h of the water at time t. This in turn will allow you to figure the water pressure.

Fig. 2.28. A water tank problem in the mountains.

First, the differential equation governing the flow of water from a tank through an orifice is given as

$$\frac{dh}{dt} = \frac{K - \alpha a\sqrt{2gh}}{A}.$$

Here K is the rate at which water is being pumped into the top of the tank. A is the cross sectional area of this tank. α is called the contraction coefficient, which measures the flow through the orifice, which has cross section a. We will assume that $\alpha = 0.63$ and that the water enters in a 6 in diameter PVC pipe.

a. Assuming that the water tank is initially full, find the minimum flow rate in the system during the first two hours.
b. What is the minimum water pressure during the first two hours? Namely, what is the gauge pressure at the house? Note that $\Delta P = \rho g H$, where ρ is the water density and H is the total height of the fluid (tank plus vertical pipe). Note that $\rho g = 0.434$ psi (pounds per square inch).
c. How long will it take for the tank to drain to 10 ft above the base of the tank?

Other information you may need is 1 gallon $= 231$ in^2 and $g = 32.2$ ft/s^2.

2.14. Initially a 200 gallon tank is filled with pure water. At time $t = 0$ a salt concentration with 3 pounds of salt per gallon is added to the container at the rate of 4 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.
b. How many minutes does it take for the concentration to reach 2 pounds per gallon?
c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?
d. Assuming that the tank holds much more than 200 gallons, and everything is the same except that the mixture is drained at 3 gallons per minute, what would the answers to parts a and b become?

2.15. You make two gallons of chili for a party. The recipe calls for two teaspoons of hot sauce per gallon, but you had accidentally put in two tablespoons per gallon. You decide to feed your guests the chili anyway. Assume that the guests take 1 cup/min of chili and you replace what was taken with beans and tomatoes without any hot sauce. [1 gal = 16 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of hot sauce as a function of time in this mixture-type problem.
b. Solve this initial value problem.
c. How long will it take to get the chili back to the recipe's suggested concentration?

2.16. Consider the chemical reaction leading to the system in (2.111). Let the rate constants be $k_1 = 0.20$ ms^{-1}, $k_2 = 0.05$ ms^{-1}, and $k_3 = 0.10$ ms^{-1}. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ μmol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

2.17. Consider the epidemic model leading to the system in (2.112). Choose the constants as $a = 2.0$ days^{-1}, $d = 3.0$ days^{-1}, and $r = 1.0$ days^{-1}. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of $1,000$ and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?