Chapter 2

Linear Systems of Differential Equations

"Do not worry too much about your difficulties in mathematics, I can assure you that mine are still greater." - Albert Einstein (1879-1955)

2.1 Coupled Systems

In this chapter we will begin our study of systems of differential equations. After defining first order systems, we will look at constant coefficient systems and the behavior of solutions for these systems. Also, most of the discussion will focus on planar, or two dimensional, systems. For such systems we will be able to look at a variety of graphical representations of the family of solutions and discuss the qualitative features of systems we can solve in preparation for the study of systems whose solutions cannot be found in an algebraic form. However, we first turn to some simple physical problems.

There are many problems in physics that can result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems for a single mass.

We have already seen reminded in the last chapter of the simple problem of a mass on a spring. This is shown in Figure 2.1. The mass slides on a frictionless surface and reacts to the restoring force of the spring attached to a wall. The restoring force of the spring given by Hooke's Law,

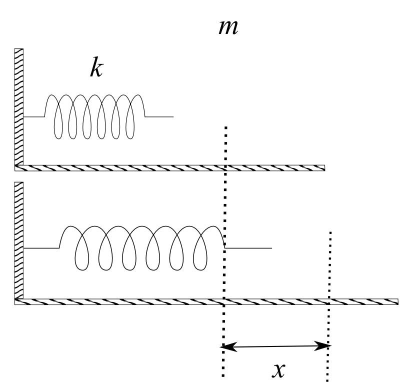


Figure 2.1: Spring-Mass system.

$$F_s = -kx,$$

where $k > 0$ is the spring constant and x is the elongation of the spring. When the spring elongation is positive, the spring force is negative and when the spring elongation is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system is found from Newton's Second Law as

$$m\ddot{x} + kx = 0.$$

This second order equation, constant coefficient equation is easily solved using the methods in the previous chapter. However, it can also be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$. Noting that $\ddot{x} = \dot{y}$, we rewrite the second order equation in terms of x and $\dot{y}$. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{2.1}$$

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.2. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as x_1 and x_2. These are shown in Figure 2.2.

For mass m_1, the forces acting on it are due to each spring. The first spring with spring constant k_1 provides a force on m_1 of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, the second spring will exert a force on m_1 of $k_2(x_2 - x_1)$.

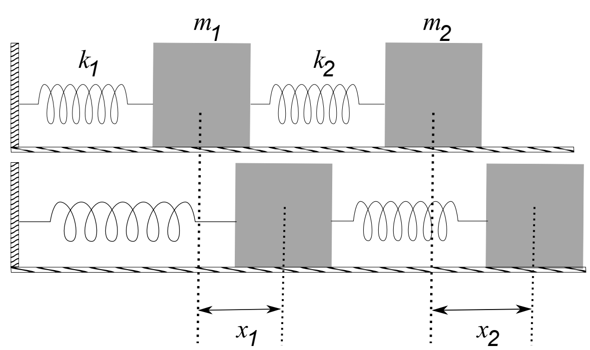


Figure 2.2: System of two masses and two springs.

Similarly, the only force acting directly on mass m_2 is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.2}$$

Thus, we see that we have a coupled system of two second order differential equations. Each equation depends on the unknowns x_1 and x_2.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1). \end{aligned} \tag{2.3}$$

As we will see in the next chapter, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.4}$$

We can solve this system of first order equations using matrix methods. However, we will first need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in planar systems.

2.2 *Planar Systems*

2.2.1 *Introduction*

WE NOW CONSIDER EXAMPLES of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems will help in the study of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= P(x,y,t) \\ y'(t) &= Q(x,y,t). \end{aligned} \tag{2.5}$$

An autonomous system is one in which there is no explicit time dependence:

Autonomous systems.

$$\begin{aligned} x'(t) &= P(x,y) \\ y'(t) &= Q(x,y). \end{aligned} \tag{2.6}$$

Otherwise the system is called nonautonomous.

A *linear system* takes the form

$$\begin{aligned} x' &= a(t)x + b(t)y + e(t) \\ y' &= c(t)x + d(t)y + f(t). \end{aligned} \tag{2.7}$$

A homogeneous linear system results when $e(t) = 0$ and $f(t) = 0$.

A linear, constant coefficient system of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \tag{2.8}$$

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

A linear, homogeneous system of constant coefficient first order differential equations in the plane.

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{2.9}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (2.9) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system (2.9) and systematically replace occurrences of y and y', since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned} \tag{2.10}$$

Rewriting the last line, we have

$$x'' - (a + d)x' + (ad - bc)x = 0. \tag{2.11}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{2.12}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (2.9):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

Example 2.1. Consider the system of differential equations

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.13}$$

Carrying out the above outlined steps, we have that $x'' + 3x' - 4x = 0$. This can be shown as follows:

$$\begin{aligned} x'' &= -x' + 6y' \\ &= -x' + 6(x - 2y) \\ &= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\ &= -3x' + 4x \end{aligned} \tag{2.14}$$

The resulting differential equation has a characteristic equation of $r^2 + 3r - 4 = 0$. The roots of this equation are $r = 1, -4$. Therefore, $x(t) = c_1 e^t + c_2 e^{-4t}$. But, we still need $y(t)$. From the first equation of the system we have

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1 e^t - 3c_2 e^{-4t}).$$

Thus, the solution to the system is

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \tfrac{1}{3} c_1 e^t - \tfrac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.15}$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Solving systems with initial conditions.

Example 2.2. Solve

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.16}$$

given $x(0) = 2$, $y(0) = 0$.

We already have the general solution of this system in (2.15). Inserting the initial conditions, we have

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \tfrac{1}{3} c_1 - \tfrac{1}{2} c_2. \end{aligned} \tag{2.17}$$

Solving for c_1 and c_2 gives $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is

$$\begin{aligned} x(t) &= \tfrac{2}{5}\left(3e^t + 2e^{-4t}\right), \\ y(t) &= \tfrac{2}{5}\left(e^t - e^{-4t}\right). \end{aligned} \tag{2.18}$$

2.2.2 *Equilibrium Solutions and Nearby Behaviors*

In studying systems of differential equations, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying equilibrium solutions of system (2.8). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let x_0 and y_0 be the (constant) equilibrium solutions. Then, x_0 and y_0 must satisfy the system

Equilibrium solutions.

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \tag{2.19}$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{2.20}$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from (x, y) to (u, v) by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

Example 2.3. Stable Node (sink)

Consider the system

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \tag{2.21}$$

This is a simple uncoupled system. Each equation is simply solved to give

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

In this case we see that all solutions tend towards the equilibrium point, $(0, 0)$. This will be called a *stable node*, or a *sink*.

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the xy-plane at the parametric curves $(x(t), y(t))$.

Solution Plots: One can plot each solution as a function of t given a set of initial conditions. Examples are shown in Figure 2.3 for several initial conditions. Note that the solutions decay for large t. Special cases result for various initial conditions. Note that for $t = 0$, $x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with c_1=0, then $x(t) = 0$ for all t. One obtains similar results when setting $y(0) = 0$.

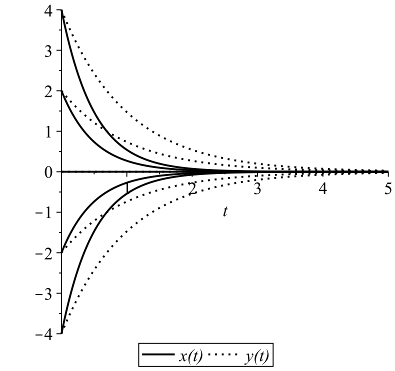


Figure 2.3: Plots of solutions of Example 2.3 for several initial conditions.

Phase Portrait: There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$ Such curves are called trajectories or orbits. The xy-plane is called the phase plane and a collection of such orbits gives a phase portrait for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter t between the known solutions to get a relationship between x and y. Since the solutions are known for the last example, we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2}\right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the xy-plane are given by dy/dx. For autonomous systems, we can write this slope just in terms of x and y. This leads to a first order differential equation, which possibly could be solved analytically or numerically.

First we will obtain the orbits for Example 2.3 by solving the corresponding slope equation. Recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Therefore,

The Slope of a parametric curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{2.22}$$

For the system in (2.21) we use Equation (2.22) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$

The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for A an arbitrary constant. Plots of these solutions in the phase plane are given in Figure 2.4. [Note that this is the same form for the orbits that we had obtained above by eliminating t from the solution of the system.]

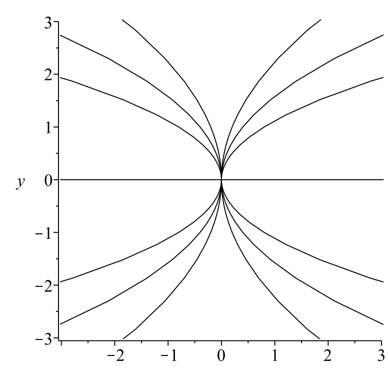


Figure 2.4: Orbits for Example 2.3.

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition

(x_0, y_0), how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

meaning that as $t \to \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicating that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \to \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

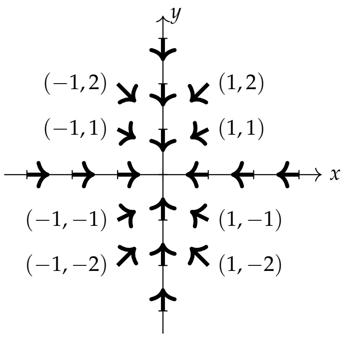


Figure 2.5: Sketch of tangent vectors using Example 2.3.

Direction Fields: Another way to determine the behavior of the solutions of the system of differential equations is to draw the direction field. A direction field is a vector field in which one plots arrows in the direction of tangents to the orbits at selected points in the plane. This is done because the slopes of the tangent lines are given by dy/dx. For the general system (2.9), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

This is a first order differential equation which can be solved as we show in the following examples.

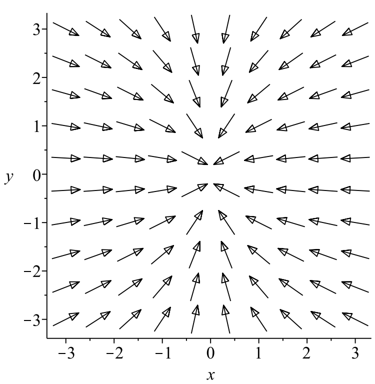


Figure 2.6: Direction field for Example 2.3.

Example 2.4. Draw the direction field for Example 2.3.

We can use software to draw direction fields. However, one can sketch these fields by hand. We have that the slope of the tangent at this point is given by

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

For each point in the plane one draws a piece of tangent line with this slope. In Figure 2.5 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (2.21), we have that x' and y' are both negative at this point. Therefore, the vector points down and to the left.

We can do this for several points, as shown in Figure 2.5. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as *isoclines* in which $\frac{dy}{dx}$ =constant.

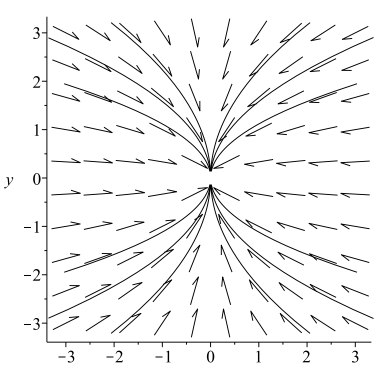


Figure 2.7: Phase portrait for Example 2.3. This is a stable node, or sink

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 2.3 the direction field is shown in Figure 2.6. Looking at this direction field, one can begin to "see" the orbits by following the tangent vectors.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 2.7. Are these the patterns you saw in Figure 2.6?

In this example we see all orbits "flow" towards the origin, or equilibrium point. Again, this is an example of what is called a *stable node* or a *sink*. (Imagine what happens to the water in a sink when the drain is unplugged.)

This is another uncoupled system. The solutions are again simply gotten by integration. We have that $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{t}$. Here we have that x decays as t gets large and y increases as t gets large. In particular, if one picks initial conditions with $c_2 = 0$, then orbits follow the x-axis towards the origin. For initial points with $c_1 = 0$, orbits originating on the y-axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.

In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small c_2 is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 2.8.

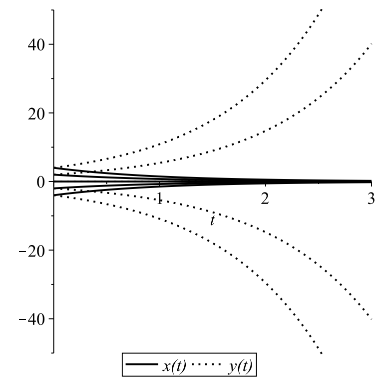


Figure 2.8: Plots of solutions of Example 2.5 for several initial conditions.

Example 2.5. Saddle Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned} \tag{2.23}$$

Similar to the first example, we can look at plots of solutions orbits in the phase plane. These are given by Figures 2.8-2.9. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

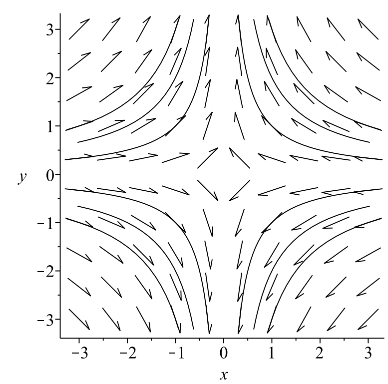


Figure 2.9: Phase portrait for Example 2.5. This is a saddle.

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a saddle point. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the x-axis. In this case, the line of initial conditions is given by the x-axis.

Example 2.6. Unstable Node (source)

$$\begin{aligned} x' &= 2x \\ y' &= y. \end{aligned} \tag{2.24}$$

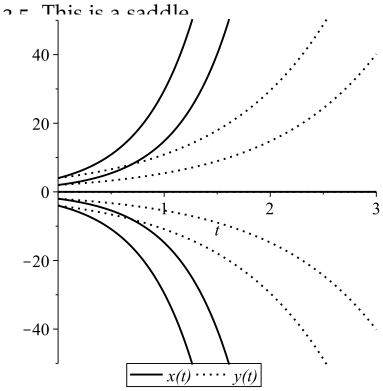


Figure 2.10: Plots of solutions of Example 2.6 for several initial conditions.

This example is similar to Example 2.3. The solutions are obtained by replacing t with $-t$. The solutions, orbits, and direction fields can be seen in Figures 2.10-2.11. This is once again a node, but all orbits lead away from the equilibrium point. It is called an *unstable node* or a *source*.

Example 2.7. Center

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{2.25}$$

This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain

$$x'' + x = 0.$$

We recognize this equation as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Performing a separation of variables and integrating, we find that

$$x^2 + y^2 = C.$$

Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 2.12-2.13 confirms this result. This type of point is called a center.

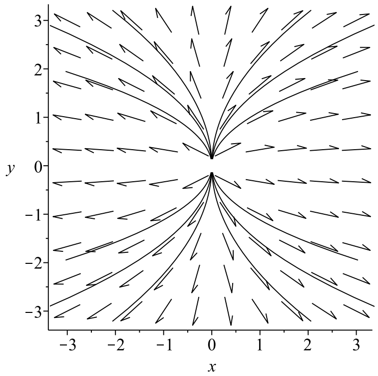


Figure 2.11: Phase portrait for Example 2.6, an unstable node or source.

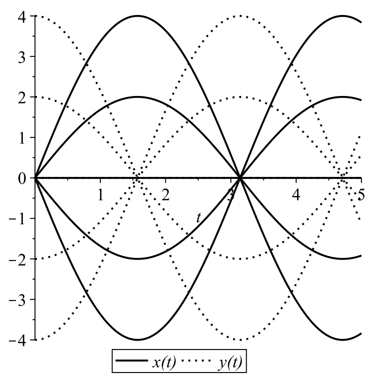


Figure 2.12: Plots of solutions of Example 2.7 for several initial conditions.

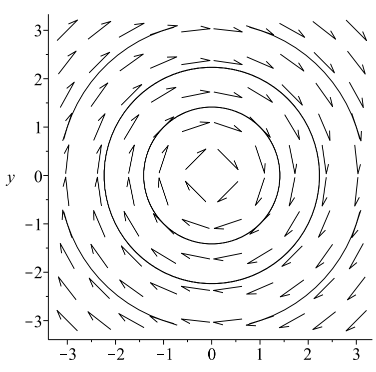


Figure 2.13: Phase portrait for Example 2.7, a center.

Example 2.8. Focus (spiral)

$$\begin{aligned} x' &= \alpha x + y \\ y' &= -x. \end{aligned} \tag{2.26}$$

In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, αx, to the system in Example 2.7. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the Figures 2.15-2.18. We see orbits that look like spirals. These orbits are stable and unstable spirals (or foci, the plural of focus.)

We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation

$$x'' - \alpha x' + x = 0.$$

We recall from our first course that this is a form of damped simple harmonic motion. The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

There are five special cases to consider as shown in the below classification.

Classification of Solutions of $x'' - \alpha x' + x = 0$

1. $\alpha = -2$. There is one real solution. This case is called *critical damping* since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.

2. $\alpha < -2$. There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called *overdamped* motion. There are no oscillations

3. $-2 < \alpha < 0$. There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an *underdamped oscillation*.

4. $\alpha = 0$. This leads to *simple harmonic motion*.

5. $0 < \alpha < 2$. This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.

6. $\alpha = 2$. There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^{t}$. It leads to unbounded growth in time.

7. For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.

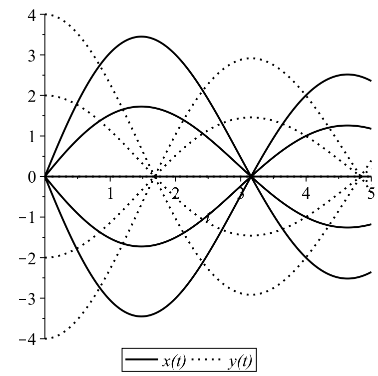


Figure 2.14: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = -0.2$.

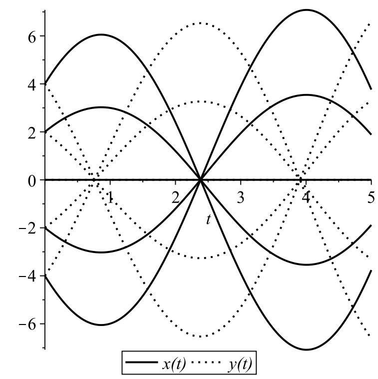


Figure 2.15: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = 0.1$.

For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 2.14.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 2.15.) Of course, there can be overdamped motion if the magnitude of α is too large.

Example 2.9. Degenerate Node For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{2.27}$$

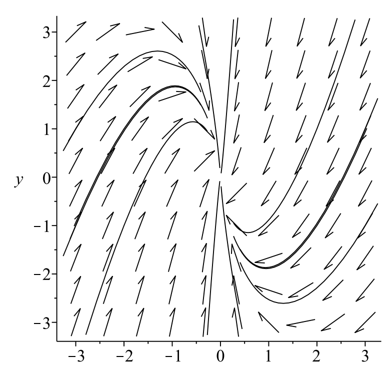


Figure 2.16: Phase portrait for 2.9. This is a degenerate node.

There are two possible approaches:

a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have

$$y' + y = -2c_1 e^{-t}.$$

This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.

b. Another method would be to proceed to rewrite this as a second order equation. Computing x'' does not get us very far. So, we look at

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{2.28}$$

Therefore, y satisfies

$$y'' + 2y' + y = 0.$$

The characteristic equation has one real root, $r = -1$. So, we write

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

This is a stable *degenerate node.* Combining this with the solution $x(t) = c_1 e^{-t}$, we can show that $y(t) = (c_2 - 2c_1 t)e^{-t}$ as before.

In Figure 2.16 we see several orbits in this system. It differs from the stable node show in Figure 2.4 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, then $x(t) = 0$ and $y(t) = c_2 e^{-t}$. This leads to orbits running along the y-axis as seen in the figure.

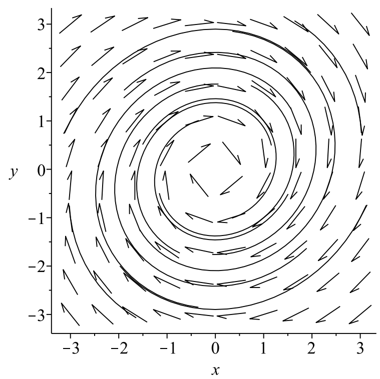


Figure 2.17: Phase portrait for Example 2.8 with $\alpha = -0.2$. This is a stable focus, or spiral.

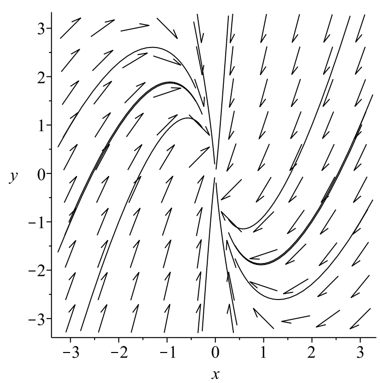


Figure 2.18: Phase portrait for Example 2.9. This is a degenerate node.

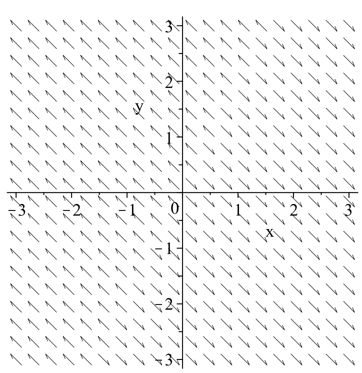


Figure 2.19: Plots of direction field of Example 2.10.

Example 2.10. A Line of Equilibria, Zero Root

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{2.29}$$

In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{2.30}$$

So, the second order equation is

$$x'' - 3x' = 0$$

and the characteristic equation is $0 = r(r - 3)$. This gives the general solution as

$$x(t) = c_1 + c_2 e^{3t}$$

and thus

$$y = 2x - x' = 2(c_1 + c_2 e^3 t) - (3c_2 e^{3t}) = 2c_1 - c_2 e^{3t}.$$

In Figure 2.19 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

Furthermore, looking at initial conditions with $y = 2x$, we have at $t = 0$,

$$2c_1 - c_2 = 2(c_1 + c_2) \quad \Rightarrow \quad c_2 = 0.$$

Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. This line of fixed points is called a line of equilibria.

2.2.3 *Polar Representation of Spirals*

IN THE EXAMPLES WITH A CENTER OR A SPIRAL, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian (x, y) or polar (r, θ) coordinates. Given the polar form, one can find the Cartesian components using

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{2.31}$$

Since x and y are functions of t, then naturally we can think of r and θ as functions of t. Converting a system of equations in the plane for x' and y' to polar form requires knowing r' and θ'. So, we first find expressions for r' and θ' in terms of x' and y'.

Differentiating the first equation in (2.31) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for x' and y' from system 2.9, we have

$$rr' = x(ax + by) + y(cx + dy).$$

In some cases this may be written entirely in terms of r's. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

Derivatives of Polar Variables

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned} \tag{2.32}$$

Example 2.11. Rewrite the following system in polar form and solve the resulting system.

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay. \end{aligned} \tag{2.33}$$

We first compute r' and θ':

$$rr' = xx' + yy' = x(ax + by) + y(-bx + ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

This leads to simpler system

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \tag{2.34}$$

This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that $r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 - bt$. Eliminating t between these solutions, we finally find the polar equation of the orbits:

$$r = r_0 e^{-a(\theta - \theta_0)t/b}.$$

If you graph this for $a \neq 0$, you will get stable or unstable spirals.

Example 2.12. Consider the specific system

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \tag{2.35}$$

In order to convert this system into polar form, we compute

$$rr' = xx' + yy' = x(-y + x) + y(x + y) = r^2.$$

$$r^2\theta' = -xy' - yx' = x(x + y) - y(-y + x) = r^2.$$

This leads to simpler system

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \tag{2.36}$$

Solving these equations yields

$$r(t) = r_0 e^t, \quad \theta(t) = t + \theta_0.$$

Eliminating t from this solution gives the orbits in the phase plane, $r(\theta) = r_0 e^{\theta - \theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

Example 2.13.

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \tag{2.37}$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

2.3 *Applications*

In this section we will describe some simple applications leading to systems of differential equations which can be solved using the methods in this chapter. These systems are left for homework problems and the as the start of further explorations for student projects.

2.3.1 *Mass-Spring Systems*

The first examples that we had seen involved masses on springs. Recall that for a simple mass on a spring we studied simple harmonic motion, which is governed by the equation

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x, \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x, \end{aligned} \tag{2.39}$$

where $\omega^2 = \frac{k}{m}$. The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

We also looked at the system of two masses and two springs as shown in Figure 2.20. The equations governing the motion of the masses is

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.40}$$

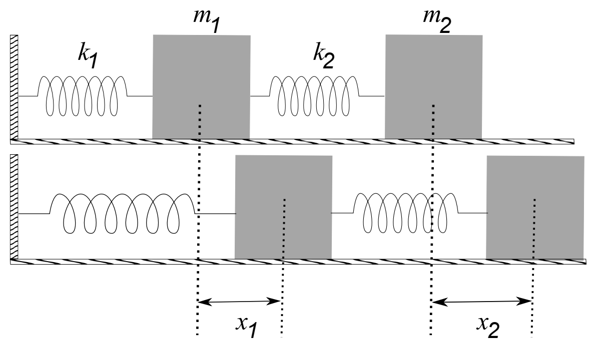


Figure 2.20: System of two masses and two springs.

We can rewrite this system as four first order equations

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1)\\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1).\end{aligned} \tag{2.41}$$

The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}.$$

We can study this system for specific values of the constants using the methods covered in the last sections.

Writing the spring-block system as a second order vector system.

Actually, one can also put the system (2.40) in the matrix form

$$\begin{pmatrix} m_1 & 0\\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1\\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_2\\ k_2 & -k_2 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}. \tag{2.42}$$

This system can then be written compactly as

$$M\ddot{\mathbf{x}} = -K\mathbf{x}, \tag{2.43}$$

where

$$M = \begin{pmatrix} m_1 & 0\\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1+k_2 & -k_2\\ -k_2 & k_2 \end{pmatrix}.$$

This system can be solved by guessing a form for the solution. We could guess

$$\mathbf{x} = \mathbf{a}e^{i\omega t}$$

or

$$\mathbf{x} = \begin{pmatrix} a_1\cos(\omega t - \delta_1)\\ a_2\cos(\omega t - \delta_2) \end{pmatrix},$$

where δ_i are phase shifts determined from initial conditions.

Inserting $\mathbf{x} = \mathbf{a}e^{i\omega t}$ into the system gives

$$(K - \omega^2 M)\mathbf{a} = \mathbf{0}.$$

This is a homogeneous system. It is a generalized eigenvalue problem for eigenvalues ω^2 and eigenvectors $\mathbf{a}$. We solve this in a similar way to the standard matrix eigenvalue problems. The eigenvalue equation is found as

$$\det\,(K - \omega^2 M) = 0.$$

Once the eigenvalues are found, then one determines the eigenvectors and constructs the solution.

Example 2.14. Let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then, we have to solve the system

$$\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The eigenvalue equation is given by

$$\begin{aligned} 0 &= \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} \\ &= (2k - m\omega^2)(k - m\omega^2) - k^2 \\ &= m^2\omega^4 - 3km\omega^2 + k^2. \end{aligned} \tag{2.44}$$

Solving this quadratic equation for ω^2, we have

$$\omega^2 = \frac{3 \pm 1}{2} \frac{k}{m}.$$

For positive values of ω, one can show that

$$\omega = \frac{1}{2}\left(\pm 1 + \sqrt{5}\right)\sqrt{\frac{k}{m}}.$$

The eigenvectors can be found for each eigenvalue by solving the homogeneous system

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

The eigenvectors are given by

$$\mathbf{a}_1 = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}.$$

We are now ready to construct the real solutions to the problem. Similar to solving two first order systems with complex roots, we take the real and imaginary parts and take a linear combination of the solutions. In this problem there are four terms, giving the solution in the form

$$\mathbf{x}(t) = c_1\mathbf{a_1} cos \omega_1 t + c_2 \mathbf{a_1} sin \omega_1 t + c_3 \mathbf{a_2} cos \omega_2 t + c_4 \mathbf{a_2} sin \omega_2 t,$$

where the ω's are the eigenvalues and the $\mathbf{a}$'s are the corresponding eigenvectors. The constants are determined from the initial conditions, $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$.

2.3.2 Circuits*

In the last chapter we investigated simple series LRC circuits. More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This results in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.21. This is not a problem that can be covered in the first year physics course.

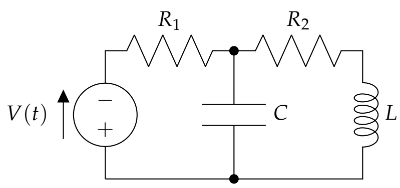


Figure 2.21: A circuit with two loops containing several different circuit elements.

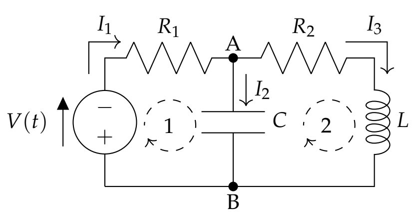


Figure 2.22: The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

There are two loops, indicated in Figure 2.22 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled I_i, $i = 1,2,3$. Corresponding to each current is a changing charge, q_i such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1,2,3.$$

We have for loop one

$$I_1 R_1 + \frac{q_2}{C} = V(t) \tag{2.45}$$

and for loop two

$$I_3 R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{2.46}$$

There are three unknown functions for the charge. Once we know the charge functions, differentiation will yield the three currents. However, we only have two equations. We need a third equation. This equation is found from Kirchoff's Point (Junction) Rule.

Consider the points A and B in Figure 2.22. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{2.47}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{2.48}$$

Equations (2.45), (2.46), and (2.48) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$\begin{aligned} R_1\dot{q}_1 + \frac{q_2}{C} &= V(t) \\ R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{2.49}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions for the other terms. $\dot{q}_3$ is gotten from the definition of q_4 and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned}
\dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} \\
\dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} - q_4 \\
\dot{q}_3 &= q_4 \\
\dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L} q_4.
\end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations.

2.3.3 *Mixture Problems*

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake. We first saw such problems in Chapter 1.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out}.$$

This can be generalized to the case of two interconnected tanks. We will provide an example, but first we review the single tank problem from Chapter 1.

Example 2.15. Single Tank Problem

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 2.23.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.

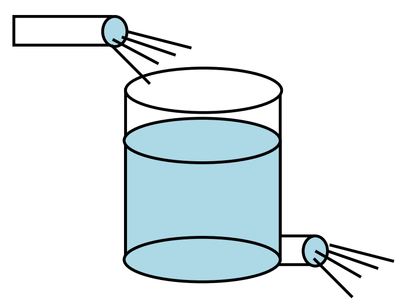


Figure 2.23: A typical mixing problem.

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = 10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x \text{ pounds}}{50 \text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = \frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is easily solved using the methods for first order equations.

Example 2.16. Double Tank Problem

One has two tanks connected together, labeled tank X and tank Y, as shown in Figure 2.24.

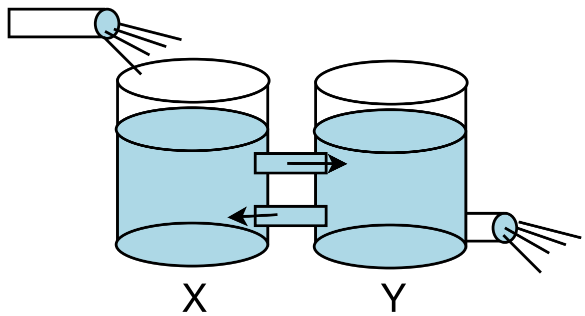


Figure 2.24: The two tank problem.

Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?

In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100} \\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}.\end{aligned} \tag{2.50}$$

This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve. The matrix form of the system is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

The eigenvalues for the problem are given by $\lambda = -3 \pm \sqrt{3}$ and the eigenvectors are

$$\begin{pmatrix} 1 \\ \pm\sqrt{3} \end{pmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is easily written down:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{\left(-3+\sqrt{3}\right)t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{\left(-3-\sqrt{3}\right)t}.$$

Finally, we need to satisfy the initial conditions. So,

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix},$$

or

$$c_1 + c_2 = 100, \quad (c_1 - c_2)\sqrt{3} = 0.$$

So, $c_2 = c_1 = 50$. The final solution is

$$\mathbf{x}(t) = 50 \left(\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{\left(-3+\sqrt{3}\right)t} + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{\left(-3-\sqrt{3}\right)t} \right),$$

or

$$\begin{aligned} x(t) &= 50\left(e^{\left(-3+\sqrt{3}\right)t} + e^{\left(-3-\sqrt{3}\right)t}\right) \\ y(t) &= 50\sqrt{3}\left(e^{\left(-3+\sqrt{3}\right)t} - e^{\left(-3-\sqrt{3}\right)t}\right). \end{aligned} \tag{2.51}$$

2.3.4 *Chemical Kinetics**

THERE ARE MANY PROBLEMS IN THE CHEMISTRY of chemical reactions which lead to systems of differential equations. The simplest reaction is when a chemical A turns into chemical B. This happens at a certain rate, $k > 0$. This reaction can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of A, $[A]$, and B, $[B]$, are given by

The chemical reactions used in these examples are first order reactions. Second order reactions have rates proportional to the square of the concentration.

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{2.52}$$

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

Here there are three concentrations and two rates of change. The system of equations governing the reaction is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.53}$$

The more complication rate of change is when [B] increases from [A] changing to [B] and decrease when [B] changes to [C]. Thus, there are two terms in the rate of change equation for concentration [B].

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$A \underset{k_1}{\overset{k_3}{\longleftrightarrow}} B \underset{k_2}{\longrightarrow} C.$$

The reverse reaction rates contribute to the reaction equations for [A] and [B]. The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A] + k_3[B], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.54}$$

Nonlinear chemical reactions will be discussed in the next chapter.

2.3.5 *Predator Prey Models**

Another common population model is that describing the coexistence of species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\begin{aligned} \frac{dR}{dt} &= aR - cF, \\ \frac{dF}{dt} &= -bF + dR, \end{aligned} \tag{2.55}$$

where all of the constants are positive numbers. Studying this coupled system would lead to a study of the dynamics of these populations. The nonlinear version of this system, the Lotka-Volterra model, will be discussed in the next chapter.

2.3.6 Love Affairs*

The next application is one that was introduced in 1988 by Strogatz as a cute system involving relationships.[1] One considers what happens to the affections that two people have for each other over time. Let R denote the affection that Romeo has for Juliet and J be the affection that Juliet has for Romeo. Positive values indicate love and negative values indicate dislike.

[1] Steven H. Strogatz introduced this problem as an interesting example of systems of differential equations in *Mathematics Magazine*, Vol. 61, No. 1 (Feb. 1988) p 35. He also describes it in his book *Nonlinear Dynamics and Chaos (1994)*.

One possible model is given by

$$\begin{aligned} \frac{dR}{dt} &= bJ \\ \frac{dJ}{dt} &= cR \end{aligned} \tag{2.56}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned} \frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= cR + dJ. \end{aligned} \tag{2.57}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

2.3.7 Epidemics*

Another interesting area of application of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let us break the population into three classes. First, we let $S(t)$ represent the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others could recover. We will consider the case that initially there is one infected person and the rest, say N, are healthy. Can we predict how many deaths have occurred by time t?

We model this problem using the compartmental analysis we had seen for mixing problems. The total rate of change of any population would be due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that a) the rate of infection is proportional to the number of healthy people, aS, and b) the number who recover is proportional to the number of infected people, rI. Thus, the rate of change of healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infected people is due to healthy people getting infected and the infected people who either recover or die. Using the corresponding terms in the other equations, we can write the rate of change of infected people as

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system of differential equations can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \tag{2.58}$$

The reader can find the solutions of this system and determine if this is a realistic model.

2.4 *First Order Matrix Differential Equations*

2.4.1 *Matrix Formulation*

We have investigated several linear systems in the plane and in the next chapter we will use some of these ideas to investigate nonlinear systems. We need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations later in this chapter.

We start with the usual homogeneous system in Equation (2.9). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix A*. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

You can verify that this is a solution by simply differentiating,

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_0 \frac{d}{dt}\left(e^{At}\right) = A\mathbf{x}_0 e^{At} = A\mathbf{x}.$$

2.4.2 *Exponentiating a Matrix*

HOWEVER, THERE REMAINS THE QUESTION, "What does it mean to exponentiate a matrix?" The exponential of a matrix is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.60}$$

In general it is difficult to sum this series, but it is doable for some simple examples.

The *exponential of a matrix* is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.59}$$

In general, it is difficult computing e^A unless A is diagonal.

Example 2.17. Evaluate e^{tA} for $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\begin{aligned}
e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^3 + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\cdots & 0 \\ 0 & 1 + 2t + \frac{2t^2}{2!} + \frac{8t^3}{3!}\cdots \end{pmatrix} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}
\end{aligned} \tag{2.61}$$

Example 2.18. Evaluate e^{tA} for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We first note that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Therefore,

$$A^n = \begin{cases} A, & n \text{ odd}, \\ I, & n \text{ even}. \end{cases}$$

Then, we have

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\ &= I + tA + \frac{t^2}{2!}I + \frac{t^3}{3!}A + \cdots. \\ &= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!}\cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!}\cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!}\cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!}\cdots \end{pmatrix} \\ &= \begin{pmatrix} \cosh & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \end{aligned} \tag{2.62}$$

Since summing these infinite series might be difficult, we will now investigate the solutions of planar systems to see if we can find other approaches for solving linear systems using matrix methods. We begin by recalling the solution to the problem in Example (2.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t}. \end{aligned} \tag{2.63}$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t \\ \frac{1}{3}c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \end{aligned} \tag{2.64}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and λ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (2.4.1) we insert this guess. Thus,

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \Rightarrow \\ \lambda \mathbf{v} e^{\lambda t} &= A\mathbf{v}e^{\lambda t}. \end{aligned} \tag{2.65}$$

For this to be true for all t, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{2.66}$$

This is an eigenvalue problem. A is a 2×2 matrix for our problem, but could easily be generalized to a system of n first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations..

2.4.3 *Eigenvalue Problems*

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{2.67}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is called a *trivial solution*. Typically, we are given the matrix A and have to determine the *eigenvalues*, λ, and the associated *eigenvectors*, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (2.67), we obtain the homogeneous algebraic system

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d-\lambda)v_2 &= 0. \end{aligned} \tag{2.68}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation (2.12) for the general constant coefficient differential equation considered in the first chapter. Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

So, the process is to

a) Write the coefficient matrix;

b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,

c) Find the eigenvectors by solving the linear system $(A - \lambda I)\mathbf{v} = 0$ for each λ.

2.5 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{2.69}$$

The type of behavior depends upon the eigenvalues of matrix A. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions[2], then the general solution is given as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Then, setting $t = 0$, we get two linear equations for c_1 and c_2:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. In Table 2.1 we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

Let's consider Case III. Note that since the original system of equations does not have any i's, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

[2] Recall that linear independence means $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

and

$$\frac{d}{dt} Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

Table 2.1: Solutions Types for Planar Systems with Constant Coefficients

Classification of the Solutions for Two Linear First Order Differential Equations

1. **Case I: Two real, distinct roots.**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

2. **Case II: One Repeated Root**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue λ, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.

3. **Case III: Two complex conjugate roots.**

   Solve the eigenvalue problem A$\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector $\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$. Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t) : \mathbf{y}_1(t) = Re(\mathbf{y}(t))$ and $\mathbf{y}_2(t) = Im(\mathbf{y}(t))$. Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a + d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned} (te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right]. \\ e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2. \\ e^{\lambda t}(\mathbf{v}_1 + \lambda\mathbf{v}_2) &= e^{\lambda t}A\mathbf{v}_2. \end{aligned} \tag{2.70}$$

Noting this is true for all t, we find that

$$\mathbf{v}_1 + \lambda \mathbf{v}_2 = A\mathbf{v}_2. \tag{2.71}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

2.6 *Examples of the Matrix Method*

Here we will give some examples for typical systems for the three cases mentioned in the last section.

Example 2.19. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

Eigenvalues: We first determine the eigenvalues.

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{2.72}$$

Therefore,

$$\begin{aligned} 0 &= (4-\lambda)(3-\lambda) - 6 \\ 0 &= \lambda^2 - 7\lambda + 6 \\ 0 &= (\lambda - 1)(\lambda - 6) \end{aligned} \tag{2.73}$$

The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.

Eigenvectors: Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda \mathbf{v}$ in each case.

Case $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.74}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.75}$$

This gives $3v_1 + 2v_2 = 0$. One possible solution yields an eigenvector of

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Case $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.76}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.77}$$

For this case we need to solve $-2v_1 + 2v_2 = 0$. This yields

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General Solution: We can now construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^{t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^{t} + c_2 e^{6t} \\ -3c_1 e^{t} + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{2.78}$$

Example 2.20. $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues: Again, one solves the eigenvalue equation.

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{2.79}$$

Therefore,

$$\begin{aligned} 0 &= (3-\lambda)(-1-\lambda) + 5 \\ 0 &= \lambda^2 - 2\lambda + 2 \\ \lambda &= \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i. \end{aligned} \tag{2.80}$$

The eigenvalues are then $\lambda = 1+i, 1-i$. This is an example of Case III.

Eigenvectors: In order to find the general solution, we need only find the eigenvector associated with $1+i$.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.81}$$

We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{2.82}$$

Complex Solution: In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.

$$e^{\lambda t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{(1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
&= e^t(\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t\begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.
\end{aligned}
$$

General Solution: Now we can construct the general solution.

$$
\begin{aligned}
\mathbf{x}(t) &= c_1 e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1\cos t + c_2\sin t \end{pmatrix}. \qquad (2.83)
\end{aligned}
$$

Note: This can be rewritten as

$$\mathbf{x}(t) = e^t\cos t\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^t\sin t\begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.$$

Example 2.21. $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} \qquad (2.84)$$

Therefore,

$$
\begin{aligned}
0 &= (7-\lambda)(1-\lambda) + 9 \\
0 &= \lambda^2 - 8\lambda + 16 \\
0 &= (\lambda - 4)^2. \qquad (2.85)
\end{aligned}
$$

There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.

Eigenvectors: In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.

$$
\begin{aligned}
\begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.86)
\end{aligned}
$$

Therefore, we have

$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Second Linearly Independent Solution:
Now we need to solve $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{2.87}$$

Expanding the matrix product, we obtain the system of equations

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{2.88}$$

The solution of this system is $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

General Solution: We construct the general solution as

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1). \\ &= c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}. \end{aligned} \tag{2.89}$$

2.7 *Planar Systems - Summary*

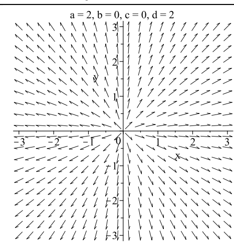
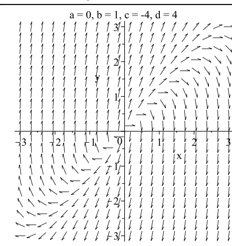
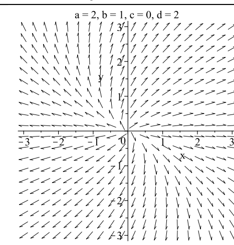
The reader should have noted by now that there is a connection between the behavior of the solutions obtained in Section 2.2.2 and the eigenvalues found from the coefficient matrices in the previous examples. In Table 2.2 we summarize some of these cases.

Type	Eigenvalues	Stability
Node	Real λ, same signs	$\lambda < 0$, stable $\lambda > 0$, unstable
Saddle	Real λ opposite signs	Mostly Unstable
Center	λ pure imaginary	—
Focus/Spiral	Complex λ, $\text{Re}(\lambda) \neq 0$	$\text{Re}(\lambda) < 0$, stable $\text{Re}(\lambda) > 0$, unstable
Degenerate Node	Repeated roots,	$\lambda > 0$, stable
Lines of Equilibria	One zero eigenvalue	$\lambda < 0$, stable

Table 2.2: List of typical behaviors in planar systems.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 2.3 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

Table 2.3: Three examples of systems with a repeated root of $\lambda = 2$.

System 1	System 2	System 3
		
$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$

Another way to look at the classification of these solution is to use the determinant and trace of the coefficient matrix. Recall that the determinant and trace of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by $\det A = ad - bc$ and $\operatorname{tr} A = a + d$.

We note that the general eigenvalue equation,

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

can be written as

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0. \tag{2.90}$$

Therefore, the eigenvalues are found from the quadratic formula as

$$\lambda_{1,2} = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}. \tag{2.91}$$

The solution behavior then depends on the sign of discriminant,

$$(\operatorname{tr} A)^2 - 4\det A.$$

If we consider a plot of where the discriminant vanishes, then we could plot

$$(\operatorname{tr} A)^2 = 4\det A$$

in the $\det A \operatorname{tr} A)$-plane. This is a parabolic cure as shown by the dashed line in Figure 2.25. The region inside the parabola have a negative discriminant, leading to complex roots. In these cases we have oscillatory solutions. If $\operatorname{tr} A = 0$, then one has centers. If $\operatorname{tr} A < 0$, the solutions are stable spirals; otherwise, they are unstable spirals. If the discriminant is positive, then the roots are real, leading to nodes or saddles in the regions indicated.

2.8 *Theory of Homogeneous Constant Coefficient Systems*

There is a general theory for solving homogeneous, constant coefficient systems of first order differential equations. We begin by once again recalling

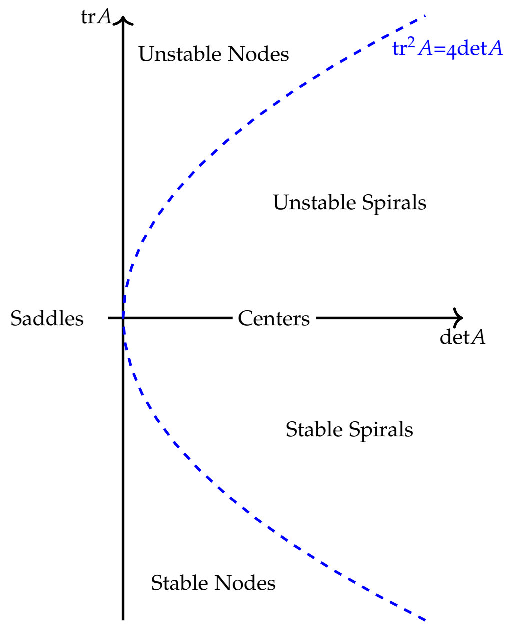


Figure 2.25: Solution Classification for Planar Systems.

the specific problem (2.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.92}$$

This time we rewrite the solution as

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{-4t} \\ \frac{1}{3} e^t & -\frac{1}{2} e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &\equiv \Phi(t)\mathbf{C}. \end{aligned} \tag{2.93}$$

Thus, we can write the general solution as a 2×2 matrix Φ times an arbitrary constant vector. The matrix Φ consists of two columns that are linearly independent solutions of the original system. This matrix is an example of what we will define as the *Fundamental Matrix* of solutions of the system. So, determining the Fundamental Matrix will allow us to find the general solution of the system upon multiplication by a constant matrix. In fact, we will see that it will also lead to a simple representation of the solution of the initial value problem for our system. We will outline the general theory.

Consider the homogeneous, constant coefficient system of first order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \end{aligned}$$

$$\begin{aligned} &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{aligned} \tag{2.94}$$

As we have seen, this can be written in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now, consider m vector solutions of this system: $\phi_1(t)$, $\phi_2(t), \ldots \phi_m(t)$. These solutions are said to be *linearly independent* on some domain if

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_m\phi_m(t) = 0$$

for all t in the domain implies that $c_1 = c_2 = \ldots = c_m = 0$.

Let $\phi_1(t), \phi_2(t), \ldots \phi_n(t)$ be a set of n linearly independent set of solutions of our system, called a *fundamental set of solutions*. We construct a matrix from these solutions using these solutions as the column of that matrix. We define this matrix to be the *fundamental matrix solution*. This matrix takes the form

$$\Phi = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}.$$

What do we mean by a "matrix" solution? We have assumed that each ϕ_k is a solution of our system. Therefore, we have that $\phi_k' = A\phi_k$, for $k = 1, \ldots, n$. We say that Φ is a matrix solution because we can show that Φ also satisfies the matrix formulation of the system of differential equations. We can show this using the properties of matrices.

$$\begin{aligned} \frac{d}{dt}\Phi &= \begin{pmatrix} \phi_1' & \ldots & \phi_n' \end{pmatrix} \\ &= \begin{pmatrix} A\phi_1 & \ldots & A\phi_n \end{pmatrix} \\ &= A\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} \\ &= A\Phi. \end{aligned} \tag{2.95}$$

Given a set of vector solutions of the system, when are they linearly independent? We consider a matrix solution $\Omega(t)$ of the system in which we have n vector solutions. Then, we define the *Wronskian* of $\Omega(t)$ to be

$$W = \det \Omega(t).$$

If $W(t) \neq 0$, then $\Omega(t)$ is a fundamental matrix solution.

Before continuing, we list the fundamental matrix solutions for the set of examples in the last section. (Refer to the solutions from those examples.) Furthermore, note that the fundamental matrix solutions are not unique as one can multiply any column by a nonzero constant and still have a fundamental matrix solution.

Example 2.19 $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix}.$$

We should note in this case that the Wronskian is found as

$$\begin{aligned} W &= \det \Phi(t) \\ &= \begin{vmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{vmatrix} \\ &= 5e^{7t} \neq 0. \end{aligned} \tag{2.96}$$

Example 2.20 $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^t(2\cos t - \sin t) & e^t(\cos t + 2\sin t) \\ e^t \cos t & e^t \sin t \end{pmatrix}.$$

Example 2.21 $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{4t}(1+t) \\ 3e^{4t} & e^{4t}(2+3t) \end{pmatrix}.$$

So far we have only determined the general solution. This is done by the following steps:

Procedure for Determining the General Solution

1. Solve the eigenvalue problem $(A - \lambda I)\mathbf{v} = 0$.
2. Construct vector solutions from $\mathbf{v}e^{\lambda t}$. The method depends if one has real or complex conjugate eigenvalues.
3. Form the fundamental solution matrix $\Phi(t)$ from the vector solution.
4. The general solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$ an arbitrary constant vector.

We are now ready to solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting with the general solution, we have that

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

As usual, we need to solve for the c_k's. Using matrix methods, this is now easy. Since the Wronskian is not zero, then we can invert Φ at any value of t. So, we have

$$\mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Putting $\mathbf{C}$ back into the general solution, we obtain the solution to the initial value problem:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0.$$

You can easily verify that this is a solution of the system and satisfies the initial condition at $t = t_0$.

The matrix combination $\Phi(t)\Phi^{-1}(t_0)$ is useful. So, we will define the resulting product to be the *principal matrix solution*, denoting it by

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0).$$

Thus, the solution of the initial value problem is $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$. Furthermore, we note that $\Psi(t)$ is a solution to the matrix initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = I,$$

where I is the $n \times n$ identity matrix.

Matrix Solution of the Homogeneous Problem

In summary, the matrix solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0,$$

where $\Phi(t)$ is the fundamental matrix solution and $\Psi(t)$ is the principal matrix solution.

Example 2.22. Let's consider the matrix initial value problem

$$\begin{aligned} x' &= 5x + 3y \\ y' &= -6x - 4y, \end{aligned} \tag{2.97}$$

satisfying $x(0) = 1$, $y(0) = 2$. Find the solution of this problem.

We first note that the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalue equation is easily found from

$$\begin{aligned} 0 &= -(5-\lambda)(4+\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned} \tag{2.98}$$

So, the eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found to be

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now we construct the fundamental matrix solution. The columns are obtained using the eigenvectors and the exponentials, $e^{\lambda t}$:

$$\phi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \qquad \phi_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

So, the fundamental matrix solution is

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}.$$

The general solution to our problem is then

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \mathbf{C}$$

for $\mathbf{C}$ is an arbitrary constant vector.

In order to find the particular solution of the initial value problem, we need the principal matrix solution. We first evaluate $\Phi(0)$, then we invert it:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

The particular solution is then

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} + 4e^{2t} \\ 6e^{-t} - 4e^{2t} \end{pmatrix} \end{aligned} \tag{2.99}$$

Thus, $x(t) = -3e^{-t} + 4e^{2t}$ and $y(t) = 6e^{-t} - 4e^{2t}$.

2.9 *Nonhomogeneous Systems*

Before leaving the theory of systems of linear, constant coefficient systems, we will discuss nonhomogeneous systems. We would like to solve systems of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t). \tag{2.100}$$

We will assume that we have found the fundamental matrix solution of the homogeneous equation. Furthermore, we will assume that $A(t)$ and $\mathbf{f}(t)$ are continuous on some common domain.

As with second order equations, we can look for solutions that are a sum of the general solution to the homogeneous problem plus a particular solution of the nonhomogeneous problem. Namely, we can write the general solution as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \mathbf{x}_p(t),$$

where $\mathbf{C}$ is an arbitrary constant vector, $\Phi(t)$ is the fundamental matrix solution of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$\mathbf{x}_p' = A(t)\mathbf{x}_p + \mathbf{f}(t).$$

Such a representation is easily verified.

We need to find the particular solution, $\mathbf{x}_p(t)$. We can do this by applying *The Method of Variation of Parameters for Systems*. We consider a solution in the form of the solution of the homogeneous problem, but replace the constant vector by unknown parameter functions. Namely, we assume that

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{c}(t).$$

Differentiating, we have that

$$\mathbf{x}_p' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = A\Phi\mathbf{c} + \Phi\mathbf{c}',$$

or

$$\mathbf{x}_p' - A\mathbf{x}_p = \Phi\mathbf{c}'.$$

But the left side is $\mathbf{f}$. So, we have that,

$$\Phi\mathbf{c}' = \mathbf{f},$$

or, since Φ is invertible (why?),

$$\mathbf{c}' = \Phi^{-1}\mathbf{f}.$$

In principle, this can be integrated to give $\mathbf{c}$. Therefore, the particular solution can be written as

$$\mathbf{x}_p(t) = \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{2.101}$$

This is the *variation of parameters formula.*

The general solution of Equation (2.100) has been found as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{2.102}$$

We can use the general solution to find the particular solution of an initial value problem consisting of Equation (2.100) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. This condition is satisfied for a solution of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.103}$$

provided

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

This can be solved for $\mathbf{C}$ as in the last section. Inserting the solution back into the general solution (2.103), we have

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.104}$$

This solution can be written a little neater in terms of the principal matrix solution, $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$:

$$\boxed{\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{f}(s)\,ds} \tag{2.105}$$

Finally, one further simplification occurs when A is a constant matrix, which are the only types of problems we have solved in this chapter. In this case, we have that $\Psi^{-1}(t) = \Psi(-t)$. So, computing $\Psi^{-1}(t)$ is relatively easy.

Example 2.23. $x'' + x = 2\cos t$, $x(0) = 4$, $x'(0) = 0$. This example can be solved using the Method of Undetermined Coefficients. However, we will use the matrix method described in this section.

First, we write the problem in matrix form. The system can be written as

$$\begin{aligned} x' &= y \\ y' &= -x + 2\cos t. \end{aligned} \tag{2.106}$$

Thus, we have a nonhomogeneous system of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}.$$

Next we need the fundamental matrix of solutions of the homogeneous problem. We have that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm i$. An eigenvector associated with $\lambda = i$ is easily found as $\begin{pmatrix} 1 \\ i \end{pmatrix}$. This leads to a complex solution

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}.$$

From this solution we can construct the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So, the general solution to the homogeneous problem is

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}.$$

Next we seek a particular solution to the nonhomogeneous problem. From Equation (2.103) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have

$$\begin{aligned}\Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\ &= \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix}. \end{aligned} \tag{2.107}$$

We now compute

$$\begin{aligned}\Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\int_{t_0}^t \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} -\sin^2 t \\ t+\frac{1}{2}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}. \end{aligned} \tag{2.108}$$

therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix},$$

or

$$\mathbf{x} = \begin{pmatrix} 4\cos t + t\sin t \\ -3\sin t + t\cos t \end{pmatrix}.$$

2.10 *Appendix: More on Exponentiation of Matrices*

We have seen that in general that exponentiating a matrix using the Maclaurin series (2.60) can be difficult. If the matrix is diagonal, then the exponentiation is simple since powers of a diagonal matrix involves the powers of the diagonal elements. In other words, if

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

then

$$e^A = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & \cdots & 0 \\ 0 & e^{\lambda_2} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{\lambda_n} \end{pmatrix}.$$

If A is not diagonal, then if it is diagonalizable, then there is an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix. Then, we have

$$\begin{aligned} e^A &= e^{PDP^{-1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{n!}(PDP^{-1})^n \\ &= I + PDP^{-1} + \frac{PDP^{-1}PDP^{-1}}{2!} + \frac{PDP^{-1}PDP^{-1}PDP^{-1}}{3!} + \cdots \\ &= I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots \\ &= P\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)P^{-1} \\ &= Pe^DP^{-1}. \end{aligned} \quad (2.109)$$

Therefore, we have a prescription for exponentiating a diagonalizable matrix.

But when are matrices diagonalizable? We learn from a linear algebra course that real symmetric matrices are diagonalizable.[3] A real symmetric matrix A is one which is a matrix of real numbers such that A equals its transpose, $A^T = A$. Such matrices satisfy several properties, which we will not prove here.

[3] This is a simple example of a more general theory where real symmetric matrices are replaced by complex Hermitian matrices which are self-adjoint and orthogonal matrices are replaced with unitary matrices. As matrices are representations of operators, the more general theory of self-adjoint operators are discussed later.

- The eigenvalues of a real symmetric matrix are real.
- The eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are orthogonal.
- Every real symmetric $n \times n$ matrix posses a complete set of orthonormal vectors.
- For every real symmetric $n \times n$ matrix A there is an $n \times n$ real orthogonal matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

A real matrix P is orthogonal if $P^{-1} = P^T$. Therefore $P^TP = I$. Such orthogonal matrices have column (and row) vectors which are orthogonal. Thus, one can use the orthonormal eigenvectors of A to form the matrix P. Namely, we let $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ and $P = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then,

$$\begin{aligned} P^{-1}AP &= (\mathbf{v}_1, \ldots, \mathbf{v}_n)^{-1}A(\mathbf{v}_1, \ldots, \mathbf{v}_n), \\ &= (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T(\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n), \\ &= \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1\mathbf{v}_1 & \ldots & \lambda_n\mathbf{v}_n \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix}. \qquad (2.110)$$

Thus, we have shown that $P^{-1}AP$ is a diagonal matrix and this is called a similarity transformation.

Similarity transformation.

Example 2.24. Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Find e^{tA} using the diagonalization of A.

First, we solve the eigenvalue problem. The eigenvalue equation is given by $0 = |A - \lambda I| = (1-\lambda)^2 - 9$. So, $\lambda = 4, -2$.

Next, we seek the eigenvectors. Letting $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)^T$, we have

$$(1-\lambda)\mathbf{v}_1 + 3\mathbf{v}_2 = 0.$$

For $\lambda = 4$, we need to solve $(-3)\mathbf{v}_1 + 3\mathbf{v}_2 = 0$, or $\mathbf{v}_1 = \mathbf{v}_2$. For $\lambda = -2$, we need to solve $(3)\mathbf{v}_1 + 3\mathbf{v}_2 = 0$, or $\mathbf{v}_1 = -\mathbf{v}_2$. Therefore, the eigenvalues and normalized eigenvectors are given by

$$\lambda = 4, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = -2, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This gives[4]

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

[4] Since P is orthogonal, it is a rotation matrix. In two dimensions a rotation of coordinate axes about the origin by angle θ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, in the example $\theta = -\frac{\pi}{4}$.

Then, we have

$$\begin{aligned} P^{-1}AP &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}. \end{aligned} \qquad (2.111)$$

Finally, we can exponentiate tA as

$$\begin{aligned} e^{tA} &= e^{tPDP^{-1}} = Pe^{tD}P^{-1} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{4t} & e^{4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{pmatrix}. \end{aligned} \qquad (2.112)$$

As a word of caution, we note that real symmetric matrices can be diagonalized. what about real nonsymmetric matrices? A real matrix can be transformed into a diagonal matrix through a similarity transformation as long as it has a full set of linearly independent eigenvectors. We need only look at Example 2.22. In that case

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalues were found as $\lambda = -1, 2$ and the corresponding eigenvectors were

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned} \tag{2.113}$$

So, we have a similarity transformation, but note that P is not an orthogonal matrix in this case.

We can use this diagonalization to compute e^{tA} as we show in the next example.

Example 2.25. Find e^{tA} for $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$.

$$\begin{aligned} e^{tA} &= e^{tPDP^{-1}} = Pe^{tD}P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -e^{-t} & -e^{-t} \\ 2e^{2t} & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}. \end{aligned} \tag{2.114}$$

From Example 2.22, the fundamental matrix solution is given as

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}$$

and

$$\Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

So, the principal matrix solution is found as

$$\begin{aligned}
\Phi(t) &= \Phi(t)\Phi^{-1}(0) \\
&= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -e^{-t}+2e^{2t} & -e^{-t}+2e^{2t} \\ 2e^{-t}-2e^{2t} & 2e^{-t}-e^{2t} \end{pmatrix}, \qquad (2.115)
\end{aligned}$$

which is the same as our evaluation of e^{tA} in the previous example.

There are other methods for exponentiating a matrix. These involve more sophisticated matrix methods and rely on the Cayley-Hamilton Theorem.

The Cayley-Hamilton Theorem Matrix A satisfies its own characteristic (eigenvalue) equation. This means that any power of A, and therefore e^A, can be expressed as a polynomial of A of degree $n-1$. We can also use this to evaluate A^{-1}. We demonstrate this by example.

Example 2.26. First, we note that

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix}.$$

Inserting these into the characteristic equation [see Example 2.22], we have

$$A^2 - A - 2I = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

Now, we rearrange the equation as $A(A-I) = 2I$ and find that $A^{-1} = \frac{1}{2}(A-I)$. Thus,

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 3 \\ -6 & -5 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ -3 & -\frac{5}{2} \end{pmatrix}.$$

This agrees with the usual method for computing A^{-1}.

We can use the Cayley Hamilton Theorem to evaluate functions of A. Let $f(A)$ be a matrix polynomial of A of degree n Also, let $d(\lambda) = \det(A - \lambda I)$. If the characteristic equation for A is $d(\lambda) = 0$, then we know by the Cayley-Hamilton Theorem that $d(A) = 0$. Given polynomials $f(x)$ and $d(x)$, we know that we can write

$$f(x) = d(x)q(x) + r(x),$$

where $r(x)$ is a polynomial of degree $n-1$. So, we also have $f(A) = d(A)q(A) + r(A) = r(A)$. Therefore, $f(A)$ is a polynomial in A of degree $n-1$. So, we can write $f(A) = \alpha_{n-1}A^{n-1} + \alpha_{n-1}A^{n-1} + \ldots \alpha_0 I$.

Example 2.27. Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. Find A^{10}.

The eigenvalues are $\lambda = 5, 2$. So, for $f(A) = A^{10}$, then $f(\lambda) = \lambda^{10}$ and $r(A)$ is a polynomial of degree 1. So, we have

$$A^{10} = r(A) = \alpha_1 A + \alpha_0 I.$$

Evaluating $f(\lambda_i) = r(\lambda_i)$, we obtain two equations for the two unknowns,

$$\begin{aligned} 5^{10} &= 5\alpha_1 + \alpha_0 \\ 2^{10} &= 2\alpha_1 + \alpha_0. \end{aligned} \tag{2.116}$$

Solving, we find

$$\alpha_1 = 3254867, \quad \alpha_0 = -6508710.$$

Then,

$$A^{10} = \begin{pmatrix} 6510758 & 3254867 \\ 6509734 & 3255891 \end{pmatrix}.$$

These methods can be generalized to evaluate arbitrary functions. As an example, we can exponentiate a matrix

Example 2.28. Find e^{tA} for $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$.

This is Example 2.25. However, in this case we let

$$e^{tA} = \alpha_1 A + \alpha_0 I.$$

The eigenvalues are $\lambda = -1, 2$. So, for $f(A) = e^{tA}$, then $f(\lambda) = e^{t\lambda}$. We have $r(A) = \alpha_1 A + \alpha_0 I$ and $r(\lambda) = \alpha_1 \lambda + \alpha_0$. Substituting the eigenvalues, we obtain

$$\begin{aligned} e^{-t} &= -\alpha_1 + \alpha_0 \\ e^{2t} &= 2\alpha_1 + \alpha_0. \end{aligned} \tag{2.117}$$

Solving, we have

$$\alpha_0 = \frac{1}{3}\left(2e^{-t} + e^{2t}\right), \alpha_1 = \frac{1}{3}\left(e^{2t} - e^{-t}\right).$$

Then,

$$e^{tA} = \alpha_1 A + \alpha_0 I = \begin{pmatrix} -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}.$$

Again, we arrive at the same result.

The Putzer Algorithm. Suppose that A is an $n \times n$ matrix with eigenvalues λ_k, $k = 1, \ldots, n$, not necessarily distinct. Then,

$$e^{tA} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j$$

where

$$P_j = \prod_{k=1}^{j} (A - \lambda_k I)$$

and the $r_j(t)$'s are the components of the solutions of

$$\mathbf{p}'(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 1 & \lambda_n \end{pmatrix} \mathbf{p}(t)$$

satisfying $\mathbf{p}(0) = (1, 0, \ldots, 0)^T$.

Example 2.29. Let's try a different example. Consider $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$. Find e^{tA} using the Putzer Algorithm.

Here $n = 2$ and the eigenvalues are $\lambda = -1, -3$. First we find the P_j's.

$$P_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 = A + I = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}.$$

Next, we solve the tridiagonal system

$$\begin{pmatrix} r_1'(t) \\ r_2'(t) \end{pmatrix} = \left(\begin{matrix} -1 & 0 \\ 1 & -3 \end{matrix} \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} \right)$$

with $r_1(0) = 1, \quad r_2(0) = 0$. Expanding into a coupled system of differential equations,

$$\begin{aligned} r_1' &= -r_1, \quad r_1(0) = 1, \\ r_2' &= r_1 - 3r_2, \quad r_2(0) = 0. \end{aligned} \tag{2.118}$$

The first equation is readily solved to find $r_1(t) = e^{-t}$. The second equation becomes

$$r_2' + 3r_2 = e^{-t}, \quad r_2(0) = 0.$$

This is solved using and integrating factor. The result is $r_2(t) = \frac{1}{2}\left(e^{-t}e^{-3t}\right)$.

Now that we have the pieces for the algorithm, we can construct the answer.

$$\begin{aligned} e^{tA} &= \sum_{j=0}^{n-1} r_{j+1}(t) P_j \\ &= r_1(t)P_0 + r_2(t)P_1 \\ &= e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\left(e^{-t}e^{-3t}\right) \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^{-3t} & -e^{-t}2e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -e^{-t} + 2e^{-3t} \end{pmatrix}. \end{aligned} \tag{2.119}$$

Problems

1. Consider the system

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.

b. Solve the differential equation for $x(t)$.

c. Using this solution, find $y(t)$.

d. Verify your solutions for $x(t)$ and $y(t)$.

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

2. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned} x' &= 3x \\ y' &= -2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= 2y \\ y' &= -3x. \end{aligned}$$

d.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

e.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

3. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

4. Consider the system of equations in Example 2.13.

a. Derive the polar form of the system.

b. Solve the radial equation, $r' = r(1 - r^2)$, for the initial values $r(0) = 0, 0.5, 1.0, 2.0$.

c. Based upon these solutions, plot and describe the behavior of all solutions to the original system in Cartesian coordinates.

5. Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 2, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

a.
$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

b.
$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.
$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

d.
$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

e.
$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

f.
$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

6. For the given matrix, evaluate e^{tA}, using the definition

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \dots,$$

and simplifying.

a. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

c. $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

d. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

e. $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

f. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

7. Find the fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$ where matrix A is given. If an initial condition is provided, find the solution of the initial value problem using the principal matrix.

a. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 12 & -15 \\ 4 & -4 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.

d. $A = \begin{pmatrix} 4 & -13 \\ 2 & -6 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

e. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

f. $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$.

g. $A = \begin{pmatrix} 8 & -5 \\ 16 & 8 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

h. $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$.

i. $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

8. Solve the following initial value problems using Equation (2.105), the solution of a nonhomogeneous system using the principal matrix solution.

a. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. $\mathbf{x}' = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

9. Add a third spring connected to mass two in the coupled system shown in Figure 2.2 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.

b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

d. Model this initial value problem with a set of two second order differential equations. Set up the system in the form $M\ddot{\mathbf{x}} = -K\mathbf{x}$ and solve using the values in part b.

10. In Example 2.14 we investigated a couple mass-spring system as a pair of second order differential equations.

a. In that problem we used $\sqrt{\frac{3\pm\sqrt{5}}{2}} = \frac{\sqrt{5}\pm 1}{2}$. Prove this result.

b. Rewrite the system as a system of four first order equations.

c. Find the eigenvalues and eigenfunctions for the system of equations in part b to arrive at the solution found in Example 2.14.

d. Let $k = 5.00$ N/m and $m = 0.250$ kg. Assume that the masses are initially at rest and plot the positions as a function of time if initially i) $x_1(0) = x_2(0) = 10.0$ cm and i) $x_1(0) = -x_2(0) = 10.0$ cm. Describe the resulting motion.

11. Consider the series circuit in Figure 1.6 with $L = 1.00$ H, $R = 1.00 \times 10^2$ Ω, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

12. Consider the series circuit in Figure 2.21 with $L = 1.00$ H, $R_1 = R_2 = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of first order differential equations for the charges and the currents in each loop.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

13. Initially a 100 gallon tank is filled with pure water. At time $t = 0$ water with a half a pound of salt per two gallons is added to the container at the rate of 3 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

14. You make two quarts of salsa for a party. The recipe calls for five teaspoons of lime juice per quart, but you had accidentally put in five tablespoons per quart. You decide to feed your guests the salsa anyway. Assume that the guests take a quarter cup of salsa per minute and that you replace what was taken with chopped tomatoes and onions without any lime juice. [1 quart = 4 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of lime juice as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the salsa back to the recipe's suggested concentration?

15. Consider the chemical reaction leading to the system in (2.54). Let the rate constants be $k_1 = 0.20\ \text{ms}^{-1}$, $k_2 = 0.05\ \text{ms}^{-1}$, and $k_3 = 0.10\ \text{ms}^{-1}$. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ μmol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

16. Find and classify any equilibrium points in the Romeo and Juliet problem for the following cases. Solve the systems and describe their affections as a function of time.

a. $a = 0, b = 2, c = -1, d = 0, R(0) = 1, J(0) = 1.$

b. $a = 0, b = 2, c = 1, d = 0, R(0) = 1, J(0) = 1.$

c. $a = -1, b = 2, c = -1, d = 0, R(0) = 1, J(0) = 1.$

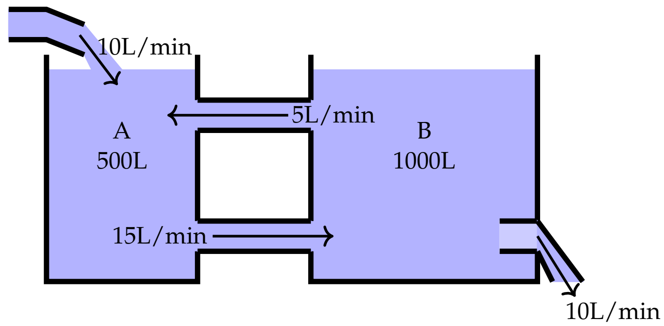


Figure 2.26: Figure for Problem 17.

17. Two tanks contain a mixture of water and alcohol with tank A containing 500 L and tank B 1000L. Initially, the concentration of alcohol in Tank A is 0% and that of tank B is 80%. Solution leaves tank A into B at a rate of 15 liter/min and the solution in tank B returns to A at a rate of 5 L/min while well mixed solution also leaves the system at 10 liter/min through an outlet. A mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 10% through an inlet. What will be the concentration of the alcohol of the solution in each tank after 10 mins?

18. Consider the tank system in Problem 17. Add a third tank (C) to tank B with a volume of 300 L. Connect C with 8 L/min from tank B and 2 L/min flow back. Let 10 L/min flow out of the system. If the initial concentration is 10% in each tank and a mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 20% through an inlet, what will be the concentration of the alcohol in each of the tanks after an hour?

19. Consider the epidemic model leading to the system in (2.58). Choose the constants as $a = 2.0$ days^{-1}, $d = 3.0$ days^{-1}, and $r = 1.0$ days^{-1}. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of $1,000$ and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?