Chapter 1

Introduction

In order to solve a differential equation, you look at it till a solution occurs to you.
- George Pólya (1887-1985)

These are notes for a second course in differential equations originally taught in the Spring semester of 2005 at the University of North Carolina Wilmington to upper level and first year graduate students and later updated in Fall 2007, Fall 2008, and Fall 2024. It is assumed that you have had an introductory course in differential equations. However, we will begin this chapter with a review of some of the material from your first course in differential equations and then give an overview of the material we are about to cover.

Typically an introductory course in differential equations introduces students to analytical solutions of first order differential equations which are separable, first order linear differential equations, and sometimes to some other special types of equations. Students then explore the theory of second order differential equations generally restricted to the study of exact solutions of constant coefficient linear differential equations or even equations of the Cauchy-Euler type. These are later followed by the study of special techniques, such as power series methods or Laplace transform methods. If time permits, ones explores a few special functions, such as Legendre polynomials and Bessel functions, while using power series methods for solving differential equations.

More recently, variations on this inventory of topics have been introduced through the early introduction of systems of differential equations, qualitative studies of these systems and a more intense use of technology for understanding the behavior of solutions of differential equations. This is typically done at the expense of not covering power series methods, special functions, or Laplace transforms. In either case, the types of problems solved are initial value problems in which the differential equation to be solved is accompanied by a set of initial conditions.

In this course we will assume some exposure to the overlap of these two approaches. We will first give a quick review of the solution of separable and linear first order equations. Then we will review second order linear differential equations and Cauchy-Euler equations. This will then be followed by an overview of some of the topics covered. As with any course in differential equations, we will emphasize analytical, graphical and (sometimes) approximate solutions of differential equations. Throughout we will present applications from physics, chemistry and biology.

1.1 *Review of the First Course*

In this section we review a few of the solution techniques encountered in a first course in differential equations. We will not review the basic theory except in possible references as reminders as to what we are doing.

We first recall that an *n-th order ordinary differential equation* is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An *initial value problem* consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.2}$$

A *linear nth order differential equation* takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{1.3}$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

1.1.1 *First Order Differential Equations*

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \tag{1.4}$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

Separable equations.

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be *autonomous*.

The *general solution* to equation (1.5) is obtained in terms of two integrals:

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx + C, \qquad (1.6)}$$

where C is an integration constant. This yields a *1-parameter family of solutions* to the differential equation corresponding to different values of C. If one can solve (1.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

Example 1.1. $y' = 2xy$, $y(0) = 2$.

Applying (1.6), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 1.1.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial conditions is $y(x) = 2e^{x^2}$.

Example 1.2. $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 1.2.

Linear first order equations.

The second type of first order equation encountered is the *linear first order differential equation* in the form

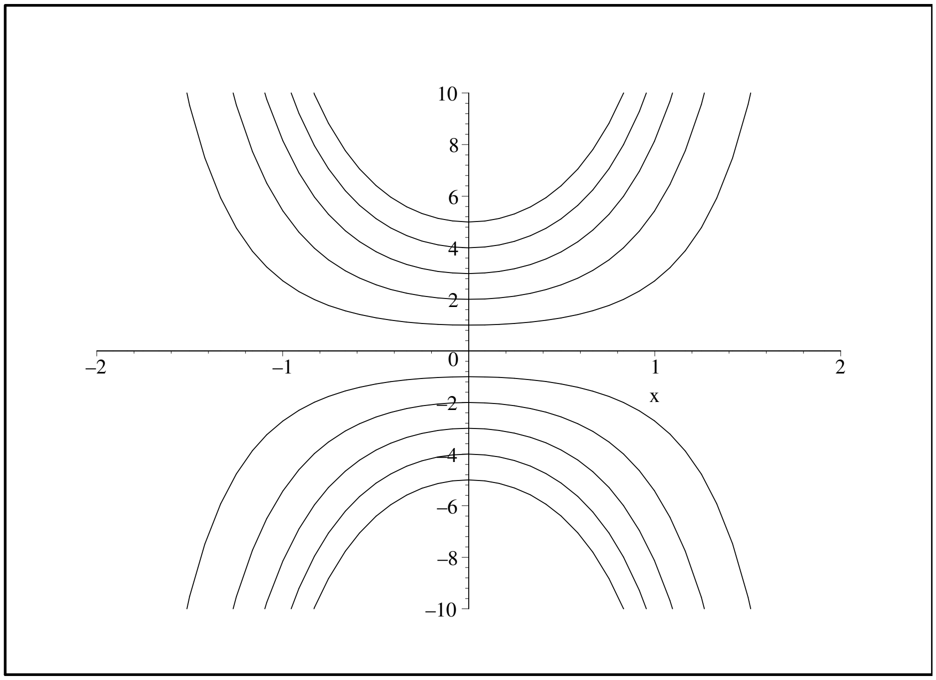
$$y'(x) + p(x)y(x) = q(x). \qquad (1.7)$$

Integrating factor.

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x). \qquad (1.8)$$

Figure 1.1: Plots of solutions from the 1-parameter family of solutions of Example 1.1 for several initial conditions.



The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\, d\xi)$. One can show this by expanding the derivative in Equation (1.8),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{1.9}$$

and comparing this equation to the one obtained from multiplying (1.7) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{1.10}$$

Note that these last two equations would be the same if

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation whose solution is the above given form for the integrating factor,

$$\boxed{\mu(x) = \exp\left(\int^x p(\xi)\, d\xi\right).} \tag{1.11}$$

Equation (1.8) is easily integrated to obtain

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\, d\xi + C\right].} \tag{1.12}$$

Example 1.3. $xy' + y = x, \quad x > 0, y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the original equation is not separable. However, it is not in the standard form. So, we first rewrite the

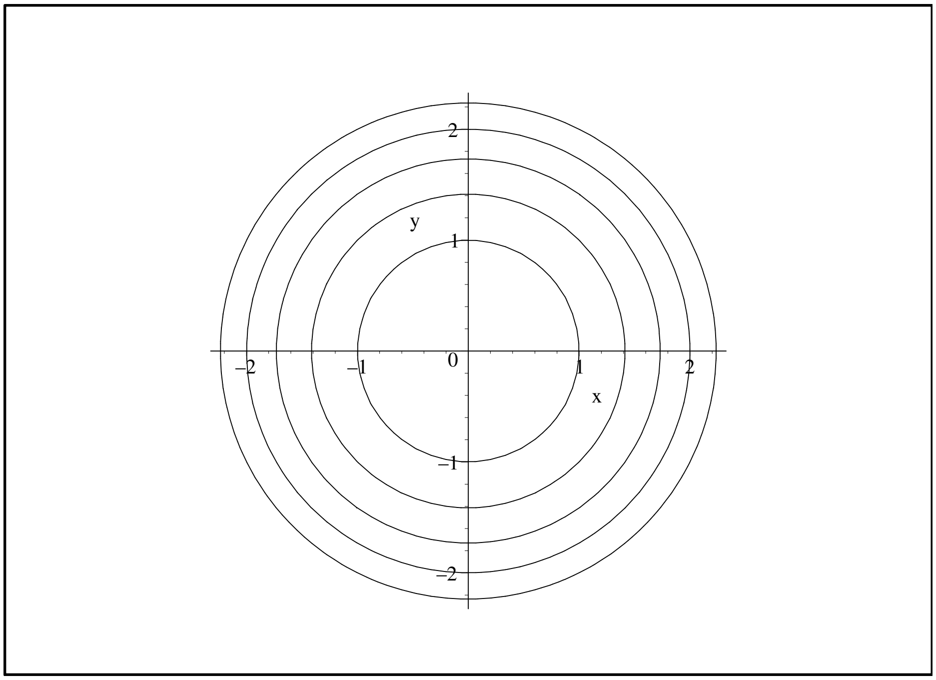


Figure 1.2: Plots of solutions of Example 1.2 for several initial conditions.

equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{1.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (1.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, equation (1.8) becomes

$$(xy)' = x.$$

Integrating one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is $y(x) = \frac{1}{2}(x - \frac{1}{x})$.

Example 1.4. $(\sin x)y' + (\cos x)y = x^2 \sin x$.

Actually, this problem is easy if you realize that

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, rewrite the original differential equation in standard form:

$$y' + (\cot x)y = x^2.$$

Then, compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \xi \, d\xi\right) = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Using the integrating factor, the original equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later in the course.

1.1.2 *Second Order Linear Differential Equations*

SECOND ORDER DIFFERENTIAL EQUATIONS ARE TYPICALLY harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.14}$$

One can rewrite this equation using operator terminology.[1] Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (1.14) becomes

$$Ly = f. \tag{1.15}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space* [2] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

[1] We note that Leibniz introduced $\frac{dy}{dx}$ and Newton used the dot notation, $\dot{y}$. About a century later Lagrange introduced y' and Arbogast introduced the operator notation D.

[2] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

a. $L(f+g) = Lf + Lg$

b. $L(af) = aLf$.

One typically solves (1.14) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{1.16}$$

There are methods for finding a particular solution of a differential equation. These range from pure guessing to the Method of Undetermined Coefficients, or by making use of the Method of Variation of Parameters. We will review some of these methods later.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, others have studied a variety of second order linear equations and have saved us the trouble for some of the differential equations that often appear in applications.

Again, linearity is useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[3] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. As you may recall, linear independence is established if the Wronskian of the solutions in not zero. In this case, we have

[3] Recall, a set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1 y(1(x) + \ldots + c_n y_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{1.17}$$

1.1.3 Constant Coefficient Equations

THE SIMPLEST AND MOST SEEN SECOND ORDER DIFFERENTIAL EQUATIONS are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{1.18}$$

where a, b, and c are constants.

Solutions to (1.18) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (1.18) leads to the *characteristic equation*

Characteristic equation.

$$ar^2 + br + c = 0. \tag{1.19}$$

The roots of this equation in turn lead to three types of solution depending upon the nature of the roots as shown below.

Example 1.5. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \qquad (1.20)$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$. [This is covered in the appendix to this chapter.]

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$. [This is covered in the appendix to this chapter.]

Example 1.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is easily obtained as $y(x) = (c_1 + c_2 x)e^{-3x}$.

Example 1.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

Example 1.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right] e^{-x}.$$

One of the most important applications of the equations in the last two examples is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$mx'' + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. Such motion is called *simple harmonic motion*.

Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. The next example shows a forced harmonic oscillator.

Example 1.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 1.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what A needs to be. Recall, this is the *Method of Undetermined Coefficients* which we review in the next section. Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

1.1.4 *Method of Undetermined Coefficients*

To date, we only know how to solve constant coefficient, homogeneous equations. How does one solve a nonhomogeneous equation like that in Equation (1.14),

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.21}$$

Recall, that one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. So, how do we find the particular solution?

General solution of the nonhomogeneous problem, $y = y_h + y_p$.

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically develop the particular solution. We will come back to this method the Method of Variation of Parameters, later in the book.

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 1.10. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{1.22}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{1.23}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (1.22). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (1.22), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining our two solutions, we have the general solution to the original nonhomogeneous equation (1.22). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine our arbitrary constants.

What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{1.24}$$

The only thing that would change is our particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (**??**), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (1.24)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{1.25}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{1.26}$$

So, our particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 1.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have your solution. This solution is then added to the general solution of the homogeneous differential equation.

Table 1.1: Educated guesses given nonhomogeneous $f(x)$.

$f(x)$	Guess
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Example 1.11. As a final example, let's consider the equation

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{1.27}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If our driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

1.1.5 Cauchy-Euler Equations

Another class of solvable linear differential equations that is of interest are the Cauchy-Euler type of equations. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{1.28}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (1.28), we have

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

Characteristic equation for the Cauchy-Euler Equation.

$$ar(r-1) + br + c = 0. \tag{1.29}$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. These are shown below. Some of the details are provided in the next section.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Example 1.12. $x^2y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{1.30}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)\right] x^{-2}$

Example 1.13. $t^2y'' + 3ty' + y = 0, \quad y(1) = 0, y'(1) = 1.$

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln|t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t|t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t|t^{-1}$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients. We will demonstrate this with a couple of examples.

Example 1.14. Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1x^{-1} + c_2x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{1.31}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3}x^2.$$

Example 1.15. Find the solution of $x^2 y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2 y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

1.2 Overview of the Course

For the most part, your first course in differential equations was about solving initial value problems. When second order equations did not fall into the above cases, then you might have learned how to obtain approximate solutions using power series methods, or even finding new functions from these methods. In this course we will explore two broad topics: systems of differential equations and boundary value problems.

We will see that there are interesting initial value problems when studying systems of differential equations. In fact, many of the second order equations that you have seen in the past can be written as a system of two first order equations. For example, the equation for simple harmonic motion,

$$x'' + \omega^2 x = 0,$$

can be written as the system

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 x \end{aligned}.$$

Just note that $x'' = y' = -\omega^2 x$. Of course, one can generalize this to systems with more complicated right hand sides. The behavior of such systems can be fairly interesting and these systems result from a variety of physical models.

In the second part of the course we will explore boundary value problems. Often these problems evolve from the study of partial differential equations. Such examples stem from vibrating strings, temperature distributions, bending beams, etc. Boundary conditions are conditions that are imposed at more than one point, while for initial value problems the conditions are specified at one point. For example, we could take the oscillation equation above and ask when solutions of the equation would satisfy the conditions $x(0) = 0$ and $x(1) = 0$. The general solution, as we have determined earlier, is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Requiring $x(0) = 0$, we find that $c_1 = 0$, leaving $x(t) = c_2 \sin \omega t$. Also imposing that $0 = x(1) = c_2 \sin \omega$, we are forced to make $\omega = n\pi$, for $n = 1, 2, \ldots$. (Making $c_2 = 0$ would not give a nonzero solution of the problem.) Thus, there are an infinite number of solutions possible, if we have the freedom to choose our ω. In the second half of the course we will investigate techniques for solving boundary value problems and look at several applications, including seeing the connections with partial differential equations and Fourier series.

1.3 Appendix: Reduction of Order and Complex Roots

In this section we provide some of the details leading to the general forms for the constant coefficient and Cauchy-Euler differential equations. In the first subsection we review how the Method of Reduction of Order is used to obtain the second linearly independent solutions for the case of one repeated root. In the second subsection we review how the complex solutions can be used to produce two linearly independent real solutions.

Method of Reduction of Order

First we consider constant coefficient equations. In the case when there is a repeated real root, one has only one independent solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second solution? Since the solutions are independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. For constant coefficient second order equations, we can write the equation as

$$(D - r)^2 y = 0,$$

where $D = \frac{d}{dx}$.

We now insert $y_2(x)$ into this equation. First we compute

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$(D - r)^2 ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, $(D - r)^2 y_2 = 0$, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$.

Deriving the solution for Case 2 for the Cauchy-Euler equations is messier, but works in the same way. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1 r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Rewriting this, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the general Cauchy-Euler equation in this case takes the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1 - 1}, \\ y_2''(x) &= (x^2 v'' + 2r_1 xv' + r_1(r_1 - 1)v)x^{r_1 - 2}. \end{aligned} \tag{1.32}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1 + 1}. \end{aligned} \tag{1.33}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln |v'| = -\ln |x| + C.$$

Exponentiating, we have one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln |x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln |x|.$$

Complex Roots

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula

$$e^{i\beta x} = \cos \beta x + i \sin \beta x. \tag{1.34}$$

Then the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos \beta x + i(A-B)\sin \beta x\right] \\ &\equiv e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \end{aligned} \tag{1.35}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$.

When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first recall that

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^{\alpha} e^{i\beta \ln x}, \quad x > 0.$$

We can now find two real, linearly independent solutions, $x^{\alpha} \cos(\beta \ln |x|)$ and $x^{\alpha} \sin(\beta \ln |x|)$ following the same steps as above for the constant coefficient case.

1.4 Applications

Differential equations arose due to the need to solve problems. In a first course you might have seen some typical applications of first and second order differential equations. For example first order equuations arise in the study of population dynamics and logistic growth, Newton's law of cooling, free fall with drag, or mixture problems. Some historically interesting problems such as finding orthogonal families of curves, pursuit curves, and tractional motion were taken up by some of the top mathematicians of the eighteenth century.

Second order differential equations appear in applications of simple harmonic motion (SHM or SHO in some texts)such as a mass on a spring, pendula (the plural of pendulum), and simple circuits. Some of these problems can be further generalized into problems involving systems of differential equations. So, we will describe some of these applications as well as some interesting first order problems which can also be found in *A First Course in Differential Equations for Scientists and Engineers* by the author.

1.4.1 Mass-Spring Systems

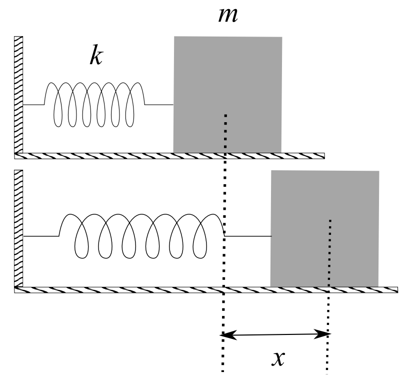


Figure 1.3: Spring-Mass system.

WE BEGIN WITH THE CASE of a single block on a spring as shown in Figure 1.3. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0. \tag{1.36}$$

Dividing by the mass, this equation can be written in the form

$$\ddot{x} + \omega^2 x = 0, \tag{1.37}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

This is the generic differential equation for simple harmonic motion. Two solutions of this equation are given by

$$\begin{aligned} x(t) &= A\cos\omega t, \\ x(t) &= A\sin\omega t, \end{aligned} \tag{1.38}$$

where ω is the angular frequency, measured in rad/s, and A is called the amplitude of the oscillation. .

The angular frequency is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

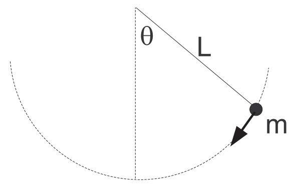


Figure 1.4: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

1.4.2 *The Simple Pendulum*

The simple pendulum consists of a point mass m hanging on a string of length L from some support. [See Figure 1.4.] One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

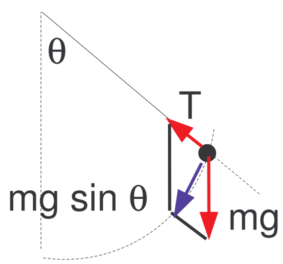


Figure 1.5: There are two forces acting on the mass, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 1.5 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Linear and nonlinear pendulum equation.

Canceling the masses, this then gives us the nonlinear pendulum equation

$$L\ddot{\theta} + g\sin\theta = 0. \tag{1.39}$$

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position ℓ, the torque is given by $\tau = -mg\ell\sin\theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell\sin\theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. For a simple pendulum, we let $\ell = L$ and $I = mL^2$, and obtain $\omega = \sqrt{g/L}$.

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

There are several variations of Equation (1.39) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (1.39) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{1.40}$$

1.4.3 *LRC Circuits*

Another typical problem often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 1.6.

The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

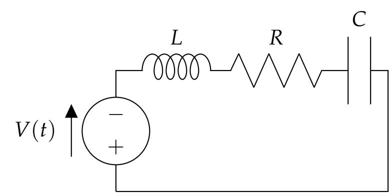


Figure 1.6: Series LRC Circuit.

Furthermore, we need to define the current as $I = \dfrac{dq}{dt}$. where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 1.7. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems in the next chapter.

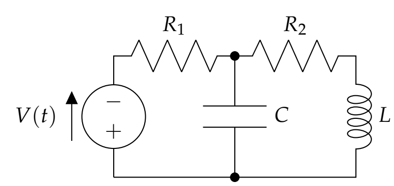


Figure 1.7: Parallel LRC Circuit.

1.4.4 *Orthogonal Trajectories of Curves**

There are many problems from geometry which have lead to the study of differential equations. One such problem is the construction of orthogonal trajectories. Give a a family of curves, $y_1(x;a)$, we seek another family of curves $y_2(x;c)$ such that the second family of curves are perpendicular the to given family. This means that the tangents of two intersecting curves at the point of intersection are perpendicular to each other. The slopes of the tangent lines are given by the derivatives $y_1'(x)$ and $y_2'(x)$. We recall from elementary geometry that the slopes of two perpendicular lines are related by

$$y_2'(x) = -\frac{1}{y_1'(x)}.$$

Example 1.16. Find a family of orthogonal trajectories to the family of parabolae $y_1(x;a) = ax^2$.

We note that the new collection of curves has to satisfy the equation

$$y_2'(x) = -\frac{1}{y_1'(x)} = -\frac{1}{2ax}.$$

Before solving for $y_2(x)$, we need to eliminate the parameter a. From the give function, we have that $a = \frac{y}{x^2}$. Inserting this into the equation for y_2', we have

$$y'(x) = -\frac{1}{2ax} = -\frac{x}{2y}.$$

Thus, to find $y_2(x)$, we have to solve the differential equation

$$2yy' + x = 0.$$

Noting that $(y^2)' = 2yy'$ and $(\frac{1}{2}x^2)' = x$,, this (exact) equation can be written as

$$\frac{d}{dx}\left(y^2 + \frac{1}{2}x^2\right) = 0.$$

Integrating, we find the family of solutions,

$$y^2 + \frac{1}{2}x^2 = k.$$

In Figure 1.8 we plot both families of orthogonal curves.

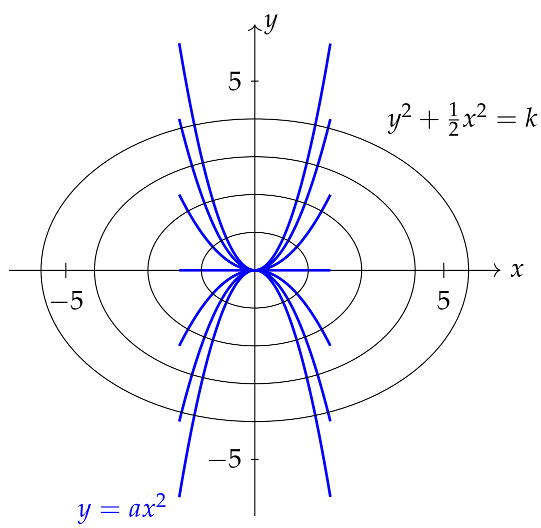


Figure 1.8: Plot of orthogonal families of curves, $y = ax^2$ and $y^2 + \frac{1}{2}x^2 = k$.

1.4.5 *Pursuit Curves**

Another application that is interesting is to find the path that a body traces out as it moves towards a fixed point or another moving body. Such curses are know as pursuit curves. These could model aircraft or submarines following targets, or predators following prey. We demonstrate this with an example.

Example 1.17. A hawk at point (x,y) sees a sparrow traveling at speed v along a straight line. The hawk flies towards the sparrow at constant speed w but always in a direction along line of sight between their positions. If the hawk starts out at the point $(a,0)$ at $t=0$, when the sparrow is at $(0,0)$, then what is the path the hawk needs to follow? Will the hawk catch the sparrow? The situation is shown in Figure 1.9. We pick the path of the sparrow to be along the $y-$axis. Therefore, the sparrow is at position $(0,vt)$.

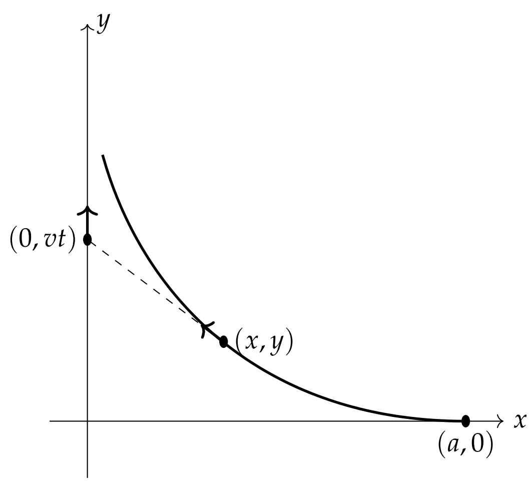


Figure 1.9: A hawk at point (x,y) sees a sparrow at point $(0,vt)$ and always follows the straight line between these points.

First we need the equation of the line of sight between the points (x,y) and $(0,vt)$. Considering that the slope of the line is the same as the slope of the tangent to the path, $y=y(x)$, we have

$$y' = \frac{y-vt}{x}.$$

The hawk is moving at a constant speed, w. Since the speed is related to the time through the distance the hawk travels. we need to find the arclength of the path between $(a,0)$ and (x,y). This is given by

$$L = \int ds = \int_x^a \sqrt{1+[y'(x)]^2}\,dx.$$

The distance is related to the speed, w, and the time, t, by $L=wt$.

Eliminating the time using $y' = \frac{y-vt}{x}$, we have

$$\int_x^a \sqrt{1+[y'(x)]^2}\,dx = \frac{w}{v}(y-xy').$$

Furthermore, we can differentiate this result with respect to x to get rid of the integral,

$$\sqrt{1+[y'(x)]^2} = \frac{w}{v}xy''.$$

Even though this is a second order differential equation for $y(x)$, it is a first order separable equation in the speed function $z(x)=y'(x)$. Namely,

$$\frac{w}{v}xz' = \sqrt{1+z^2}.$$

Separating variables, we find

$$\frac{w}{v}\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{x}.$$

The integrals can be computed using standard methods from calculus. We can easily integrate the right hand side,

$$\int \frac{dx}{x} = \ln|x| + c_1.$$

The left hand side takes a little extra work, or looking the value up in Tables or using a CAS package. Recall a trigonometric substitution is in order. [See the Appendix.] We let $z = \tan\theta$. Then $dz = \sec^2\theta\, d\theta$. The methods proceeds as follows:

$$\begin{aligned}
\int \frac{dz}{\sqrt{1+z^2}} &= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln(\tan\theta + \sec\theta) + c_2 \\
&= \ln(z + \sqrt{1+z^2}) + c_2. \qquad (1.41)
\end{aligned}$$

Putting these together, we have for $x > 0$,

$$\ln(z + \sqrt{1+z^2}) = \frac{v}{w}\ln x + C.$$

Using the initial condition $z = y' = 0$ and $x = a$ at $t = 0$,

$$0 = \frac{v}{w}\ln a + C,$$

or $C = -\frac{v}{w}\ln a$.

Using this value for c, we find

$$\begin{aligned}
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln x - \frac{v}{w}\ln a \\
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln\frac{x}{a} \\
\ln(z + \sqrt{1+z^2}) &= \ln\left(\frac{x}{a}\right)^{\frac{v}{w}} \\
z + \sqrt{1+z^2} &= \left(\frac{x}{a}\right)^{\frac{v}{w}}. \qquad (1.42)
\end{aligned}$$

We can solve for $z = y'$, to find

$$y' = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{v}{w}} - \left(\frac{x}{a}\right)^{-\frac{v}{w}}\right]$$

Integrating,

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + k.$$

The integration constant, k, can be found knowing $y(a) = 0$. This gives

$$\begin{aligned} 0 &= \frac{a}{2}\left[\frac{1}{1+\frac{v}{w}} - \frac{1}{1-\frac{v}{w}}\right] + k \\ k &= \frac{a}{2}\left[\frac{1}{1-\frac{v}{w}} - \frac{1}{1+\frac{v}{w}}\right] \\ &= \frac{avw}{w^2 - v^2}. \end{aligned} \tag{1.43}$$

The full solution for the path is given by

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + \frac{avw}{w^2 - v^2}.$$

Can the hawk catch the sparrow? This would happen if there is a time when $y(0) = vt$. Inserting $x = 0$ into the solution, we have $y(0) = \frac{avw}{w^2 - v^2} = vt$. This is possible if $w > v$.

A related problem is one that was posed in 1676 by the French physician Claude Perrault (1613-1688), who was the brother of Charles Perrault, who published stories like *Cinderella* and *Little Red Riding Hood.* Claude Perrault was at a meeting in Paris and placed his watch in the middle of the table and pulled the end of the watch chain along the edge of the table. He demonstrated that when the end of the watch chain followed a straight line perpendicular to the starting point, the watch would be dragged along a curve. He asked. "What is the shape of the curve traced by the watch?" This curve was first studied by Christiaan Huygens (1629-1695) in 1692, who gave it the name tractrix.

Such a problem is an inverse tangent problem. One of the first such problems was originally posed in a letter from Florimond de Beaune (1601 - 1652) to Marin Mersenne (1588 - 1648) in 1638. Instead of seeking the tangent to a known curve, he asked for the curve which had a specific property. This led to being one of the first differential equations which in modern day notation is given by

$$\frac{dy}{dx} = \frac{x - y}{a}, \tag{1.44}$$

for some constant a. It is interesting to note that this was discussed in Leibniz's first publication in calculus in 1684, *Nova methodus pro maximis et minimis itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus.* After the work carried out by Bernoulli, Leibniz, Newton, and Euler on solution methods for differential equations, we now can easily solve this problem to obtain the curve $y = x + a\left(e^{-x/a} - 1\right)$.

The Perrault's tractrix problem is another inverse tangent problem. In Figure 1.10 is given the setup for the tractrix problem. Perrault's problem can be stated that one seeks the shape of a curve whose tangent (segment) has a constant length, a. From the diagram, we have the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}. \tag{1.45}$$

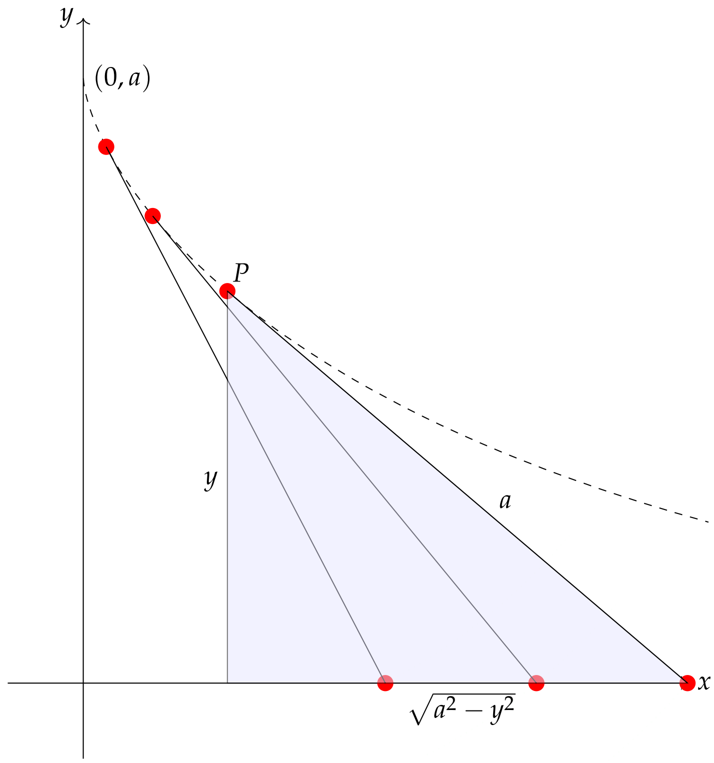


Figure 1.10: The tractrix problem: the end of the watch chain follows a straight line along the x-axis perpendicular to the starting point and the watch would be dragged along a curve coinded by Huygens as a tractrix.

This problem can be solved using several methods. Since it is separable, we have merely to evaluate an integral,

$$x + c = -\int \frac{\sqrt{a^2 - y^2}}{y}\, dy.$$

Due to the square root, nowadays one might think to use a trigonometric substitution. However, we could also use a hyperbolic trigonometric function substitution. Considering the argument of the square root is $a^2 - y^2$, we could let $y = a \operatorname{sech} u$. Then,

$$a^2 - y^2 = a^2(1 - \operatorname{sech}^2 u) = a^2 \tanh^2 u$$

and $dy = -a \operatorname{sech} u \tanh u \, du$ So, the integral becomes,

$$\begin{aligned} -\int \frac{\sqrt{a^2 - y^2}}{y}\, dy &= \int \frac{a \tanh u}{a \operatorname{sech} u}\, a \operatorname{sech} u \tanh u \, du \\ &= a \int \tanh^2 u \, du \\ &= a \int (1 - \operatorname{sech}^2 u)\, du \\ &= a(\tanh u - u) = x + C \qquad (1.46) \end{aligned}$$

Therefore, we have a parametric solution

$$x(u) = a(u - \tanh u) + C, \quad y(u) = a \operatorname{sech} u.$$

Requiring $(x(0), y(0)) = (0, a)$, then $C = 0$.

Finally, one can show that the evolute (the curve formed by the centers of curvature) of a tractrix is a catenary, which is the shape a hanging chain takes under its own weight. Also, the tractrix played a role in non-Euclidean geometry since rotating a tractrix around its asymptote generates a surface called a pseudosphere.

1.5 Other First Order Equations*

There are several nonlinear first order equations whose solution can be obtained using special techniques. We conclude this chapter by looking at a few of these equations named after famous mathematicians of the 17-18th century inspired by various applications.

1.5.1 Bernoulli Equation*

We begin with the Bernoulli equation, named after Jacob Bernoulli (1655-1705). The Bernoulli equation is of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 0, 1.$$

The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler (1707-1783) and Daniel Bernoulli, who is known for his work in hydrodynamics. See Figure 1.11.

Note that when $n = 0, 1$ the equation is linear and can be solved using an integrating factor. The key to solving this equation is using the transformation $z(x) = \frac{1}{y^{n-1}(x)}$ to make the equation for $z(x)$ linear. We demonstrate the procedure using an example.

Example 1.18. Solve the Bernoulli equation $xy' + y = y^2 \ln x$ for $x > 0$.

In this example $p(x) = 1$, $q(x) = \ln x$, and $n = 2$. Therefore, we let $z = \frac{1}{y}$. Then,

$$z' = -\frac{1}{y^2}y' = z^2 y'.$$

Inserting $z = y^{-1}$ and $z' = z^2 y'$ into the differential equation, we have

$$\begin{aligned} xy' + y &= y^2 \ln x \\ -x\frac{z'}{z^2} + \frac{1}{z} &= \frac{\ln x}{z^2} \\ -xz' + z &= \ln x \\ z' - \frac{1}{x}z = -\frac{\ln x}{x}. & \end{aligned} \tag{1.47}$$

Thus, the resulting equation is a linear first order differential equation. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \frac{1}{x}.$$

Multiplying the differential equation by the integrating factor, we have

$$\left(\frac{z}{x}\right)' = \frac{\ln x}{x^2}.$$

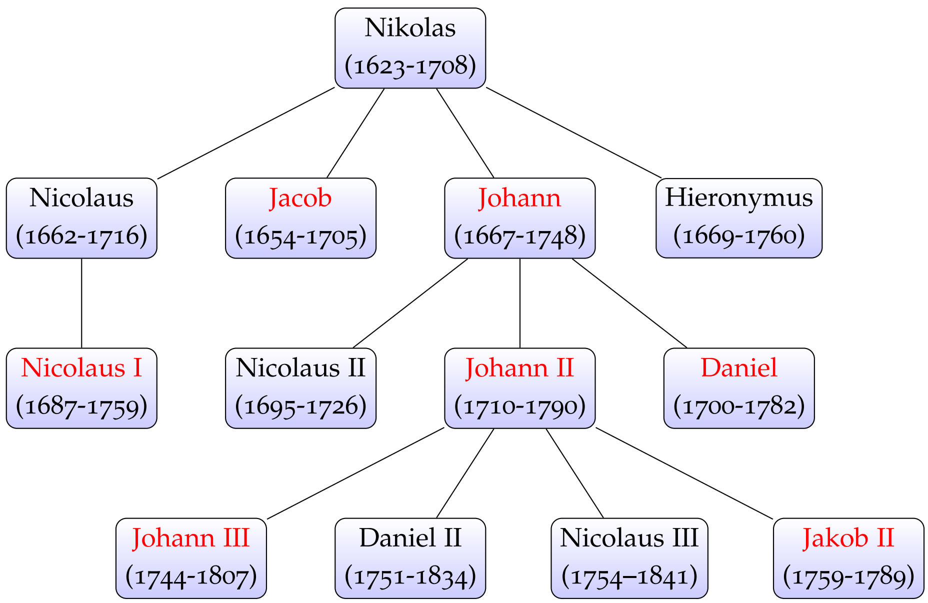
Integrating, we obtain

$$\begin{aligned} \frac{z}{x} &= -\int \frac{\ln x}{x^2} + C \\ &= \frac{\ln x}{x} + \int \frac{dx}{x^2} + C \\ &= \frac{\ln x}{x} + \frac{1}{x} + C. \end{aligned} \tag{1.48}$$

Multiplying by x, we have $z = \ln x + 1 + Cx$. Since $z = y^{-1}$, the general solution to the problem is

$$y = \frac{1}{\ln x + 1 + Cx}.$$

Figure 1.11: The Bernoulli's were a family of Swiss mathematicians spanning at least three generations starting with the brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748).



1.5.2 *Lagrange and Clairaut Equations**

ALEXIS CLAUDE CLAIRAUT (1713-1765) SOLVED the differential equation

$$y = xy' + g(y').$$

This is a special case of the family of Lagrange equations,

$$y = xf(y') + g(y'),$$

named after Joseph Louis Lagrange (1736-1813). These equations also have solutions called singular solutions. Singular solution are solutions for which there is a failure of uniqueness to the initial value problem at every point on the curve. A singular solution is often one that is tangent to every solution in a family of solutions.

First, we consider solving the more general Lagrange equation. Let $p = y'$ in the Lagrange equation, giving

$$y = xf(p) + g(p). \tag{1.49}$$

Next, we differentiate with respect to x to find

$$y' = p = f(p) + xf'(p)p' + g'(p)p'.$$

Lagrange equations, $y = xf(y') + g(y')$.

Here we used the Chain Rule. For example,

$$\frac{dg(p)}{dx} = \frac{dg}{dp}\frac{dp}{dx}.$$

Solving for p', we have

$$\frac{dp}{dx} = \frac{p - f(p)}{xf'(p) + g'(p)}. \tag{1.50}$$

We have introduced $p = p(x)$, viewed as a function of x. Let's assume that we can invert this function to find $x = x(p)$. Then, from introductory calculus, we know that the derivatives of a function and its inverse are related,

$$\frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}.$$

Applying this to Equation (1.50), we have

$$\begin{aligned} \frac{dx}{dp} &= \frac{xf'(p) + g'(p)}{p - f(p)} \\ x' - \frac{f'(p)}{p - f(p)}x &= \frac{g'(p)}{p - f(p)}, \end{aligned} \tag{1.51}$$

assuming that $p - f(p) \neq 0$.

As can be seen, we have transformed the Lagrange equation into a first order linear differential equation (1.51) for $x(p)$. Using methods from earlier in the chapter, we can in principle obtain a family of solutions

$$x = F(p, C),$$

where C is an arbitrary integration constant. Using Equation (1.49), one might be able to eliminate p in Equation (1.51) to obtain a family of solutions of the Lagrange equation in the form

$$\varphi(x, y, C) = 0.$$

If it is not possible to eliminate p from Equations (1.49) and (1.51), then one could report the family of solutions as a parametric family of solutions with p the parameter. So, the parametric solutions would take the form

$$\begin{aligned} x &= F(p, C), \\ y &= F(p, C)f(p) + g(p). \end{aligned} \tag{1.52}$$

We had also assumed the $p - f(p) \neq 0$. However, there might also be solutions of Lagrange's equation for which $p - f(p) = 0$. Such solutions are called singular solutions.

Singular solutions are possible for Lagrange equations.

Example 1.19. Solve the Lagrange equation $y = 2xy' - y'^2$.

We will start with Equation (1.51). Noting that $f(p) = 2p$, $g(p) = -p^2$, we have

$$\begin{aligned} x' - \frac{f'(p)}{p - f(p)} x &= \frac{g'(p)}{p - f(p)} \\ x' - \frac{2}{p - 2p} x &= \frac{-2p}{p - 2p} \\ x' + \frac{2}{p} x &= 2. \end{aligned} \tag{1.53}$$

This first order linear differential equation can be solved using an integrating factor. Namely,

$$\mu(p) = \exp\left(\int \frac{2}{p}\, dp\right) = e^{2\ln p} = p^2.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dp}\left(xp^2\right) = 2p^2.$$

Integrating,

$$xp^2 = \frac{2}{3}p^3 + C.$$

This gives the general solution

$$x(p) = \frac{2}{3}p + \frac{C}{p^2}.$$

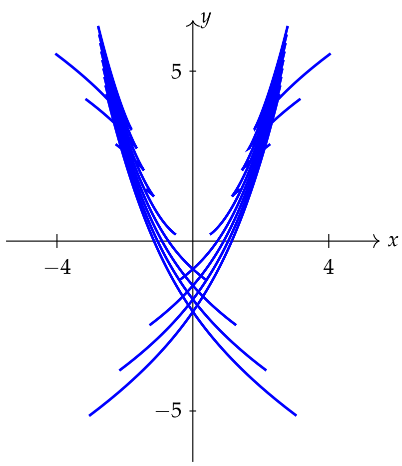


Figure 1.12: Family of solutions of the Lagrange equation $y = 2xy' - y'^2$.

Replacing $y' = p$ in the original differential equation, we have $y = 2xp - p^2$. The family of solutions is then given by the parametric equations

$$\begin{aligned} x &= \frac{2}{3}p + \frac{C}{p^2}, \\ y &= 2\left(\frac{2}{3}p + \frac{C}{p^2}\right)p - p^2 \\ &= \frac{1}{3}p^2 + \frac{2C}{p}. \end{aligned} \tag{1.54}$$

The plots of these solutions is shown in Figure 1.12.

We also need to check for a singular solution. We solve the equation $p - f(p) = 0$, or $p = 0$. This gives the solution $y(x) = (2xp - p^2)_{p=0} = 0$.

Clairaut equations, $y = xy' + g(y')$.

The Clairaut differential equation is given by

$$y = xy' + g(y').$$

Letting $p = y'$, we have

$$y = xp + g(p).$$

This is the Lagrange equation with $f(p) = p$. Differentiating with respect to x,

$$p = p + xp' + g'(p)p'.$$

Rearranging, we find

$$x = -g'(p)$$

So, we have the parametric solution

$$\begin{aligned} x &= -g'(p), \\ y &= -pg'(p) + g(p). \end{aligned} \qquad (1.55)$$

For the case that $y' = C$, it can be seen that $y = Cx + g(C)$ is a general solution solution.

Example 1.20. Find the solutions of $y = xy' - y'^2$.

As noted, there is a family of straight line solutions $y = Cx - C^2$, since $g(p) = -p^2$. There might also by a parametric solution not contained n this family. It would be given by the set of equations

$$\begin{aligned} x &= -g'(p) = 2p, \\ y &= -pg'(p) + g(p) = 2p^2 - p^2 = p^2. \end{aligned} \qquad (1.56)$$

Eliminating p, we have the parabolic curve $y = x^2/4$.

In Figure 1.13 we plot these solutions. The family of straight line solutions are shown in blue. The limiting curve traced out, much like string figures one might create, is the parametric curve.

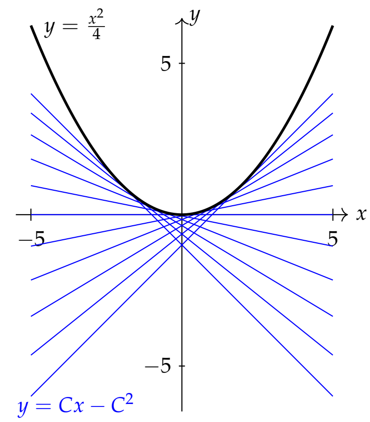


Figure 1.13: Plot of solutions to the Clairaut equation $y = xy' - y'^2$. The straight line solutions are a family of curves whose limit is the parametric slution.

1.5.3 *Riccati Equation**

Jacopo Francesco Riccati (1676-1754) studied curves with some specified curvature. He proposed an equation of the form

$$y' + a(x)y^2 + b(x)y + c(x) = 0$$

around 1720. He communicated this to the Bernoulli's. It was Daniel Bernoulli who had actually solved this equation. As noted by Ranjan Roy (2011), Riccati had published his equation in 1722 with a note that D. Bernoulli giving the solution in terms of an anagram. Furthermore, when $a \equiv 0$, the Riccati equation reduces to a Bernoulli equation.

In Section 3.2.1, we will show that the Ricatti equation can be transformed into a second order linear differential equation. However, there are special cases in which we can get our hands on the solutions. For example, if a, b, and c are constants, then the differential equation can be integrated directly. We have

$$\frac{dy}{dx} = -(ay^2 + by + c).$$

This equation is separable and we obtain

$$x - C = -\int \frac{dy}{ay^2 + by + c}.$$

When a differential equation is left in this form, it is said to be solved by quadrature when the resulting integral in principle can be computed in terms of elementary functions.[4]

[4] By elementary functions we mean well known functions like polynomials, trigonometric, hyperbolic, and some not so well know to undergraduates, such as Jacobi or Weierstrass elliptic functions.

If a particular solution is known, then one can obtain a solution to the Riccati equation. Let the known solution be $y_1(x)$ and assume that the general solution takes the form $y(x) = y_1(x) + z(x)$ for some unknown function $z(x)$. Substituting this form into the differential equation, we can show that $v(x) = 1/z(x)$ satisfies a first order linear differential equation.

Inserting $y = y_1 + z$ into the general Riccati equation, we have

$$\begin{aligned}
0 &= \frac{dy}{dx} + a(x)y^2 + b(x)y + c \\
&= \frac{dz}{dx} + az^2 + 2azy_1 + bz + \\
&\quad + \frac{dy_1}{dx} + ay_1^2 + by_1 + c \\
&= \frac{dz}{dx} + a(x)[2y_1 z + z^2] + b(x)z \\
-a(x)z^2 &= \frac{dz}{dx} + [2a(x)y_1 + b(x)]z.
\end{aligned} \tag{1.57}$$

The last equation is a Bernoulli equation with $n = 2$. So, we can make it a linear equation with the substitution $z = \frac{1}{v}$, $z' = -\frac{z'}{v^2}$. Then, we obtain a differential equation for $v(x)$. It is given by

$$v' - (2a(x)y_1(x) + b(x))v = a(x).$$

Example 1.21. Find the general solution of the Riccati equation, $y' - y^2 + 2e^x y - e^{2x} - e^x = 0$, using the particular solution $y_1(x) = e^x$.

We let the sought solution take the form $y(x) = z(x) + e^x$. Then, the equation for $z(x)$ is found as

$$\frac{dz}{dx} = z^2.$$

This equation is simple enough to integrate directly to obtain $z = \frac{1}{C-x}$. Then, the solution to the problem becomes

$$y(x) = \frac{1}{C-x} + e^x.$$

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2.$

e. $y' - (\sin x)y = \sin x.$

f. $xy' - 2y = x^2, y(1) = 1.$

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1.$

h. $x' - 2x = te^{2t}.$

i. $\dfrac{dy}{dx} + y = \sin x, \, y(0) = 0.$

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3, \, y(1) = 4.$

2. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

3. Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}.$

b. $y = 2xy' + \ln y'.$

c. $y' + 2xy = 2xy^2.$

d. $y' + 2xy = y^2 e^{x^2}.$

4. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in our review. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

5. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

6. Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

7. Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.

b. $x^2y'' - 3xy' + 3y = 0$.

c. $x^2y'' + 5xy' + 4y = 0$.

d. $x^2y'' - 2xy' + 3y = 0$.

e. $x^2y'' + 3xy' - 3y = x^2$.

8. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

9. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

b. $y'' + y' = 3x^2$. Method of Variation of Parameters.

10. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

11. Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

12. Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6$.

b. $y'' + y = 2\sin 3x$.

c. $y'' + y = 1 + 2\cos x$.

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad x > 0$.

13. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4$.

b. $xy'' - y' + 4x^3 y = 0, \quad y_1(x) = \sin(x^2).$

14. A certain model of the motion of a tossed whiffle ball is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, g=9.8 m/s^2 is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 10\ \mathrm{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?