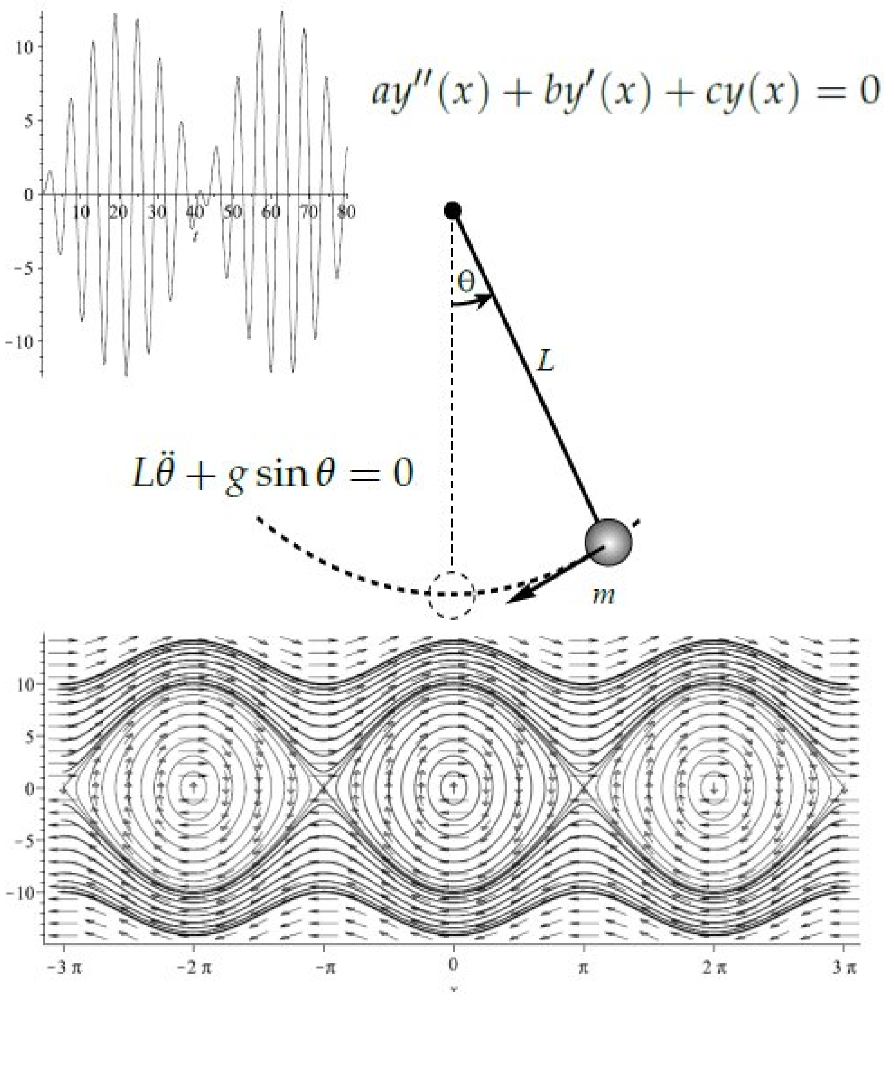
RUSSELL L. HERMAN

# A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS

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A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS by Russell Herman. Versions of these notes have been posted online since Spring 2005.
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*December 2024*

# Contents

*Dedicated to those students who have endured previous versions of my notes and to current students who take the time to read this revised edition.*

# *Chapter 1*

# *Introduction*

*In order to solve a differential equation, you look at it till a solution occurs to you.*
- George Pólya (1887-1985)

THESE ARE NOTES FOR A SECOND COURSE in differential equations originally taught in the Spring semester of 2005 at the University of North Carolina Wilmington to upper level and first year graduate students and later updated in Fall 2007, Fall 2008, and Fall 2024. It is assumed that you have had an introductory course in differential equations. However, we will begin this chapter with a review of some of the material from your first course in differential equations and then give an overview of the material we are about to cover.

Typically an introductory course in differential equations introduces students to analytical solutions of first order differential equations which are separable, first order linear differential equations, and sometimes to some other special types of equations. Students then explore the theory of second order differential equations generally restricted to the study of exact solutions of constant coefficient linear differential equations or even equations of the Cauchy-Euler type. These are later followed by the study of special techniques, such as power series methods or Laplace transform methods. If time permits, ones explores a few special functions, such as Legendre polynomials and Bessel functions, while using power series methods for solving differential equations.

More recently, variations on this inventory of topics have been introduced through the early introduction of systems of differential equations, qualitative studies of these systems and a more intense use of technology for understanding the behavior of solutions of differential equations. This is typically done at the expense of not covering power series methods, special functions, or Laplace transforms. In either case, the types of problems solved are initial value problems in which the differential equation to be solved is accompanied by a set of initial conditions.

In this course we will assume some exposure to the overlap of these two approaches. We will first give a quick review of the solution of separable and linear first order equations. Then we will review second order linear differential equations and Cauchy-Euler equations. This will then be followed by an overview of some of the topics covered. As with any course in differential equations, we will emphasize analytical, graphical and (sometimes) approximate solutions of differential equations. Throughout we will present applications from physics, chemistry and biology.

## 1.1 *Review of the First Course*

In this section we review a few of the solution techniques encountered in a first course in differential equations. We will not review the basic theory except in possible references as reminders as to what we are doing.

We first recall that an *n-th order ordinary differential equation* is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first $n$ derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.1}$$

Here $y^{(n)}(x)$ represents the $n$th derivative of $y(x)$.

An *initial value problem* consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say $x_0$:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.2}$$

A *linear nth order differential equation* takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{1.3}$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

### 1.1.1 *First Order Differential Equations*

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \tag{1.4}$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

Separable equations.

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be *autonomous*.

The *general solution* to equation (1.5) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C, \tag{1.6}$$

where $C$ is an integration constant. This yields a *1-parameter family of solutions* to the differential equation corresponding to different values of $C$. If one can solve (1.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

**Example 1.1.** $y' = 2xy$, $y(0) = 2$.

Applying (1.6), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since $C$ is an arbitrary constant, $A$ is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 1.1.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial conditions is $y(x) = 2e^{x^2}$.

**Example 1.2.** $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 1.2.

Linear first order equations.

The second type of first order equation encountered is the *linear first order differential equation* in the form

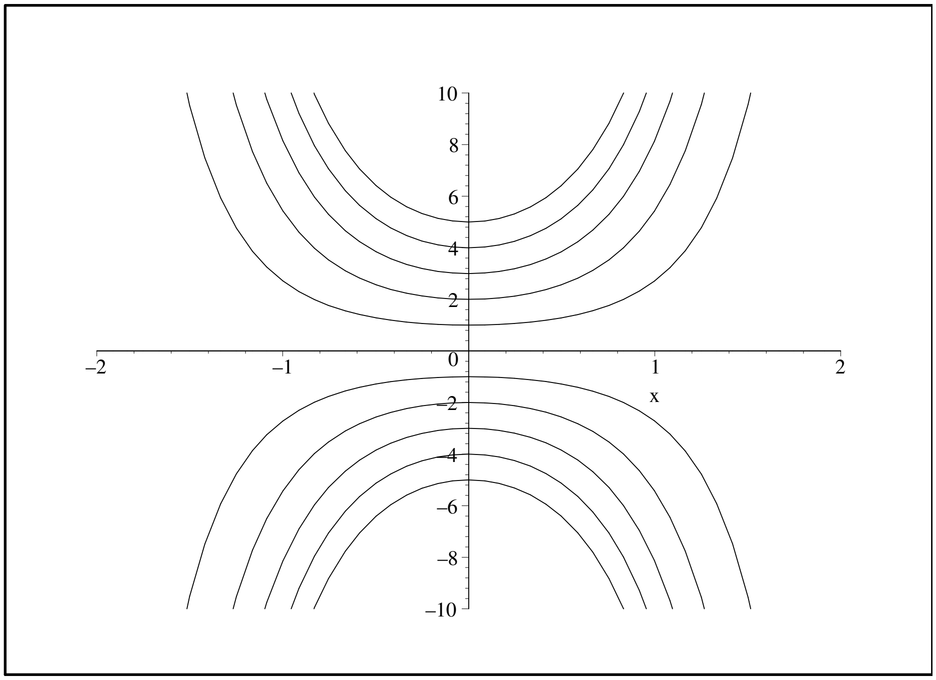
$$y'(x) + p(x)y(x) = q(x). \tag{1.7}$$

Integrating factor.

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)q(x). \tag{1.8}$$

Figure 1.1: Plots of solutions from the 1-parameter family of solutions of Example 1.1 for several initial conditions.




The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\,d\xi)$. One can show this by expanding the derivative in Equation (1.8),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{1.9}$$

and comparing this equation to the one obtained from multiplying (1.7) by $\mu(x)$ :

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{1.10}$$

Note that these last two equations would be the same if

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation whose solution is the above given form for the integrating factor,

$$\boxed{\mu(x) = \exp\left(\int^x p(\xi)\,d\xi\right).} \tag{1.11}$$

Equation (1.8) is easily integrated to obtain

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right].} \tag{1.12}$$

**Example 1.3.** $xy' + y = x, \quad x > 0, y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for $y'$, one can see that the original equation is not separable. However, it is not in the standard form. So, we first rewrite the

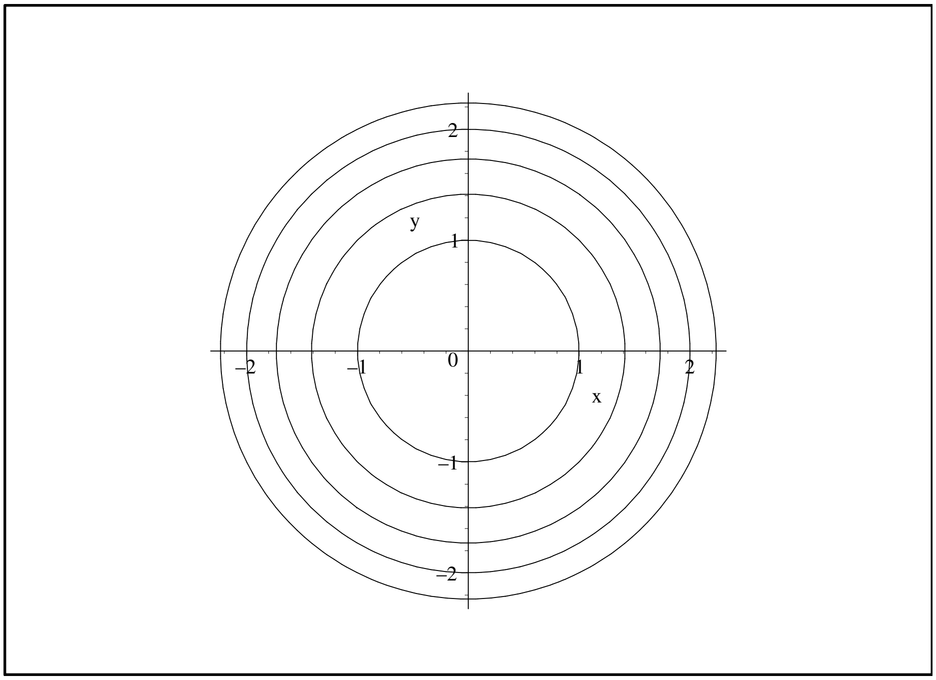



Figure 1.2: Plots of solutions of Example 1.2 for several initial conditions.

equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{1.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (1.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, equation (1.8) becomes

$$(xy)' = x.$$

Integrating one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is $y(x) = \frac{1}{2}(x - \frac{1}{x})$.

**Example 1.4.** $(\sin x)y' + (\cos x)y = x^2 \sin x$.

Actually, this problem is easy if you realize that

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, rewrite the original differential equation in standard form:

$$y' + (\cot x)y = x^2.$$

Then, compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot\xi \, d\xi\right) = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Using the integrating factor, the original equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later in the course.

### 1.1.2 *Second Order Linear Differential Equations*

SECOND ORDER DIFFERENTIAL EQUATIONS ARE TYPICALLY harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.14}$$

One can rewrite this equation using operator terminology.[1] Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (1.14) becomes

$$Ly = f. \tag{1.15}$$

The solutions of linear differential equations are found by making use of the linearity of $L$. Namely, we consider the *vector space* [2] consisting of real-valued functions over some domain. Let $f$ and $g$ be vectors in this function space. $L$ is a *linear operator* if for two vectors $f$ and $g$ and scalar $a$, we have that

[1] We note that Leibniz introduced $\frac{dy}{dx}$ and Newton used the dot notation, $\dot{y}$. About a century later Lagrange introduced $y'$ and Arbogast introduced the operator notation $D$.

[2] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

a. $L(f+g) = Lf + Lg$

b. $L(af) = aLf$.

One typically solves (1.14) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. This is true because of the linearity of $L$. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{1.16}$$

There are methods for finding a particular solution of a differential equation. These range from pure guessing to the Method of Undetermined Coefficients, or by making use of the Method of Variation of Parameters. We will review some of these methods later.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, others have studied a variety of second order linear equations and have saved us the trouble for some of the differential equations that often appear in applications.

Again, linearity is useful in producing the general solution of a homogeneous linear differential equation. If $y_1$ and $y_2$ are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if $y_1$ and $y_2$ are *linearly independent*,[3] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. As you may recall, linear independence is established if the Wronskian of the solutions in not zero. In this case, we have

[3] Recall, a set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1 y(1(x) + \ldots + c_n y_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{1.17}$$

### 1.1.3 *Constant Coefficient Equations*

THE SIMPLEST AND MOST SEEN SECOND ORDER DIFFERENTIAL EQUATIONS are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{1.18}$$

where $a$, $b$, and $c$ are constants.

Solutions to (1.18) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (1.18) leads to the *characteristic equation*

Characteristic equation.

$$ar^2 + br + c = 0. \tag{1.19}$$

The roots of this equation in turn lead to three types of solution depending upon the nature of the roots as shown below.

**Example 1.5.** $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating $y$ and $y'$ at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{1.20}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

**Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs**

1. **Real, distinct roots** $r_1, r_2$. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as $xe^{rx}$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$. [This is covered in the appendix to this chapter.]

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$. [This is covered in the appendix to this chapter.]

**Example 1.6.** $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is easily obtained as $y(x) = (c_1 + c_2 x)e^{-3x}$.

**Example 1.7.** $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1\cos(2x) + c_2\sin(2x)$.

**Example 1.8.** $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]e^{-x}.$$

One of the most important applications of the equations in the last two examples is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass $m$ on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$mx'' + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. Such motion is called *simple harmonic motion*.

Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. The next example shows a forced harmonic oscillator.

**Example 1.9.** $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 1.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what $A$ needs to be. Recall, this is the *Method of Undetermined Coefficients* which we review in the next section. Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

### *1.1.4 Method of Undetermined Coefficients*

TO DATE, WE ONLY KNOW HOW TO SOLVE constant coefficient, homogeneous equations. How does one solve a nonhomogeneous equation like that in Equation (1.14),

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.21}$$

Recall, that one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. So, how do we find the particular solution?

General solution of the nonhomogeneous problem, $y = y_h + y_p$.

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically develop the particular solution. We will come back to this method the Method of Variation of Parameters, later in the book.

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

**Example 1.10.** Consider the equation

$$y'' + 2y' - 3y = 4. \tag{1.22}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{1.23}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (1.22). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to $x$, then we are left with a linear function after inserting $x$ and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (1.22), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining our two solutions, we have the general solution to the original nonhomogeneous equation (1.22). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine our arbitrary constants.

What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{1.24}$$

The only thing that would change is our particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (**??**), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the $x$ terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (1.24)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of $x$ on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{1.25}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{1.26}$$

So, our particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 1.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have your solution. This solution is then added to the general solution of the homogeneous differential equation.

Table 1.1: Educated guesses given nonhomogeneous $f(x)$.

| $f(x)$ | Guess |
|---|---|
| $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ | $A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$ |
| $ae^{bx}$ | $Ae^{bx}$ |
| $a \cos \omega x + b \sin \omega x$ | $A \cos \omega x + B \sin \omega x$ |

**Example 1.11.** As a final example, let's consider the equation

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{1.27}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, $A$, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that $e^x$ and $e^{-3x}$ are solutions to the homogeneous problem. So, a multiple of $e^{-3x}$ will not get us anywhere. It turns out that there is one further modification of the method. If our driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of $x$ times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

**Modified Method of Undetermined Coefficients**

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by $x^k$, where $k$ is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

### 1.1.5 *Cauchy-Euler Equations*

Another class of solvable linear differential equations that is of interest are the Cauchy-Euler type of equations. These are given by

Cauchy-Euler Equation.

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{1.28}$$

Note that in such equations the power of $x$ in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (1.28), we have

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since this has to be true for all $x$ in the problem domain, we obtain the characteristic equation

Characteristic equation for the Cauchy-Euler Equation.

$$ar(r-1) + br + c = 0. \tag{1.29}$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. These are shown below. Some of the details are provided in the next section.

**Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations**

1. Real, distinct roots $r_1, r_2$. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha (c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

**Example 1.12.** $x^2 y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{1.30}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2} \ln|x|) + c_2 \sin(2\sqrt{2} \ln|x|)\right] x^{-2}$

**Example 1.13.** $t^2 y'' + 3ty' + y = 0, \quad y(1) = 0, y'(1) = 1.$

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln|t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of $y$ and $y'$. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t| t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t| t^{-1}$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients. We will demonstrate this with a couple of examples.

**Example 1.14.** Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1x^{-1} + c_2x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{1.31}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1x^{-1} + c_2x^3 - \frac{2}{3}x^2.$$

**Example 1.15.** Find the solution of $x^2y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y'' + bxy' + cy = dx^r,$$

where $r$ is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1x^{-1} + c_2x^3 + \frac{1}{2}x^3 \ln x.$$

## 1.2 *Overview of the Course*

For the most part, your first course in differential equations was about solving initial value problems. When second order equations did not fall into the above cases, then you might have learned how to obtain approximate solutions using power series methods, or even finding new functions from these methods. In this course we will explore two broad topics: systems of differential equations and boundary value problems.

We will see that there are interesting initial value problems when studying systems of differential equations. In fact, many of the second order equations that you have seen in the past can be written as a system of two first order equations. For example, the equation for simple harmonic motion,

$$x'' + \omega^2 x = 0,$$

can be written as the system

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 x \end{aligned}.$$

Just note that $x'' = y' = -\omega^2 x$. Of course, one can generalize this to systems with more complicated right hand sides. The behavior of such systems can be fairly interesting and these systems result from a variety of physical models.

In the second part of the course we will explore boundary value problems. Often these problems evolve from the study of partial differential equations. Such examples stem from vibrating strings, temperature distributions, bending beams, etc. Boundary conditions are conditions that are imposed at more than one point, while for initial value problems the conditions are specified at one point. For example, we could take the oscillation equation above and ask when solutions of the equation would satisfy the conditions $x(0) = 0$ and $x(1) = 0$. The general solution, as we have determined earlier, is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Requiring $x(0) = 0$, we find that $c_1 = 0$, leaving $x(t) = c_2 \sin \omega t$. Also imposing that $0 = x(1) = c_2 \sin \omega$, we are forced to make $\omega = n\pi$, for $n = 1, 2, \ldots$. (Making $c_2 = 0$ would not give a nonzero solution of the problem.) Thus, there are an infinite number of solutions possible, if we have the freedom to choose our $\omega$. In the second half of the course we will investigate techniques for solving boundary value problems and look at several applications, including seeing the connections with partial differential equations and Fourier series.

## 1.3 *Appendix: Reduction of Order and Complex Roots*

In this section we provide some of the details leading to the general forms for the constant coefficient and Cauchy-Euler differential equations. In the first subsection we review how the Method of Reduction of Order is used to obtain the second linearly independent solutions for the case of one repeated root. In the second subsection we review how the complex solutions can be used to produce two linearly independent real solutions.

## *Method of Reduction of Order*

FIRST WE CONSIDER CONSTANT COEFFICIENT EQUATIONS. In the case when there is a repeated real root, one has only one independent solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second solution? Since the solutions are independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. For constant coefficient second order equations, we can write the equation as

$$(D-r)^2 y = 0,$$

where $D = \frac{d}{dx}$.

We now insert $y_2(x)$ into this equation. First we compute

$$(D-r)ve^{rx} = v'e^{rx}.$$

Then,

$$(D-r)^2 ve^{rx} = (D-r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, $(D-r)^2 y_2 = 0$, then $v''(x)e^{rx} = 0$ for all $x$. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax+b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$.

Deriving the solution for Case 2 for the Cauchy-Euler equations is messier, but works in the same way. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1 r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Rewriting this, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the general Cauchy-Euler equation in this case takes the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1-1}, \\ y_2''(x) &= (x^2v'' + 2r_1xv' + r_1(r_1-1)v)x^{r_1-2}. \end{aligned} \tag{1.32}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2y'' + (1-2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1+1}. \end{aligned} \tag{1.33}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln|v'| = -\ln|x| + C.$$

Exponentiating, we have one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A\ln|x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1}\ln|x|.$$

---

### *Complex Roots*

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula

$$e^{i\beta x} = \cos\beta x + i\sin\beta x. \tag{1.34}$$

Then the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{1.35}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first recall that

$$x^y = e^{\ln x^y} = e^{y\ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^{\alpha}e^{i\beta\ln x}, \quad x > 0.$$

We can now find two real, linearly independent solutions, $x^{\alpha}\cos(\beta\ln|x|)$ and $x^{\alpha}\sin(\beta\ln|x|)$ following the same steps as above for the constant coefficient case.

## *Problems*

**1.** Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\dfrac{ds}{dt} + 2s = st^2, \quad ,s(0) = 1$.

h. $x' - 2x = te^{2t}$.

i. $\dfrac{dy}{dx} + y = \sin x$, $y(0) = 0$.

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3$, $y(1) = 4$.

**2.** Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

**3.** Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}$.

b. $y = 2xy' + \ln y'$.

c. $y' + 2xy = 2xy^2$.

d. $y' + 2xy = y^2 e^{x^2}$.

**4.** The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in our review. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

**5.** Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0$.

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$.

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$.

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

**6.** Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of $y_1$ and $y_2$.

**7.** Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.

b. $x^2y'' - 3xy' + 3y = 0$.

c. $x^2y'' + 5xy' + 4y = 0$.

d. $x^2y'' - 2xy' + 3y = 0$.

e. $x^2y'' + 3xy' - 3y = x^2$.

**8.** Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

**9.** Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

b. $y'' + y' = 3x^2$. Method of Variation of Parameters.

**10.** Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

**11.** Instead of assuming that $c_1' y_1 + c_2' y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1' y_1 + c_2' y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

**12.** Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6$.

b. $y'' + y = 2\sin 3x$.

c. $y'' + y = 1 + 2\cos x$.

d. $x^2 y'' - 2xy' + 2y = 3x^2 - x, \quad x > 0$.

**13.** Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2 y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4$.

b. $xy'' - y' + 4x^3 y = 0, \quad y_1(x) = \sin(x^2)$.

**14.** A certain model of the motion of a tossed whiffle ball is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here $m$ is the mass of the ball, $g$=9.8 m/s$^2$ is the acceleration due to gravity and $c$ is a measure of the damping. Since there is no $x$ term, we can write this as a first order equation for the velocity $v(t) = x'(t)$ :

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 10\ \text{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

# Chapter 2

# Linear Systems of Differential Equations

*"Do not worry too much about your difficulties in mathematics, I can assure you that mine are still greater." - Albert Einstein* (1879-1955)

## 2.1 Coupled Systems

In this chapter we will begin our study of systems of differential equations. After defining first order systems, we will look at constant coefficient systems and the behavior of solutions for these systems. Also, most of the discussion will focus on planar, or two dimensional, systems. For such systems we will be able to look at a variety of graphical representations of the family of solutions and discuss the qualitative features of systems we can solve in preparation for the study of systems whose solutions cannot be found in an algebraic form. However, we first turn to some simple physical problems.

There are many problems in physics that can result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems for a single mass.

We have already seen reminded in the last chapter of the simple problem of a mass on a spring. This is shown in Figure 2.1. The mass slides on a frictionless surface and reacts to the restoring force of the spring attached to a wall. The restoring force of the spring given by Hooke's Law,

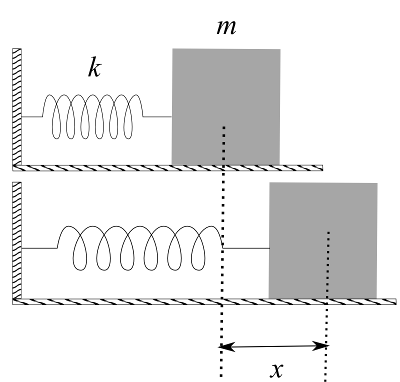


Figure 2.1: Spring-Mass system.

$$F_s = -kx,$$

where $k > 0$ is the spring constant and $x$ is the elongation of the spring. When the spring elongation is positive, the spring force is negative and when the spring elongation is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system is found from Newton's Second Law as

$$m\ddot{x} + kx = 0.$$

This second order equation, constant coefficient equation is easily solved using the methods in the previous chapter. However, it can also be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$. Noting that $\ddot{x} = \dot{y}$, we rewrite the second order equation in terms of $x$ and $\dot{y}$. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{2.1}$$

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.2. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as $x_1$ and $x_2$. These are shown in Figure 2.2.

For mass $m_1$, the forces acting on it are due to each spring. The first spring with spring constant $k_1$ provides a force on $m_1$ of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, the second spring will exert a force on $m_1$ of $k_2(x_2 - x_1)$.

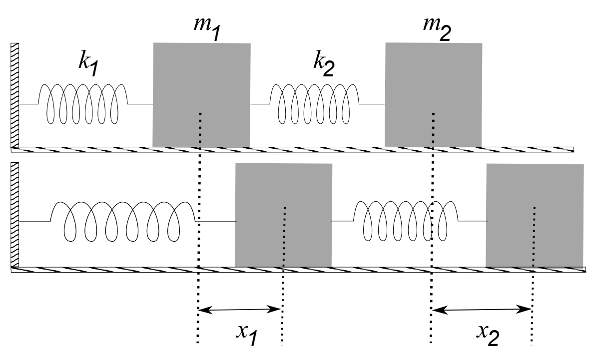



Figure 2.2: System of two masses and two springs.

Similarly, the only force acting directly on mass $m_2$ is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.2}$$

Thus, we see that we have a coupled system of two second order differential equations. Each equation depends on the unknowns $x_1$ and $x_2$.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\
\dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1).
\end{aligned} \tag{2.3}$$

As we will see in the next chapter, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.4}$$

We can solve this system of first order equations using matrix methods. However, we will first need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in planar systems.

## 2.2 *Planar Systems*

### 2.2.1 *Introduction*

We now consider examples of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems will help in the study of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$ :

$$\begin{aligned}
x'(t) &= P(x, y, t) \\
y'(t) &= Q(x, y, t).
\end{aligned} \tag{2.5}$$

An autonomous system is one in which there is no explicit time dependence:

Autonomous systems.

$$\begin{aligned}
x'(t) &= P(x, y) \\
y'(t) &= Q(x, y).
\end{aligned} \tag{2.6}$$

Otherwise the system is called nonautonomous.

A *linear system* takes the form

$$\begin{aligned}
x' &= a(t)x + b(t)y + e(t) \\
y' &= c(t)x + d(t)y + f(t).
\end{aligned} \tag{2.7}$$

A homogeneous linear system results when $e(t) = 0$ and $f(t) = 0$.

A linear, constant coefficient system of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \tag{2.8}$$

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

A linear, homogeneous system of constant coefficient first order differential equations in the plane.

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{2.9}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (2.9) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system (2.9) and systematically replace occurrences of $y$ and $y'$, since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned} \tag{2.10}$$

Rewriting the last line, we have

$$x'' - (a + d)x' + (ad - bc)x = 0. \tag{2.11}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{2.12}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (2.9):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

**Example 2.1.** Consider the system of differential equations

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.13}$$

Carrying out the above outlined steps, we have that $x'' + 3x' - 4x = 0$. This can be shown as follows:

$$\begin{aligned} x'' &= -x' + 6y' \\ &= -x' + 6(x - 2y) \\ &= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\ &= -3x' + 4x \end{aligned} \tag{2.14}$$

The resulting differential equation has a characteristic equation of $r^2 + 3r - 4 = 0$. The roots of this equation are $r = 1, -4$. Therefore, $x(t) = c_1 e^t + c_2 e^{-4t}$. But, we still need $y(t)$. From the first equation of the system we have

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1 e^t - 3c_2 e^{-4t}).$$

Thus, the solution to the system is

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \tfrac{1}{3} c_1 e^t - \tfrac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.15}$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Solving systems with initial conditions.

**Example 2.2.** Solve

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.16}$$

given $x(0) = 2$, $y(0) = 0$.

We already have the general solution of this system in (2.15). Inserting the initial conditions, we have

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \tfrac{1}{3} c_1 - \tfrac{1}{2} c_2. \end{aligned} \tag{2.17}$$

Solving for $c_1$ and $c_2$ gives $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is

$$\begin{aligned} x(t) &= \tfrac{2}{5}\left(3e^t + 2e^{-4t}\right), \\ y(t) &= \tfrac{2}{5}\left(e^t - e^{-4t}\right). \end{aligned} \tag{2.18}$$

### 2.2.2 *Equilibrium Solutions and Nearby Behaviors*

IN STUDYING SYSTEMS OF DIFFERENTIAL EQUATIONS, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying equilibrium solutions of system (2.8). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let $x_0$ and $y_0$ be the (constant) equilibrium solutions. Then, $x_0$ and $y_0$ must satisfy the system

Equilibrium solutions.

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \tag{2.19}$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{2.20}$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from $(x, y)$ to $(u, v)$ by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

**Example 2.3. Stable Node (sink)**

Consider the system

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \tag{2.21}$$

This is a simple uncoupled system. Each equation is simply solved to give

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

In this case we see that all solutions tend towards the equilibrium point, $(0, 0)$. This will be called a *stable node*, or a *sink*.

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the $xy$-plane at the parametric curves $(x(t), y(t))$.

**Solution Plots:** One can plot each solution as a function of $t$ given a set of initial conditions. Examples are shown in Figure 2.3 for several initial conditions. Note that the solutions decay for large $t$. Special cases result for various initial conditions. Note that for $t = 0$, $x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with $c_1$=0, then $x(t) = 0$ for all $t$. One obtains similar results when setting $y(0) = 0$.

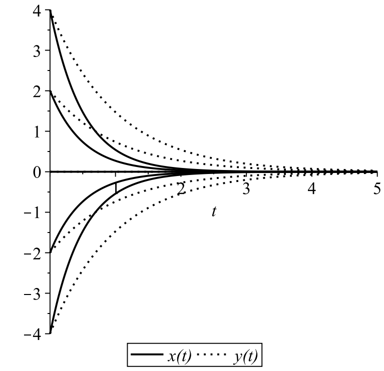



Figure 2.3: Plots of solutions of Example 2.3 for several initial conditions.

**Phase Portrait:** There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$ Such curves are called trajectories or orbits. The $xy$-plane is called the phase plane and a collection of such orbits gives a phase portrait for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter $t$ between the known solutions to get a relationship between $x$ and $y$. Since the solutions are known for the last example, we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2}\right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the $xy$-plane are given by $dy/dx$. For autonomous systems, we can write this slope just in terms of $x$ and $y$. This leads to a first order differential equation, which possibly could be solved analytically or numerically.

First we will obtain the orbits for Example 2.3 by solving the corresponding slope equation. Recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

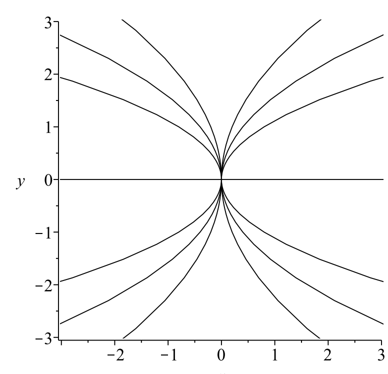
Therefore,

The Slope of a parametric curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{2.22}$$

For the system in (2.21) we use Equation (2.22) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$




The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for $A$ an arbitrary constant. Plots of these solutions in the phase plane are given in Figure 2.4. [Note that this is the same form for the orbits that we had obtained above by eliminating $t$ from the solution of the system.]

Figure 2.4: Orbits for Example 2.3.

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition

$(x_0, y_0)$, how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

meaning that as $t \to \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicating that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \to \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

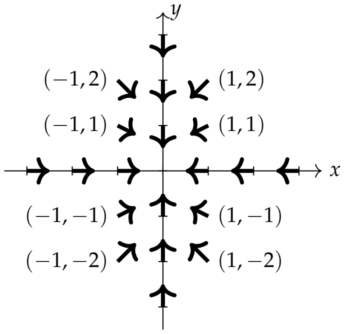



Figure 2.5: Sketch of tangent vectors using Example 2.3.

**Direction Fields:** Another way to determine the behavior of the solutions of the system of differential equations is to draw the direction field. A direction field is a vector field in which one plots arrows in the direction of tangents to the orbits at selected points in the plane. This is done because the slopes of the tangent lines are given by $dy/dx$. For the general system (2.9), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

This is a first order differential equation which can be solved as we show in the following examples.

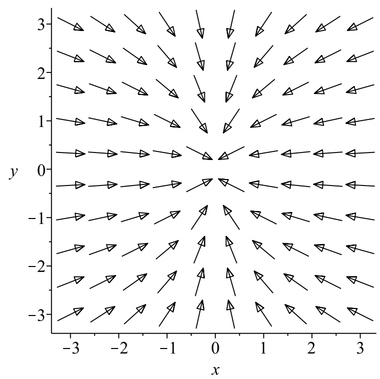



Figure 2.6: Direction field for Example 2.3.

**Example 2.4.** Draw the direction field for Example 2.3.

We can use software to draw direction fields. However, one can sketch these fields by hand. We have that the slope of the tangent at this point is given by

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

For each point in the plane one draws a piece of tangent line with this slope. In Figure 2.5 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (2.21), we have that $x'$ and $y'$ are both negative at this point. Therefore, the vector points down and to the left.

We can do this for several points, as shown in Figure 2.5. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as *isoclines* in which $\frac{dy}{dx}$ =constant.

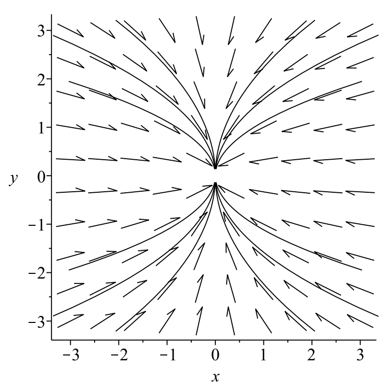



Figure 2.7: Phase portrait for Example 2.3. This is a stable node, or sink

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 2.3 the direction field is shown in Figure 2.6. Looking at this direction field, one can begin to "see" the orbits by following the tangent vectors.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 2.7. Are these the patterns you saw in Figure 2.6?

In this example we see all orbits "flow" towards the origin, or equilibrium point. Again, this is an example of what is called a *stable node* or a *sink*. (Imagine what happens to the water in a sink when the drain is unplugged.)

This is another uncoupled system. The solutions are again simply gotten by integration. We have that $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{t}$. Here we have that $x$ decays as $t$ gets large and $y$ increases as $t$ gets large. In particular, if one picks initial conditions with $c_2 = 0$, then orbits follow the $x$-axis towards the origin. For initial points with $c_1 = 0$, orbits originating on the $y$-axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.

In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small $c_2$ is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 2.8.

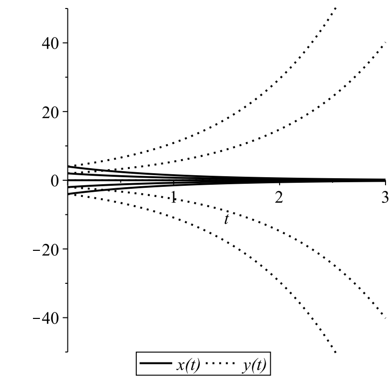



Figure 2.8: Plots of solutions of Example 2.5 for several initial conditions.

**Example 2.5. Saddle** Consider the system

$$
\begin{aligned}
x' &= -x \\
y' &= y. 
\end{aligned} \tag{2.23}
$$

Similar to the first example, we can look at plots of solutions orbits in the phase plane. These are given by Figures 2.8-2.9. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

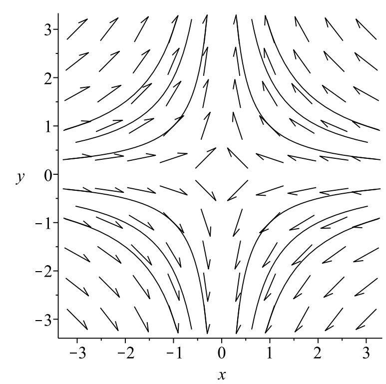



Figure 2.9: Phase portrait for Example 2.5. This is a saddle.

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a saddle point. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the $x$-axis. In this case, the line of initial conditions is given by the $x$-axis.

**Example 2.6. Unstable Node (source)**

$$
\begin{aligned}
x' &= 2x \\
y' &= y. 
\end{aligned} \tag{2.24}
$$

This example is similar to Example 2.3. The solutions are obtained by replacing $t$ with $-t$. The solutions, orbits, and direction fields can be seen in Figures 2.10-2.11. This is once again a node, but all orbits lead away from the equilibrium point. It is called an *unstable node* or a *source*.

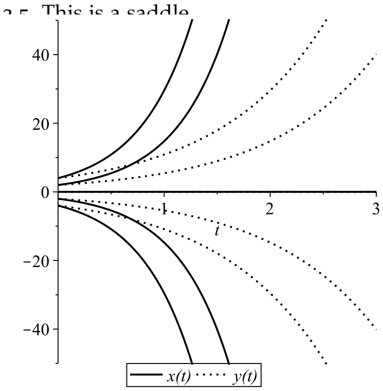



Figure 2.10: Plots of solutions of Example 2.6 for several initial conditions.

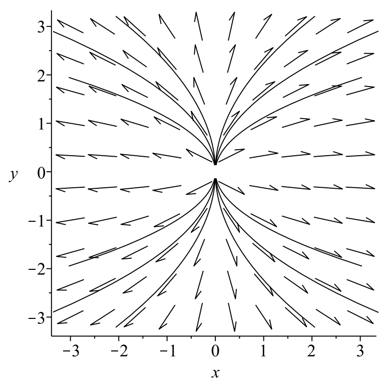



Figure 2.11: Phase portrait for Example 2.6, an unstable node or source.

**Example 2.7. Center**

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{2.25}$$

This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain

$$x'' + x = 0.$$

We recognize this equation as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

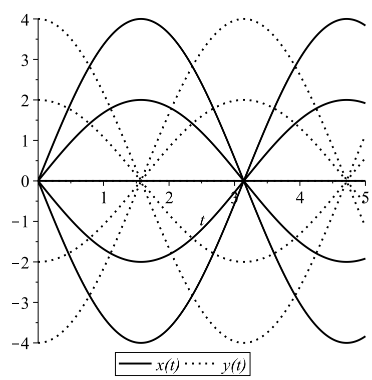



Figure 2.12: Plots of solutions of Example 2.7 for several initial conditions.

In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Performing a separation of variables and integrating, we find that

$$x^2 + y^2 = C.$$

Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 2.12-2.13 confirms this result. This type of point is called a center.

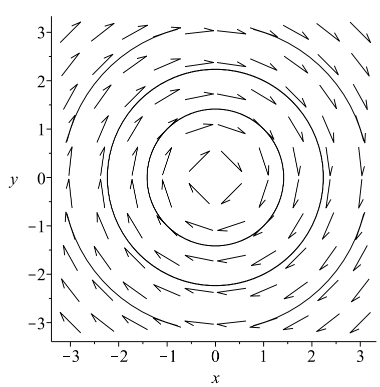



Figure 2.13: Phase portrait for Example 2.7, a center.

**Example 2.8. Focus (spiral)**

$$\begin{aligned} x' &= \alpha x + y \\ y' &= -x. \end{aligned} \tag{2.26}$$

In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, $\alpha x$, to the system in Example 2.7. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the Figures 2.15-2.18. We see orbits that look like spirals. These orbits are stable and unstable spirals (or foci, the plural of focus.)

We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation

$$x'' - \alpha x' + x = 0.$$

We recall from our first course that this is a form of damped simple harmonic motion. The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

There are five special cases to consider as shown in the below classification.

**Classification of Solutions of $x'' - \alpha x' + x = 0$**

1. $\alpha = -2$. There is one real solution. This case is called *critical damping* since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.
2. $\alpha < -2$. There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called *overdamped* motion. There are no oscillations
3. $-2 < \alpha < 0$. There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an *underdamped oscillation*.
4. $\alpha = 0$. This leads to *simple harmonic motion*.
5. $0 < \alpha < 2$. This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.
6. $\alpha = 2$. There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^{t}$. It leads to unbounded growth in time.
7. For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.

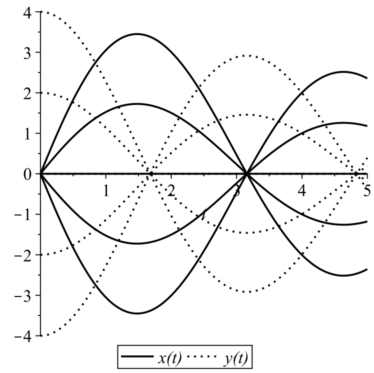



Figure 2.14: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = -0.2$.

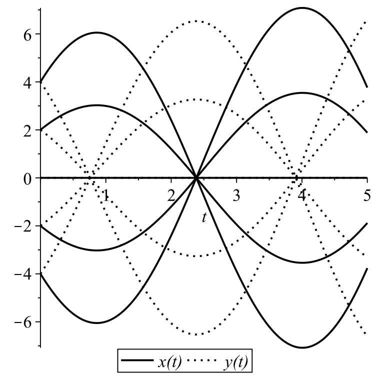



Figure 2.15: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = 0.1$.

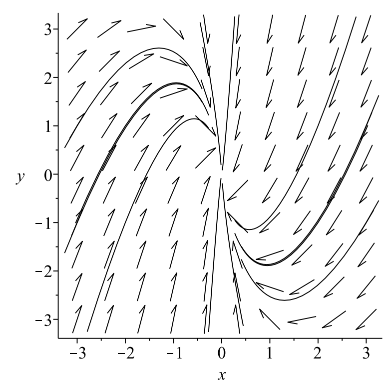



Figure 2.16: Phase portrait for 2.9. This is a degenerate node.

For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 2.14.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 2.15.) Of course, there can be overdamped motion if the magnitude of $\alpha$ is too large.

**Example 2.9. Degenerate Node** For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{2.27}$$

There are two possible approaches:

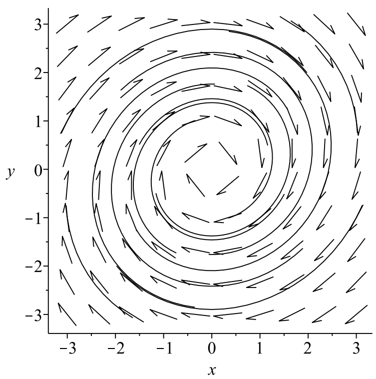



Figure 2.17: Phase portrait for Example 2.8 with $\alpha = -0.2$. This is a stable focus, or spiral.

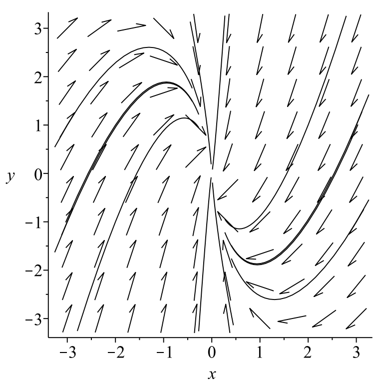



Figure 2.18: Phase portrait for Example 2.9. This is a degenerate node.

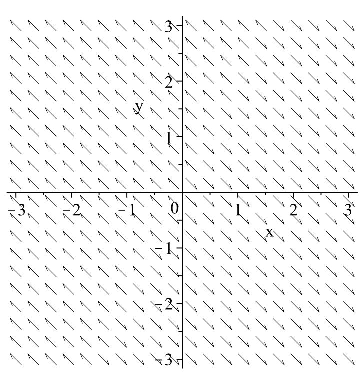



Figure 2.19: Plots of direction field of Example 2.10.

a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have

$$y' + y = -2c_1 e^{-t}.$$

This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.

b. Another method would be to proceed to rewrite this as a second order equation. Computing $x''$ does not get us very far. So, we look at

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{2.28}$$

Therefore, $y$ satisfies

$$y'' + 2y' + y = 0.$$

The characteristic equation has one real root, $r = -1$. So, we write

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

This is a stable *degenerate node.* Combining this with the solution $x(t) = c_1 e^{-t}$, we can show that $y(t) = (c_2 - 2c_1 t)e^{-t}$ as before.

In Figure 2.16 we see several orbits in this system. It differs from the stable node show in Figure 2.4 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, then $x(t) = 0$ and $y(t) = c_2 e^{-t}$. This leads to orbits running along the $y$-axis as seen in the figure.

**Example 2.10. A Line of Equilibria, Zero Root**

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{2.29}$$

In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{2.30}$$

So, the second order equation is

$$x'' - 3x' = 0$$

and the characteristic equation is $0 = r(r-3)$. This gives the general solution as

$$x(t) = c_1 + c_2 e^{3t}$$

and thus

$$y = 2x - x' = 2(c_1 + c_2 e^3 t) - (3c_2 e^{3t}) = 2c_1 - c_2 e^{3t}.$$

In Figure 2.19 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

Furthermore, looking at initial conditions with $y = 2x$, we have at $t = 0$,

$$2c_1 - c_2 = 2(c_1 + c_2) \quad \Rightarrow \quad c_2 = 0.$$

Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. This line of fixed points is called a line of equilibria.

### 2.2.3 *Polar Representation of Spirals*

IN THE EXAMPLES WITH A CENTER OR A SPIRAL, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian $(x, y)$ or polar $(r, \theta)$ coordinates. Given the polar form, one can find the Cartesian components using

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{2.31}$$

Since $x$ and $y$ are functions of $t$, then naturally we can think of $r$ and $\theta$ as functions of $t$. Converting a system of equations in the plane for $x'$ and $y'$ to polar form requires knowing $r'$ and $\theta'$. So, we first find expressions for $r'$ and $\theta'$ in terms of $x'$ and $y'$.

Differentiating the first equation in (2.31) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for $x'$ and $y'$ from system 2.9, we have

$$rr' = x(ax + by) + y(cx + dy).$$

In some cases this may be written entirely in terms of $r$'s. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

**Derivatives of Polar Variables**

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned} \tag{2.32}$$

**Example 2.11.** Rewrite the following system in polar form and solve the resulting system.

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay. \end{aligned} \tag{2.33}$$

We first compute $r'$ and $\theta'$:

$$rr' = xx' + yy' = x(ax + by) + y(-bx + ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

This leads to simpler system

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \tag{2.34}$$

This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that $r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 - bt$. Eliminating $t$ between these solutions, we finally find the polar equation of the orbits:

$$r = r_0 e^{-a(\theta - \theta_0)t/b}.$$

If you graph this for $a \neq 0$, you will get stable or unstable spirals.

**Example 2.12.** Consider the specific system

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \tag{2.35}$$

In order to convert this system into polar form, we compute

$$rr' = xx' + yy' = x(-y + x) + y(x + y) = r^2.$$

$$r^2\theta' = -xy' - yx' = x(x + y) - y(-y + x) = r^2.$$

This leads to simpler system

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \tag{2.36}$$

Solving these equations yields

$$r(t) = r_0 e^t, \quad \theta(t) = t + \theta_0.$$

Eliminating $t$ from this solution gives the orbits in the phase plane, $r(\theta) = r_0 e^{\theta - \theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

**Example 2.13.**

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \quad (2.37)$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

## 2.3 *Applications*

In this section we will describe some simple applications leading to systems of differential equations which can be solved using the methods in this chapter. These systems are left for homework problems and the as the start of further explorations for student projects.

### 2.3.1 *Mass-Spring Systems*

The first examples that we had seen involved masses on springs. Recall that for a simple mass on a spring we studied simple harmonic motion, which is governed by the equation

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x, \end{aligned} \quad (2.38)$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x, \end{aligned} \quad (2.39)$$

where $\omega^2 = \frac{k}{m}$. The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

We also looked at the system of two masses and two springs as shown in Figure 2.20. The equations governing the motion of the masses is

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \quad (2.40)$$

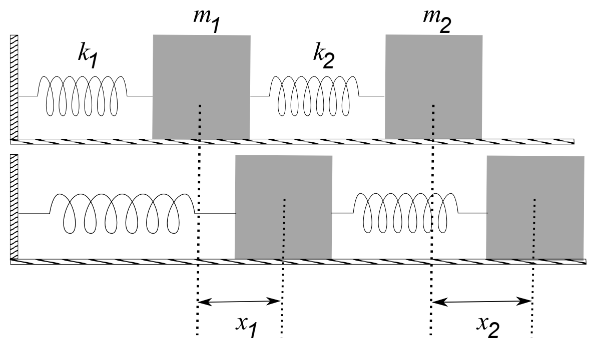



Figure 2.20: System of two masses and two springs.

We can rewrite this system as four first order equations

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1)\\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1).\end{aligned} \tag{2.41}$$

The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}.$$

We can study this system for specific values of the constants using the methods covered in the last sections.

Writing the spring-block system as a second order vector system.

Actually, one can also put the system (2.40) in the matrix form

$$\begin{pmatrix} m_1 & 0\\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1\\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_2\\ k_2 & -k_2 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}. \tag{2.42}$$

This system can then be written compactly as

$$M\ddot{\mathbf{x}} = -K\mathbf{x}, \tag{2.43}$$

where

$$M = \begin{pmatrix} m_1 & 0\\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1+k_2 & -k_2\\ -k_2 & k_2 \end{pmatrix}.$$

This system can be solved by guessing a form for the solution. We could guess

$$\mathbf{x} = \mathbf{a}e^{i\omega t}$$

or

$$\mathbf{x} = \begin{pmatrix} a_1\cos(\omega t - \delta_1)\\ a_2\cos(\omega t - \delta_2) \end{pmatrix},$$

where $\delta_i$ are phase shifts determined from initial conditions.

Inserting $\mathbf{x} = \mathbf{a}e^{i\omega t}$ into the system gives

$$(K - \omega^2 M)\mathbf{a} = \mathbf{0}.$$

This is a homogeneous system. It is a generalized eigenvalue problem for eigenvalues $\omega^2$ and eigenvectors $\mathbf{a}$. We solve this in a similar way to the standard matrix eigenvalue problems. The eigenvalue equation is found as

$$\det\,(K - \omega^2 M) = 0.$$

Once the eigenvalues are found, then one determines the eigenvectors and constructs the solution.

**Example 2.14.** Let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then, we have to solve the system

$$\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The eigenvalue equation is given by

$$\begin{aligned} 0 &= \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} \\ &= (2k - m\omega^2)(k - m\omega^2) - k^2 \\ &= m^2\omega^4 - 3km\omega^2 + k^2. \end{aligned} \tag{2.44}$$

Solving this quadratic equation for $\omega^2$, we have

$$\omega^2 = \frac{3 \pm 1}{2} \frac{k}{m}.$$

For positive values of $\omega$, one can show that

$$\omega = \frac{1}{2}\left(\pm 1 + \sqrt{5}\right)\sqrt{\frac{k}{m}}.$$

The eigenvectors can be found for each eigenvalue by solving the homogeneous system

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

The eigenvectors are given by

$$\mathbf{a}_1 = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}.$$

We are now ready to construct the real solutions to the problem. Similar to solving two first order systems with complex roots, we take the real and imaginary parts and take a linear combination of the solutions. In this problem there are four terms, giving the solution in the form

$$\mathbf{x}(t) = c_1\mathbf{a_1} cos \omega_1 t + c_2 \mathbf{a_1} sin \omega_1 t + c_3 \mathbf{a_2} cos \omega_2 t + c_4 \mathbf{a_2} sin \omega_2 t,$$

where the $\omega$'s are the eigenvalues and the $\mathbf{a}$'s are the corresponding eigenvectors. The constants are determined from the initial conditions, $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$.

### 2.3.2 Circuits*

In the last chapter we investigated simple series LRC circuits. More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This results in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.21. This is not a problem that can be covered in the first year physics course.

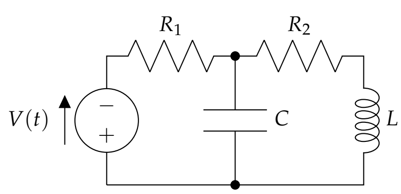



Figure 2.21: A circuit with two loops containing several different circuit elements.

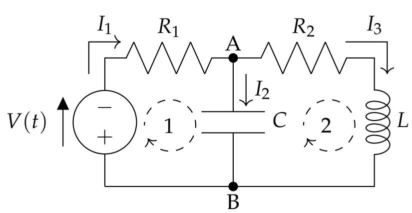



Figure 2.22: The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

There are two loops, indicated in Figure 2.22 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled $I_i$, $i = 1,2,3$. Corresponding to each current is a changing charge, $q_i$ such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1,2,3.$$

We have for loop one

$$I_1R_1 + \frac{q_2}{C} = V(t) \tag{2.45}$$

and for loop two

$$I_3R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{2.46}$$

There are three unknown functions for the charge. Once we know the charge functions, differentiation will yield the three currents. However, we only have two equations. We need a third equation. This equation is found from Kirchoff's Point (Junction) Rule.

Consider the points A and B in Figure 2.22. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{2.47}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{2.48}$$

Equations (2.45), (2.46), and (2.48) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$\begin{aligned} R_1\dot{q}_1 + \frac{q_2}{C} &= V(t) \\ R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{2.49}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions for the other terms. $\dot{q}_3$ is gotten from the definition of $q_4$ and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned}
\dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} \\
\dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} - q_4 \\
\dot{q}_3 &= q_4 \\
\dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L} q_4.
\end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations.

### 2.3.3 *Mixture Problems*

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake. We first saw such problems in Chapter 1.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out}.$$

This can be generalized to the case of two interconnected tanks. We will provide an example, but first we review the single tank problem from Chapter 1.

**Example 2.15. Single Tank Problem**

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 2.23.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time $t$. In all such problems one assumes that the solution is well mixed at each instant of time.

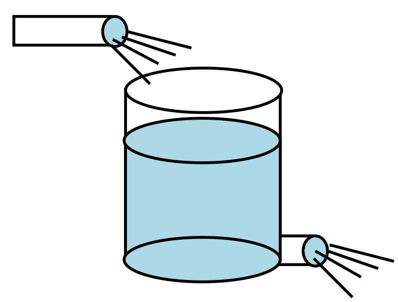


Figure 2.23: A typical mixing problem.

Let $x(t)$ be the amount of salt at time $t$. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that $dx/dt$ has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = 10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x\ \text{pounds}}{50\ \text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = \frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is easily solved using the methods for first order equations.

**Example 2.16. Double Tank Problem**

One has two tanks connected together, labeled tank X and tank Y, as shown in Figure 2.24.

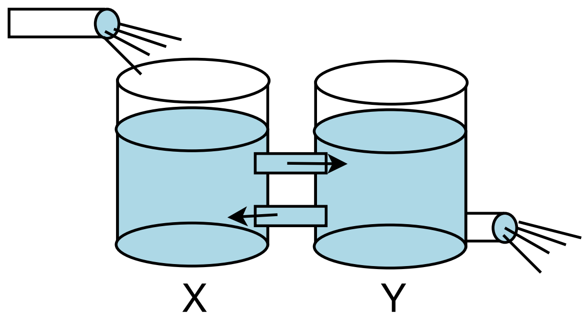



Figure 2.24: The two tank problem.

Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?

In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank $Y$. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100} \\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}.\end{aligned} \tag{2.50}$$

This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve. The matrix form of the system is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

The eigenvalues for the problem are given by $\lambda = -3 \pm \sqrt{3}$ and the eigenvectors are

$$\begin{pmatrix} 1 \\ \pm\sqrt{3} \end{pmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is easily written down:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t}.$$

Finally, we need to satisfy the initial conditions. So,

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix},$$

or

$$c_1 + c_2 = 100, \quad (c_1 - c_2)\sqrt{3} = 0.$$

So, $c_2 = c_1 = 50$. The final solution is

$$\mathbf{x}(t) = 50 \left( \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t} \right),$$

or

$$\begin{aligned} x(t) &= 50 \left( e^{(-3+\sqrt{3})t} + e^{(-3-\sqrt{3})t} \right) \\ y(t) &= 50\sqrt{3} \left( e^{(-3+\sqrt{3})t} - e^{(-3-\sqrt{3})t} \right). \end{aligned} \tag{2.51}$$

### 2.3.4 *Chemical Kinetics**

THERE ARE MANY PROBLEMS IN THE CHEMISTRY of chemical reactions which lead to systems of differential equations. The simplest reaction is when a chemical $A$ turns into chemical $B$. This happens at a certain rate, $k > 0$. This reaction can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of $A$, $[A]$, and $B$, $[B]$, are given by

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{2.52}$$

The chemical reactions used in these examples are first order reactions. Second order reactions have rates proportional to the square of the concentration.

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

Here there are three concentrations and two rates of change. The system of equations governing the reaction is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.53}$$

The more complication rate of change is when [B] increases from [A] changing to [B] and decrease when [B] changes to [C]. Thus, there are two terms in the rate of change equation for concentration [B].

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$A \underset{k_1}{\overset{k_3}{\rightleftarrows}} B \xrightarrow[k_2]{} C.$$

The reverse reaction rates contribute to the reaction equations for [A] and [B]. The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A] + k_3[B], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.54}$$

Nonlinear chemical reactions will be discussed in the next chapter.

### 2.3.5 *Predator Prey Models**

ANOTHER COMMON POPULATION MODEL is that describing the coexistence of species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\begin{aligned} \frac{dR}{dt} &= aR - cF, \\ \frac{dF}{dt} &= -bF + dR, \end{aligned} \tag{2.55}$$

where all of the constants are positive numbers. Studying this coupled system would lead to a study of the dynamics of these populations. The nonlinear version of this system, the Lotka-Volterra model, will be discussed in the next chapter.

### 2.3.6 Love Affairs*

The next application is one that was introduced in 1988 by Strogatz as a cute system involving relationships.[1] One considers what happens to the affections that two people have for each other over time. Let $R$ denote the affection that Romeo has for Juliet and $J$ be the affection that Juliet has for Romeo. Positive values indicate love and negative values indicate dislike.

[1] Steven H. Strogatz introduced this problem as an interesting example of systems of differential equations in *Mathematics Magazine*, Vol. 61, No. 1 (Feb. 1988) p 35. He also describes it in his book *Nonlinear Dynamics and Chaos (1994)*.

One possible model is given by

$$\begin{aligned} \frac{dR}{dt} &= bJ \\ \frac{dJ}{dt} &= cR \end{aligned} \tag{2.56}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned} \frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= cR + dJ. \end{aligned} \tag{2.57}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

### 2.3.7 Epidemics*

Another interesting area of application of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let us break the population into three classes. First, we let $S(t)$ represent the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others could recover. We will consider the case that initially there is one infected person and the rest, say $N$, are healthy. Can we predict how many deaths have occurred by time t?

We model this problem using the compartmental analysis we had seen for mixing problems. The total rate of change of any population would be due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that a) the rate of infection is proportional to the number of healthy people, $aS$, and b) the number who recover is proportional to the number of infected people, $rI$. Thus, the rate of change of healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infected people is due to healthy people getting infected and the infected people who either recover or die. Using the corresponding terms in the other equations, we can write the rate of change of infected people as

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system of differential equations can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \qquad (2.58)$$

The reader can find the solutions of this system and determine if this is a realistic model.

## 2.4 *First Order Matrix Differential Equations*

### 2.4.1 *Matrix Formulation*

WE HAVE INVESTIGATED SEVERAL LINEAR SYSTEMS in the plane and in the next chapter we will use some of these ideas to investigate nonlinear systems. We need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations later in this chapter.

We start with the usual homogeneous system in Equation (2.9). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix* $A$. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

You can verify that this is a solution by simply differentiating,

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_0 \frac{d}{dt}\left(e^{At}\right) = A\mathbf{x}_0 e^{At} = A\mathbf{x}.$$

### 2.4.2 *Exponentiating a Matrix*

HOWEVER, THERE REMAINS THE QUESTION, "What does it mean to exponentiate a matrix?" The exponential of a matrix is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.60}$$

In general it is difficult to sum this series, but it is doable for some simple examples.

The *exponential of a matrix* is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.59}$$

In general, it is difficult computing $e^A$ unless $A$ is diagonal.

**Example 2.17.** Evaluate $e^{tA}$ for $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\begin{aligned}
e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^3 + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots & 0 \\ 0 & 1 + 2t + \frac{2t^2}{2!} + \frac{8t^3}{3!} \cdots \end{pmatrix} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}
\end{aligned} \tag{2.61}$$

**Example 2.18.** Evaluate $e^{tA}$ for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We first note that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Therefore,

$$A^n = \begin{cases} A, & n \text{ odd}, \\ I, & n \text{ even}. \end{cases}$$

Then, we have

$$\begin{aligned}
e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\
&= I + tA + \frac{t^2}{2!}I + \frac{t^3}{3!}A + \cdots. \\
&= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!}\cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!}\cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!}\cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!}\cdots \end{pmatrix} \\
&= \begin{pmatrix} \cosh & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \qquad (2.62)
\end{aligned}$$

Since summing these infinite series might be difficult, we will now investigate the solutions of planar systems to see if we can find other approaches for solving linear systems using matrix methods. We begin by recalling the solution to the problem in Example (2.16). We obtained the solution to this system as

$$\begin{aligned}
x(t) &= c_1 e^t + c_2 e^{-4t}, \\
y(t) &= \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t}. \qquad (2.63)
\end{aligned}$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^t \\ \frac{1}{3}c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \qquad (2.64)
\end{aligned}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and $\lambda$ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (2.4.1) we insert this guess. Thus,

$$\begin{aligned}
\mathbf{x}' &= A\mathbf{x} \Rightarrow \\
\lambda \mathbf{v} e^{\lambda t} &= A\mathbf{v}e^{\lambda t}. \qquad (2.65)
\end{aligned}$$

For this to be true for all $t$, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \qquad (2.66)$$

This is an eigenvalue problem. $A$ is a $2 \times 2$ matrix for our problem, but could easily be generalized to a system of $n$ first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations..

### 2.4.3 *Eigenvalue Problems*

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2.67}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is called a *trivial solution*. Typically, we are given the matrix $A$ and have to determine the *eigenvalues*, $\lambda$, and the associated *eigenvectors*, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (2.67), we obtain the homogeneous algebraic system

$$\begin{aligned} (a - \lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d - \lambda)v_2 &= 0. \end{aligned} \tag{2.68}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a + d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation (2.12) for the general constant coefficient differential equation considered in the first chapter. Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

So, the process is to

a) Write the coefficient matrix;

b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,

c) Find the eigenvectors by solving the linear system $(A - \lambda I)\mathbf{v} = 0$ for each $\lambda$.

## 2.5 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{2.69}$$

The type of behavior depends upon the eigenvalues of matrix $A$. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions[2], then the general solution is given as

[2] Recall that linear independence means $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Then, setting $t = 0$, we get two linear equations for $c_1$ and $c_2$:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. In Table 2.1 we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

Let's consider Case III. Note that since the original system of equations does not have any $i$'s, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

and

$$\frac{d}{dt} Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

Table 2.1: Solutions Types for Planar Systems with Constant Coefficients

**Classification of the Solutions for Two Linear First Order Differential Equations**

1. **Case I: Two real, distinct roots.**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

2. **Case II: One Repeated Root**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue $\lambda$, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.

3. **Case III: Two complex conjugate roots.**

   Solve the eigenvalue problem $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector $\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$. Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t) : \mathbf{y}_1(t) = Re(\mathbf{y}(t))$ and $\mathbf{y}_2(t) = Im(\mathbf{y}(t))$. Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a+d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned} (te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right]. \\ e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2. \\ e^{\lambda t}\left(\mathbf{v}_1 + \lambda\mathbf{v}_2\right) &= e^{\lambda t}A\mathbf{v}_2. \end{aligned} \tag{2.70}$$

Noting this is true for all $t$, we find that

$$\mathbf{v}_1 + \lambda \mathbf{v}_2 = A\mathbf{v}_2. \tag{2.71}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

## 2.6 *Examples of the Matrix Method*

Here we will give some examples for typical systems for the three cases mentioned in the last section.

**Example 2.19.** $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

**Eigenvalues:** We first determine the eigenvalues.

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{2.72}$$

Therefore,

$$\begin{aligned} 0 &= (4-\lambda)(3-\lambda) - 6 \\ 0 &= \lambda^2 - 7\lambda + 6 \\ 0 &= (\lambda - 1)(\lambda - 6) \end{aligned} \tag{2.73}$$

The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.

**Eigenvectors:** Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda \mathbf{v}$ in each case.

Case $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.74}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.75}$$

This gives $3v_1 + 2v_2 = 0$. One possible solution yields an eigenvector of

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Case $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.76}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.77}$$

For this case we need to solve $-2v_1 + 2v_2 = 0$. This yields

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**General Solution:** We can now construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^t \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^t + c_2 e^{6t} \\ -3c_1 e^t + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{2.78}$$

**Example 2.20.** $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}.$

**Eigenvalues:** Again, one solves the eigenvalue equation.

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{2.79}$$

Therefore,

$$\begin{aligned} 0 &= (3-\lambda)(-1-\lambda) + 5 \\ 0 &= \lambda^2 - 2\lambda + 2 \\ \lambda &= \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i. \end{aligned} \tag{2.80}$$

The eigenvalues are then $\lambda = 1 + i, 1 - i$. This is an example of Case III.

**Eigenvectors:** In order to find the general solution, we need only find the eigenvector associated with $1 + i$.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.81}$$

We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{2.82}$$

**Complex Solution:** In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.

$$e^{\lambda t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{(1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\begin{aligned} &= e^t(\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^t\begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

**General Solution:** Now we can construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\ &= e^t\begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1\cos t + c_2\sin t \end{pmatrix}. \end{aligned} \tag{2.83}$$

Note: This can be rewritten as

$$\mathbf{x}(t) = e^t\cos t\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^t\sin t\begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.$$

**Example 2.21.** $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

**Eigenvalues:**

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} \tag{2.84}$$

Therefore,

$$\begin{aligned} 0 &= (7-\lambda)(1-\lambda) + 9 \\ 0 &= \lambda^2 - 8\lambda + 16 \\ 0 &= (\lambda - 4)^2. \end{aligned} \tag{2.85}$$

There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.

**Eigenvectors:** In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.86}$$

Therefore, we have

$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

**Second Linearly Independent Solution:**
Now we need to solve $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{2.87}$$

Expanding the matrix product, we obtain the system of equations

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{2.88}$$

The solution of this system is $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

**General Solution:** We construct the general solution as

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1). \\ &= c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}. \end{aligned} \tag{2.89}$$

## 2.7 *Planar Systems - Summary*

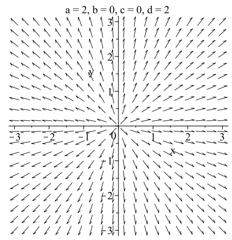
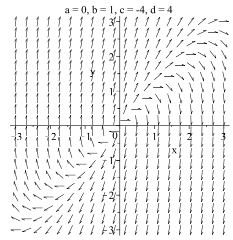
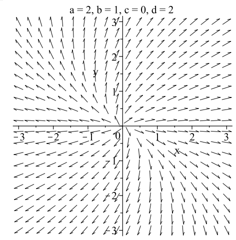
The reader should have noted by now that there is a connection between the behavior of the solutions obtained in Section 2.2.2 and the eigenvalues found from the coefficient matrices in the previous examples. In Table 2.2 we summarize some of these cases.

| Type | Eigenvalues | Stability |
|---|---|---|
| Node | Real $\lambda$, same signs | $\lambda < 0$, stable<br>$\lambda > 0$, unstable |
| Saddle | Real $\lambda$ opposite signs | Mostly Unstable |
| Center | $\lambda$ pure imaginary | — |
| Focus/Spiral | Complex $\lambda$, $\mathrm{Re}(\lambda) \neq 0$ | $\mathrm{Re}(\lambda) < 0$, stable<br>$\mathrm{Re}(\lambda) > 0$, unstable |
| Degenerate Node | Repeated roots, | $\lambda > 0$, stable |
| Lines of Equilibria | One zero eigenvalue | $\lambda < 0$, stable |

Table 2.2: List of typical behaviors in planar systems.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 2.3 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

Table 2.3: Three examples of systems with a repeated root of $\lambda = 2$.

| System 1 | System 2 | System 3 |
|---|---|---|
|   |   |   |
| $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ | $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$ | $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ |

Another way to look at the classification of these solution is to use the determinant and trace of the coefficient matrix. Recall that the determinant and trace of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by $\det A = ad - bc$ and $\operatorname{tr} A = a + d$.

We note that the general eigenvalue equation,

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

can be written as

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0. \tag{2.90}$$

Therefore, the eigenvalues are found from the quadratic formula as

$$\lambda_{1,2} = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}. \tag{2.91}$$

The solution behavior then depends on the sign of discriminant,

$$(\operatorname{tr} A)^2 - 4\det A.$$

If we consider a plot of where the discriminant vanishes, then we could plot

$$(\operatorname{tr} A)^2 = 4\det A$$

in the $\det A \operatorname{tr} A$)-plane. This is a parabolic cure as shown by the dashed line in Figure 2.25. The region inside the parabola have a negative discriminant, leading to complex roots. In these cases we have oscillatory solutions. If $\operatorname{tr} A = 0$, then one has centers. If $\operatorname{tr} A < 0$, the solutions are stable spirals; otherwise, they are unstable spirals. If the discriminant is positive, then the roots are real, leading to nodes or saddles in the regions indicated.

## 2.8 *Theory of Homogeneous Constant Coefficient Systems*

There is a general theory for solving homogeneous, constant coefficient systems of first order differential equations. We begin by once again recalling

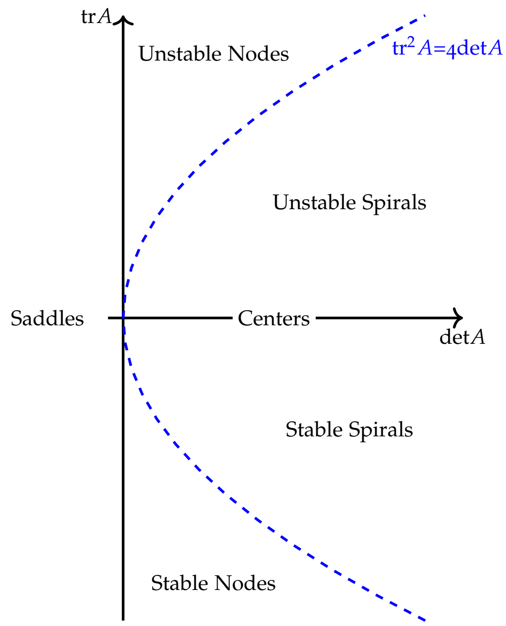



Figure 2.25: Solution Classification for Planar Systems.

the specific problem (2.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.92}$$

This time we rewrite the solution as

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{-4t} \\ \frac{1}{3} e^t & -\frac{1}{2} e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &\equiv \Phi(t)\mathbf{C}. \end{aligned} \tag{2.93}$$

Thus, we can write the general solution as a $2 \times 2$ matrix $\Phi$ times an arbitrary constant vector. The matrix $\Phi$ consists of two columns that are linearly independent solutions of the original system. This matrix is an example of what we will define as the *Fundamental Matrix* of solutions of the system. So, determining the Fundamental Matrix will allow us to find the general solution of the system upon multiplication by a constant matrix. In fact, we will see that it will also lead to a simple representation of the solution of the initial value problem for our system. We will outline the general theory.

Consider the homogeneous, constant coefficient system of first order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \end{aligned}$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \tag{2.94}$$

As we have seen, this can be written in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now, consider $m$ vector solutions of this system: $\phi_1(t)$, $\phi_2(t), \ldots \phi_m(t)$. These solutions are said to be *linearly independent* on some domain if

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_m\phi_m(t) = 0$$

for all $t$ in the domain implies that $c_1 = c_2 = \ldots = c_m = 0$.

Let $\phi_1(t), \phi_2(t), \ldots \phi_n(t)$ be a set of $n$ linearly independent set of solutions of our system, called a *fundamental set of solutions*. We construct a matrix from these solutions using these solutions as the column of that matrix. We define this matrix to be the *fundamental matrix solution*. This matrix takes the form

$$\Phi = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}.$$

What do we mean by a "matrix" solution? We have assumed that each $\phi_k$ is a solution of our system. Therefore, we have that $\phi_k' = A\phi_k$, for $k = 1, \ldots, n$. We say that $\Phi$ is a matrix solution because we can show that $\Phi$ also satisfies the matrix formulation of the system of differential equations. We can show this using the properties of matrices.

$$\begin{aligned} \frac{d}{dt}\Phi &= \begin{pmatrix} \phi_1' & \ldots & \phi_n' \end{pmatrix} \\ &= \begin{pmatrix} A\phi_1 & \ldots & A\phi_n \end{pmatrix} \\ &= A\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} \\ &= A\Phi. \end{aligned} \tag{2.95}$$

Given a set of vector solutions of the system, when are they linearly independent? We consider a matrix solution $\Omega(t)$ of the system in which we have $n$ vector solutions. Then, we define the *Wronskian* of $\Omega(t)$ to be

$$W = \det \Omega(t).$$

If $W(t) \neq 0$, then $\Omega(t)$ is a fundamental matrix solution.

Before continuing, we list the fundamental matrix solutions for the set of examples in the last section. (Refer to the solutions from those examples.) Furthermore, note that the fundamental matrix solutions are not unique as one can multiply any column by a nonzero constant and still have a fundamental matrix solution.

**Example 2.19** $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix}.$$

We should note in this case that the Wronskian is found as

$$\begin{aligned} W &= \det \Phi(t) \\ &= \begin{vmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{vmatrix} \\ &= 5e^{7t} \neq 0. \end{aligned} \tag{2.96}$$

**Example 2.20** $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^t(2\cos t - \sin t) & e^t(\cos t + 2\sin t) \\ e^t \cos t & e^t \sin t \end{pmatrix}.$$

**Example 2.21** $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{4t}(1+t) \\ 3e^{4t} & e^{4t}(2+3t) \end{pmatrix}.$$

So far we have only determined the general solution. This is done by the following steps:

**Procedure for Determining the General Solution**

1. Solve the eigenvalue problem $(A - \lambda I)\mathbf{v} = 0$.
2. Construct vector solutions from $\mathbf{v}e^{\lambda t}$. The method depends if one has real or complex conjugate eigenvalues.
3. Form the fundamental solution matrix $\Phi(t)$ from the vector solution.
4. The general solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$ an arbitrary constant vector.

We are now ready to solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting with the general solution, we have that

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

As usual, we need to solve for the $c_k$'s. Using matrix methods, this is now easy. Since the Wronskian is not zero, then we can invert $\Phi$ at any value of $t$. So, we have

$$\mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Putting $\mathbf{C}$ back into the general solution, we obtain the solution to the initial value problem:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0.$$

You can easily verify that this is a solution of the system and satisfies the initial condition at $t = t_0$.

The matrix combination $\Phi(t)\Phi^{-1}(t_0)$ is useful. So, we will define the resulting product to be the *principal matrix solution*, denoting it by

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0).$$

Thus, the solution of the initial value problem is $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$. Furthermore, we note that $\Psi(t)$ is a solution to the matrix initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = I,$$

where $I$ is the $n \times n$ identity matrix.

**Matrix Solution of the Homogeneous Problem**

In summary, the matrix solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0,$$

where $\Phi(t)$ is the fundamental matrix solution and $\Psi(t)$ is the principal matrix solution.

**Example 2.22.** Let's consider the matrix initial value problem

$$\begin{aligned} x' &= 5x + 3y \\ y' &= -6x - 4y, \end{aligned} \tag{2.97}$$

satisfying $x(0) = 1$, $y(0) = 2$. Find the solution of this problem.

We first note that the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalue equation is easily found from

$$\begin{aligned} 0 &= -(5-\lambda)(4+\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned} \tag{2.98}$$

So, the eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found to be

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now we construct the fundamental matrix solution. The columns are obtained using the eigenvectors and the exponentials, $e^{\lambda t}$ :

$$\phi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \qquad \phi_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

So, the fundamental matrix solution is

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}.$$

The general solution to our problem is then

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \mathbf{C}$$

for $\mathbf{C}$ is an arbitrary constant vector.

In order to find the particular solution of the initial value problem, we need the principal matrix solution. We first evaluate $\Phi(0)$, then we invert it:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

The particular solution is then

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} + 4e^{2t} \\ 6e^{-t} - 4e^{2t} \end{pmatrix} \end{aligned} \tag{2.99}$$

Thus, $x(t) = -3e^{-t} + 4e^{2t}$ and $y(t) = 6e^{-t} - 4e^{2t}$.

## 2.9 *Nonhomogeneous Systems*

Before leaving the theory of systems of linear, constant coefficient systems, we will discuss nonhomogeneous systems. We would like to solve systems of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t). \tag{2.100}$$

We will assume that we have found the fundamental matrix solution of the homogeneous equation. Furthermore, we will assume that $A(t)$ and $\mathbf{f}(t)$ are continuous on some common domain.

As with second order equations, we can look for solutions that are a sum of the general solution to the homogeneous problem plus a particular solution of the nonhomogeneous problem. Namely, we can write the general solution as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \mathbf{x}_p(t),$$

where $\mathbf{C}$ is an arbitrary constant vector, $\Phi(t)$ is the fundamental matrix solution of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$\mathbf{x}_p' = A(t)\mathbf{x}_p + \mathbf{f}(t).$$

Such a representation is easily verified.

We need to find the particular solution, $\mathbf{x}_p(t)$. We can do this by applying *The Method of Variation of Parameters for Systems*. We consider a solution in the form of the solution of the homogeneous problem, but replace the constant vector by unknown parameter functions. Namely, we assume that

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{c}(t).$$

Differentiating, we have that

$$\mathbf{x}_p' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = A\Phi\mathbf{c} + \Phi\mathbf{c}',$$

or

$$\mathbf{x}_p' - A\mathbf{x}_p = \Phi\mathbf{c}'.$$

But the left side is $\mathbf{f}$. So, we have that,

$$\Phi\mathbf{c}' = \mathbf{f},$$

or, since $\Phi$ is invertible (why?),

$$\mathbf{c}' = \Phi^{-1}\mathbf{f}.$$

In principle, this can be integrated to give $\mathbf{c}$. Therefore, the particular solution can be written as

$$\mathbf{x}_p(t) = \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{2.101}$$

This is the *variation of parameters formula.*

The general solution of Equation (2.100) has been found as

$$\boxed{\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds.} \tag{2.102}$$

We can use the general solution to find the particular solution of an initial value problem consisting of Equation (2.100) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. This condition is satisfied for a solution of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.103}$$

provided

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

This can be solved for $\mathbf{C}$ as in the last section. Inserting the solution back into the general solution (2.103), we have

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.104}$$

This solution can be written a little neater in terms of the principal matrix solution, $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$ :

$$\boxed{\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{f}(s)\,ds} \tag{2.105}$$

Finally, one further simplification occurs when $A$ is a constant matrix, which are the only types of problems we have solved in this chapter. In this case, we have that $\Psi^{-1}(t) = \Psi(-t)$. So, computing $\Psi^{-1}(t)$ is relatively easy.

**Example 2.23.** $x'' + x = 2\cos t$, $x(0) = 4$, $x'(0) = 0$. This example can be solved using the Method of Undetermined Coefficients. However, we will use the matrix method described in this section.

First, we write the problem in matrix form. The system can be written as

$$\begin{aligned} x' &= y \\ y' &= -x + 2\cos t. \end{aligned} \tag{2.106}$$

Thus, we have a nonhomogeneous system of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}.$$

Next we need the fundamental matrix of solutions of the homogeneous problem. We have that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm i$. An eigenvector associated with $\lambda = i$ is easily found as $\begin{pmatrix} 1 \\ i \end{pmatrix}$. This leads to a complex solution

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}.$$

From this solution we can construct the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So, the general solution to the homogeneous problem is

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}.$$

Next we seek a particular solution to the nonhomogeneous problem. From Equation (2.103) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have

$$\begin{aligned}
\Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\
&= \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix}. \qquad (2.107)
\end{aligned}$$

We now compute

$$\begin{aligned}
\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_{t_0}^{t} \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix} ds \\
&= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin^2 t \\ t + \frac{1}{2}\sin(2t) \end{pmatrix} \\
&= \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}. \qquad (2.108)
\end{aligned}$$

therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix},$$

or

$$\mathbf{x} = \begin{pmatrix} 4\cos t + t\sin t \\ -3\sin t + t\cos t \end{pmatrix}.$$

## *Problems*

**1.** Consider the system

$$x' = -4x - y$$
$$y' = x - 2y.$$

a. Determine the second order differential equation satisfied by $x(t)$.

b. Solve the differential equation for $x(t)$.

c. Using this solution, find $y(t)$.

d. Verify your solutions for $x(t)$ and $y(t)$.

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

**2.** Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned} x' &= 3x \\ y' &= -2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= 2y \\ y' &= -3x. \end{aligned}$$

d.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

e.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

**3.** Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

4. Consider the system of equations in Example 2.13.

   a. Derive the polar form of the system.

   b. Solve the radial equation, $r' = r(1 - r^2)$, for the initial values $r(0) = 0, 0.5, 1.0, 2.0$.

   c. Based upon these solutions, plot and describe the behavior of all solutions to the original system in Cartesian coordinates.

5. Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 2, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

   a.
$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

   b.
$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

   c.
$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

   d.
$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

   e.
$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

   f.
$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

**6.** For the given matrix, evaluate $e^{tA}$, using the definition

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \ldots,$$

and simplifying.

a. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$

b. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}.$

c. $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$

d. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

e. $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$

f. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$

**7.** Find the fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$ where matrix $A$ is given. If an initial condition is provided, find the solution of the initial value problem using the principal matrix.

a. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}.$

b. $A = \begin{pmatrix} 12 & -15 \\ 4 & -4 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}.$

d. $A = \begin{pmatrix} 4 & -13 \\ 2 & -6 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

e. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}.$

f. $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}.$

g. $A = \begin{pmatrix} 8 & -5 \\ 16 & 8 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

h. $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}.$

i. $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

**8.** Solve the following initial value problems using Equation (2.105), the solution of a nonhomogeneous system using the principal matrix solution.

a. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. $\mathbf{x}' = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

**9.** Add a third spring connected to mass two in the coupled system shown in Figure 2.2 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.

b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

d. Model this initial value problem with a set of two second order differential equations. Set up the system in the form $M\ddot{\mathbf{x}} = -K\mathbf{x}$ and solve using the values in part b.

**10.** In Example 2.14 we investigated a couple mass-spring system as a pair of second order differential equations.

a. In that problem we used $\sqrt{\frac{3\pm\sqrt{5}}{2}} = \frac{\sqrt{5}\pm 1}{2}$. Prove this result.

b. Rewrite the system as a system of four first order equations.

c. Find the eigenvalues and eigenfunctions for the system of equations in part b to arrive at the solution found in Example 2.14.

d. Let $k = 5.00$ N/m and $m = 0.250$ kg. Assume that the masses are initially at rest and plot the positions as a function of time if initially i) $x_1(0) = x_2(0) = 10.0$ cm and i) $x_1(0) = -x_2(0) = 10.0$ cm. Describe the resulting motion.

**11.** Consider the series circuit in Figure **??** with $L = 1.00$ H, $R = 1.00 \times 10^2$ $\Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when $V_0$ is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

**12.** Consider the series circuit in Figure 2.21 with $L = 1.00$ H, $R_1 = R_2 = 1.00 \times 10^2$ $\Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of first order differential equations for the charges and the currents in each loop.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when $V_0$ is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

**13.** Initially a 100 gallon tank is filled with pure water. At time $t = 0$ water with a half a pound of salt per two gallons is added to the container at the rate of 3 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

**14.** You make two quarts of salsa for a party. The recipe calls for five teaspoons of lime juice per quart, but you had accidentally put in five tablespoons per quart. You decide to feed your guests the salsa anyway. Assume that the guests take a quarter cup of salsa per minute and that you replace what was taken with chopped tomatoes and onions without any lime juice. [1 quart = 4 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of lime juice as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the salsa back to the recipe's suggested concentration?

**15.** Consider the chemical reaction leading to the system in (2.54). Let the rate constants be $k_1 = 0.20$ ms$^{-1}$, $k_2 = 0.05$ ms$^{-1}$, and $k_3 = 0.10$ ms$^{-1}$. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ $\mu$mol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

**16.** Find and classify any equilibrium points in the Romeo and Juliet problem for the following cases. Solve the systems and describe their affections as a function of time.

a. $a = 0$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

b. $a = 0$, $b = 2$, $c = 1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

c. $a = -1$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

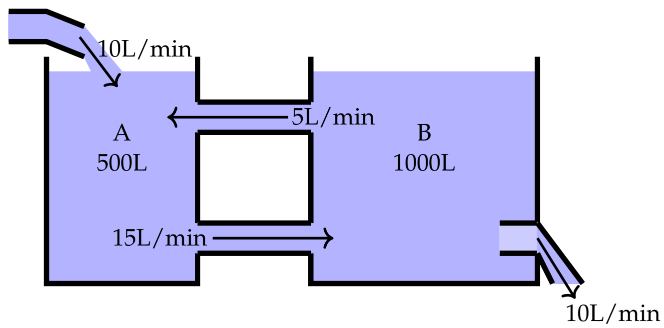



Figure 2.26: Figure for Problem 17.

**17.** Two tanks contain a mixture of water and alcohol with tank A containing 500 L and tank B 1000L. Initially, the concentration of alcohol in Tank A is 0% and that of tank B is 80%. Solution leaves tank A into B at a rate of 15 liter/min and the solution in tank B returns to A at a rate of 5 L/min while well mixed solution also leaves the system at 10 liter/min through an outlet. A mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 10% through an inlet. What will be the concentration of the alcohol of the solution in each tank after 10 mins?

**18.** Consider the tank system in Problem 17. Add a third tank (C) to tank B with a volume of 300 L. Connect C with 8 L/min from tank B and 2 L/min flow back. Let 10 L/min flow out of the system. If the initial concentration is 10% in each tank and a mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 20% through an inlet, what will be the concentration of the alcohol in each of the tanks after an hour?

**19.** Consider the epidemic model leading to the system in (2.58). Choose the constants as $a = 2.0$ days$^{-1}$, $d = 3.0$ days$^{-1}$, and $r = 1.0$ days$^{-1}$. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of 1,000 and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?

# Chapter 3

# Nonlinear Systems

*"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful." - Jules Henri Poincaré (1854-1912)*

## 3.1 Introduction

Some of the most interesting phenomena in the world are modeled by nonlinear systems. These systems can be modeled by differential equations when time is considered as a continuous variable or difference equations when time is treated in discrete steps. Applications involving differential equations can be found in many physical systems such as planetary systems, weather prediction, electrical circuits, and kinetics. Even in some simple dynamical systems a combination of damping and a driving force can lead to chaotic behavior. Namely, small changes in initial conditions could lead to very different outcomes. In this chapter we will explore a few different nonlinear systems and introduce some of the tools needed to investigate them. These tools are based on some of the material in Chapters 2 and 3 for linear systems of differential equations.

Nonlinear differential equations are either integrable, but difficult to solve, or they are not integrable and can only be solved numerically. We will see that we can sometimes approximate the solutions of nonlinear systems with linear systems in small regions of phase space and determine the qualitative behavior of the system without knowledge of the exact solution.

Nonlinear problems occur naturally. We will see problems from many of the same fields we explored in Section 2.3. We will concentrate mainly on continuous dynamical systems. We will begin with a simple population model and look at the behavior of equilibrium solutions of first order autonomous differential equations. We will then look at nonlinear systems in the plane, such as the nonlinear pendulum and other nonlinear oscillations. We will conclude by discussing a few other interesting physical examples stressing some of the key ideas of nonlinear dynamics.

## 3.2 *The Logistic Equation*

In this section we will explore a simple nonlinear population model. Typically, we want to model the growth of a given population, $y(t)$, and the differential equation governing the growth behavior of this population is developed in a manner similar to that used previously for mixing problems. Namely, we note that the rate of change of the population is given by an equation of the form

$$\frac{dy}{dt} = \text{Rate In } - \text{ Rate Out}.$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. While there are other potential contributions to these rates we will consider the birth and death rates in the simplest examples.

A simple population model can be obtained if one assumes that these rates are linear in the population. Thus, we assume that the

$$\text{Rate In } = by \text{ and the } \text{ Rate Out } = my.$$

Here we have denoted the birth rate as $b$ and the mortality rate as $m$. This gives the rate of change of population as

$$\frac{dy}{dt} = by - my. \tag{3.1}$$

Generally, these rates could depend on the time. In the case that they are both constant rates, we can define $k = b - m$ and obtain the familiar exponential model of population growth:

Malthusian population growth.

$$\frac{dy}{dt} = ky.$$

This is easily solved and one obtains exponential growth $(k > 0)$ or decay $(k < 0)$. This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

When populations get large enough, there is competition for resources, such as space and food, which can lead to a higher mortality rate. Thus, the mortality rate may be a function of the population size, $m = m(y)$. The simplest model would be a linear dependence, $m = \tilde{m} + cy$. Then, the previous exponential model takes the form

$$\frac{dy}{dt} = ky - cy^2, \tag{3.2}$$

where $k = b - \tilde{m}$. This is known as the *logistic model* of population growth. Typically, $c$ is small and the added nonlinear term does not really kick in until the population gets large enough.

The logistic model was first published in 1838 by Pierre François Verhulst (1804-1849) in the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where $N$ is the population at time $t$, $r$ is the growth rate, and $K$ is what is called the carrying capacity. Note that in our model $r = k = Kc$.

**Example 3.1.** Show that Equation (3.2) can be written in the form

$$z' = kz(1-z)$$

which has only one parameter.

We carry this out be rescaling the population, $y(t) = \alpha z(t)$, where $\alpha$ is to be determined. Inserting this transformation, we have

$$\begin{aligned} y' &= ky - cy^2 \\ \alpha z' &= \alpha kz - c\alpha^2 z^2, \end{aligned}$$

or

$$z' = kz\left(1 - \alpha\frac{c}{k}z\right).$$

Thus, we obtain the result, $z' = kz(1-z)$, if we pick $\alpha = \frac{k}{c}$.

Before we obtain the exact solution, it is instructive to study the qualitative behavior of the solutions without actually writing down any explicit solutions. Such methods are useful for more difficult nonlinear equations as we will see later in this chapter.

We will demonstrate this analysis with a simple logistic equation example. We will first look for constant solutions, called equilibrium solutions, satisfying $y'(t) = 0$. Then, we will look at the behavior of solutions near the equilibrium solutions, or fixed points, and determine the stability of the equilibrium solutions. In the next section we will extend these ideas to other first order differential equations.

**Example 3.2.** Find and classify the equilibrium solutions of the logistic equation,

$$\frac{dy}{dt} = y - y^2. \tag{3.3}$$

First, we determine the equilibrium, or constant, solutions given by $y' = 0$. For this case, we have $y - y^2 = 0$. So, the equilibrium solutions are $y = 0$ and $y = 1$.

These solutions divide the $ty$-plane into three regions, $y < 0$, $0 < y < 1$, and $y > 1$. Solutions that originate in one of these regions at $t = t_0$ will remain in that region for all $t > t_0$ since solutions of this differential equation cannot intersect.

Note: If two solutions of the differential equation intersect then they have common values $y_1$ at time $t_1$. Using this information, we could set up an initial value problem for which the initial condition is $y(t_1) = y_1$. Since the two different solutions intersect at this point in the phase plane, we would have an initial value problem with two different solutions. This would violate the uniqueness theorem for initial value problems.

Next, we determine the behavior of solutions in the three regions. Noting that $y'(t)$ gives the slope of any solution in the plane, then we find that the solutions are monotonic in each region. Namely, in regions where $y'(t) > 0$, we have monotonically increasing functions and in regions where $y'(t) < 0$, we have monotonically decreasing functions. We determine the sign of $y'(t)$ from the right-hand side of the differential equation.

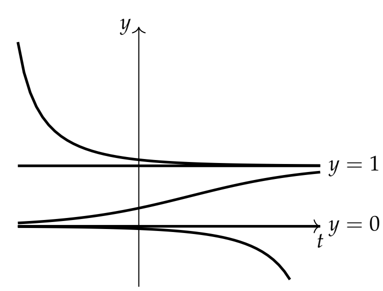


Figure 3.1: Representative solution behavior for $y' = y - y^2$.

For example, in this problem $y - y^2 > 0$ only for the middle region and $y - y^2 < 0$ for the other two regions. Thus, the slope is positive in the middle region, giving a rising solution as shown in Figure 3.1. Note that this solution does not cross the equilibrium solutions. Similar statements can be made about the solutions in the other regions.

Stable and unstable equilibria.

We further note that the solutions on either side of the equilibrium solution $y = 1$ tend to approach this equilibrium solution for large values of $t$. In fact, no matter how far these solutions are from $y = 1$, as long as $y(t) > 0$, the solutions will eventually approach this equilibrium solution as $t \to \infty$. We then say that the equilibrium solution, $y = 1$, is a *stable equilibrium*.

Similarly, we note that the solutions on either side of the equilibrium solution $y = 0$ tend away from $y = 0$ for large values of $t$. No matter how close a solution is to $y = 0$ at some given time, eventually these solutions will diverge as $t \to \infty$. We say that such equilibrium solutions are *unstable equilibria*.

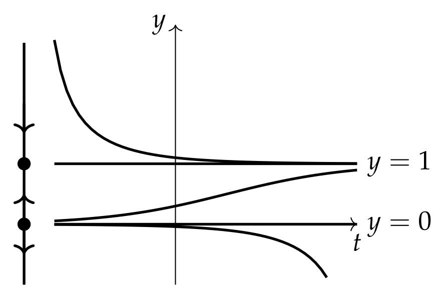



Figure 3.2: Representative solution behavior and the phase line for $y' = y - y^2$.

Phase lines.

If we are only interested in the behavior of the equilibrium solutions, we could just display a *phase line*. In Figure 3.2 we place a vertical line to the right of the $ty$-plane plot. On this line we first place dots at the corresponding equilibrium solutions and label the solutions. These points divide the phase line into three intervals.

In each interval we then place arrows pointing upward or downward indicating solutions with positive or negative slopes, respectively. For example, for the interval $y > 1$ there is a downward pointing arrow indicating that the slope is negative in that region.

Looking at the resulting phase line we can determine if a given equilibrium is stable (arrows pointing towards the point) or unstable (arrows pointing away from the point). In Figure 3.3 we draw the final phase line by itself. We see that $y = 1$ is a stable equilibrium point and $y = 0$ is an unstable equilibrium point.

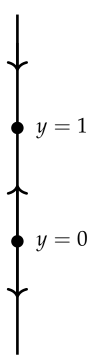



Figure 3.3: Phase line for $y' = y - y^2$.

### 3.2.1 The Riccati Equation*

We have seen that one does not need an explicit solution of the logistic equation (3.2) in order to study the behavior of its solutions. However, the logistic equation is an example of a nonlinear first order equation that is solvable. It is also an example of a general Riccati equation, a first order differential equation quadratic in the unknown function.

The Riccati equation is named after the Italian mathematician Jacopo Francesco Riccati (1676-1754). When $a(t) = 0$, the equation becomes a Bernoulli equation.

The general form of the *Riccati equation* is

$$\frac{dy}{dt} = a(t) + b(t)y + c(t)y^2. \tag{3.4}$$

As long as $c(t) \neq 0$, this equation can be reduced to a second order linear differential equation through the transformation

$$y(t) = -\frac{1}{c(t)}\frac{x'(t)}{x(t)}.$$

We will demonstrate the use of this transformation in obtaining the solution of the logistic equation.

**Example 3.3.** Solve the logistic equation

$$\frac{dy}{dt} = ky - cy^2 \tag{3.5}$$

using the transformation

$$y = \frac{1}{c}\frac{x'}{x}.$$

differentiating this transformation with respect to $t$, we obtain

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{c}\left[\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right] \\ &= \frac{1}{c}\left[\frac{x''}{x} - (cy)^2\right] \\ &= \frac{1}{c}\frac{x''}{x} - cy^2. \end{aligned} \tag{3.6}$$

Inserting this result into the logistic equation (3.5), we have

$$\frac{1}{c}\frac{x''}{x} - cy^2 = k\frac{1}{c}\left(\frac{x'}{x}\right) - cy^2.$$

Simplifying, we see that the logistic equation has been reduced to a second order linear, differential equation,

$$x'' = kx'.$$

This equation is readily solved. One integration gives

$$x'(t) = Be^{kt}.$$

A second integration gives

$$x(t) = A + Be^{kt},$$

where $A$ and $B$ are two arbitrary constants.

Inserting this result into the Riccati transformation, we obtain

$$y(t) = \frac{1}{c}\frac{x'}{x} = \frac{kBe^{kt}}{c(A + Be^{kt})}.$$

It appears that we have two arbitrary constants. However, we started out with a first order differential equation and so we expect only one arbitrary constant. We can resolve this dilemma by dividing[1] the numerator and denominator by $Be^{kt}$ and defining $C = \frac{A}{B}$. Then, we have the solution

$$y(t) = \frac{k/c}{1 + Ce^{-kt}}, \tag{3.7}$$

showing that there really is only one arbitrary constant in the solution.

[1] This general solution holds for $B \neq 0$. If $B = 0$, then we have $x(t) = A$ and, thus, $y(t)$ is the constant equilibrium solution.

Plots of the solution (3.7) of the logistic equation for different initial conditions gives the solutions seen in the last section. In particular, setting all of the constants to unity, we have the sigmoid function,

$$y(t) = \frac{1}{1+e^{-t}}.$$

This is the signature $S$-shaped curve of the logistic model as shown in Figure 3.4. We should note that this is not the only way to obtain the solution to the logistic equation, though this approach has provided us with an introduction to Riccati equations. A more direct approach would be to use separation of variables on the logistic equation, which is Problem 1.

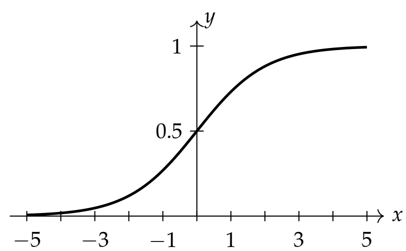


Figure 3.4: Plot of the sigmoid function.

## 3.3 *Autonomous First Order Equations*

In this section we will study the stability of nonlinear first order autonomous equations. We will then extend this study in the next section to looking at families of first order equations which are connected through a parameter.

Recall that a first order autonomous equation is given in the form

$$\frac{dy}{dt} = f(y). \tag{3.8}$$

We will assume that $f$ and $\frac{\partial f}{\partial y}$ are continuous functions of $y$, so that we know that solutions of initial value problems exist and are unique.

A solution $y(t)$ of Equation (3.8) is called an *equilibrium solution*, or a *fixed point* solution, if it is a constant solution satisfying $y'(t) = 0$. Such solutions are the roots of the right-hand side of the differential equation, $f(y) = 0$.

**Example 3.4.** Find the equilibrium solutions of $y' = 1 - y^2$.

The equilibrium solutions are the roots of $f(y) = 1 - y^2 = 0$. The equilibria are found to be $y = \pm 1$.

Once we have determined the equilibrium solutions, we would like to classify them. Are they stable or unstable? As we had seen previously, we are interested in the behavior of solutions near the equilibria. This classification can be determined using a linearization of the given equation. This will provide an analytic criteria to establish the stability of equilibrium solutions without geometrically drawing the phase lines as we had done previously.

Linearization of first order equations.

Let $y^*$ be an equilibrium solution of Equation (3.8). Then, any solution can be written in the form

$$y(t) = y^* + \xi(t),$$

where $\xi(t)$ measures how far the solution is from the equilibrium at any given time.

Inserting this form into Equation (3.8), we have

$$\frac{d\xi}{dt} = f(y^* + \xi).$$

We now consider small $\xi(t)$ in order to study solutions near the equilibrium solution. For such solutions, we can expand $f(y)$ about the equilibrium solution,

$$f(y^* + \xi) = f(y^*) + f'(y^*)\xi + \frac{1}{2!}f''(y^*)\xi^2 + \cdots .$$

Since $y^*$ is an equilibrium solution, $f(y^*) = 0$, the first term in the Taylor series vanishes. If the first derivative does not vanish, then for solutions close to equilibrium, we can neglect higher order terms in the expansion. Then, $\xi(t)$ approximately satisfies the differential equation

$$\frac{d\xi}{dt} = f'(y^*)\xi. \tag{3.9}$$

This is called a linearization of the original nonlinear equation about the equilibrium point. This equation has exponential solutions for $f'(y^*) \neq 0$,

$$\xi(t) = \xi_0 e^{f'(y^*)t}.$$

Now we see how the stability criteria arise. If $f'(y^*) > 0$, $\xi(t)$ grows in time. Therefore, nearby solutions stray from the equilibrium solution for large times. On the other hand, if $f'(y^*) < 0$, $\xi(t)$ decays in time and nearby solutions approach the equilibrium solution for large $t$. Thus, we have the results:

The stability criteria for equilibrium solutions of a first order differential equation.

$$\begin{array}{ll} f'(y^*) < 0, & y^* \text{ is stable.} \\ f'(y^*) > 0, & y^* \text{ is unstable.} \end{array} \tag{3.10}$$

**Example 3.5.** Determine the stability of the equilibrium solutions of $y' = 1 - y^2$.

In the last example we found the equilibrium solutions, $y^* = \pm 1$. The stability criteria require computing

$$f'(y^*) = -2y^*.$$

For this problem we have $f'(\pm 1) = \mp 2$. Therefore, $y^* = 1$ is a stable equilibrium and $y^* = -1$ is an unstable equilibrium.

**Example 3.6.** Find and classify the equilibria for the logistic equation $y' = y - y^2$.

We had already investigated this problem using phase lines. There are two equilibria, $y = 0$ and $y = 1$.

We next apply the stability criteria. Noting that $f'(y) = 1 - 2y$, the first equilibrium solution gives $f'(0) = 1$. So, $y = 0$ is an unstable equilibrium. Since $f'(1) = -1 < 0$, we see that $y = 1$ is a stable equilibrium. These results are the same as we hade determined earlier using phase lines.

## 3.4 *Bifurcations for First Order Equations*

We now consider families of first order autonomous differential equations of the form

$$\frac{dy}{dt} = f(y; \mu).$$

Here $\mu$ is a parameter that we can change and then observe the resulting behaviors of the solutions of the differential equation. When a small change in the parameter leads to changes in the behavior of the solution, then the system is said to undergo a *bifurcation*. The value of the parameter, $\mu$, at which the bifurcation occurs is called a *bifurcation point*.

Bifurcations and bifurcation points.

We will consider several generic examples, leading to special classes of bifurcations of first order autonomous differential equations. We will study the stability of equilibrium solutions using both phase lines and the stability criteria developed in the last section

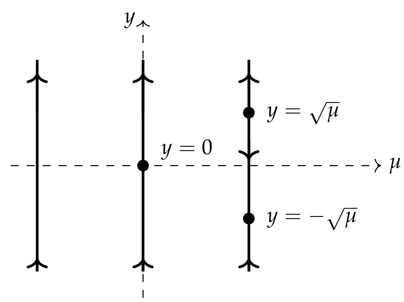



Figure 3.5: Phase lines for $y' = y^2 - \mu$. On the right $\mu > 0$ and on the left $\mu < 0$.

**Example 3.7.** $y' = y^2 - \mu$.

First note that equilibrium solutions occur for $y^2 = \mu$. In this problem, there are three cases to consider.

1. $\mu > 0$.

   In this case there are two real solutions of $y^2 = \mu$, $y = \pm\sqrt{\mu}$. Note that $y^2 - \mu < 0$ for $|y| < \sqrt{\mu}$. So, we have the right phase line in Figure 3.5.

2. $\mu = 0$.

   There is only one equilibrium point at $y = 0$. The equation becomes $y' = y^2$. It is obvious that the right side of this equation is never negative. So, the phase line, which is shown as the middle line in Figure 3.5, has upward pointing arrows.

3. $\mu < 0$.

   In this case there are no equilibrium solutions. Since $y^2 - \mu > 0$, the slopes for all solutions are positive as indicated by the last phase line in Figure 3.5.

We can also confirm the behaviors of the equilibrium points by noting that $f'(y) = 2y$. Then, $f'(\pm\sqrt{\mu}) = \pm 2\sqrt{\mu}$ for $\mu \geq 0$. Therefore, the equilibria $y = +\sqrt{\mu}$ are unstable equilibria for $\mu > 0$. Similarly, the equilibria $y = -\sqrt{\mu}$ are stable equilibria for $\mu > 0$.

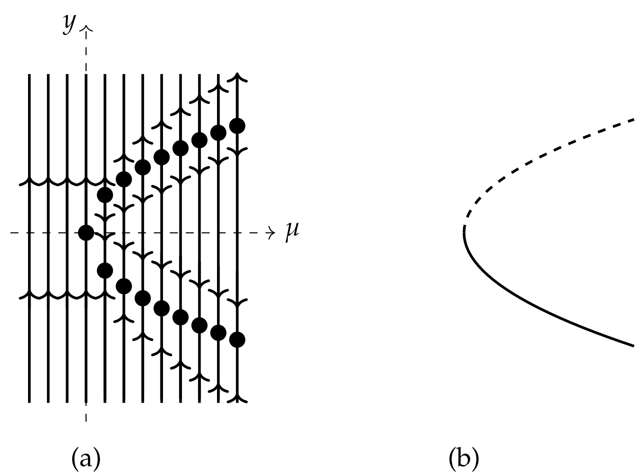



Figure 3.6: (a) The typical phase lines for $y' = y^2 - \mu$. (b) Bifurcation diagram for $y' = y^2 - \mu$. This is an example of a saddle-node bifurcation.

We can combine these results for the phase lines into one diagram known as a bifurcation diagram. We will plot the equilibrium solutions and their phase lines $y = \pm\sqrt{\mu}$ in the $\mu y$-plane. We begin by lining up the phase lines for various $\mu$'s. These are shown on the left side of Figure 3.6. Note the pattern of equilibrium points lies on the parabolic curve $y^2 = \mu$. The upper branch of this curve is a collection of unstable equilibria and the bottom is a stable branch. So, we can dispose of the phase lines and just keep the equilibria. However, we will draw the unstable branch as a dashed line and the stable branch as a solid line.

The bifurcation diagram is displayed on the right side of Figure 3.6. This type of bifurcation is called a *saddle-node bifurcation*. The point $\mu = 0$ at which the behavior changes is the *bifurcation point. As $\mu$ changes from negative to positive values, the system goes from having no equilibria to having one stable and one unstable equilibrium point.*

**Example 3.8.** $y' = y^2 - \mu y$.

Writing this equation in factored form, $y' = y(y - \mu)$, we see that there are two equilibrium points, $y = 0$ and $y = \mu$. The behavior of the solutions depends upon the sign of $y' = y(y - \mu)$. This leads to four cases with the indicated signs of the derivative. The regions indicating the signs of $y'$ are shown in Figure 3.7.

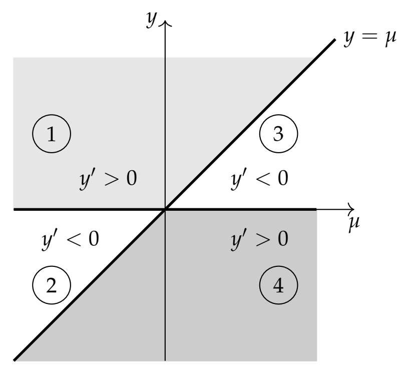


Figure 3.7: The regions indicating the different signs of the derivative for $y' = y^2 - \mu y$.

1. $y > 0, y - \mu > 0 \Rightarrow y' > 0$.
2. $y < 0, y - \mu > 0 \Rightarrow y' < 0$.
3. $y > 0, y - \mu < 0 \Rightarrow y' < 0$.
4. $y < 0, y - \mu < 0 \Rightarrow y' > 0$.

The corresponding phase lines and superimposed bifurcation diagram are shown in figure 3.8. The bifurcation diagram is on the right side of Figure 3.8 and this type of bifurcation is called a *transcritical bifurcation*.

Again, the stability can be determined from the derivative $f'(y) = 2y - \mu$ evaluated at $y = 0, \mu$. From $f'(0) = -\mu$, we see that $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. Similarly, $f'(\mu) = \mu$ implies that $y = \mu$ is unstable for $\mu > 0$ and stable for $\mu < 0$. These results are consistent with the phase line plots.

**Example 3.9.** $y' = y^3 - \mu y$.

For this last example, we find from $y^3 - \mu y = y(y^2 - \mu) = 0$ that there are two cases.

1. $\mu < 0$. In this case there is only one equilibrium point at $y = 0$. For positive values of $y$ we have that $y' > 0$ and for negative values of $y$ we have that $y' < 0$. Therefore, this is an unstable equilibrium point.
2. $\mu > 0$. Here we have three equilibria, $y = 0, \pm\sqrt{\mu}$. A careful investigation shows that $y = 0$ is a stable equilibrium point and that the other two equilibria are unstable.

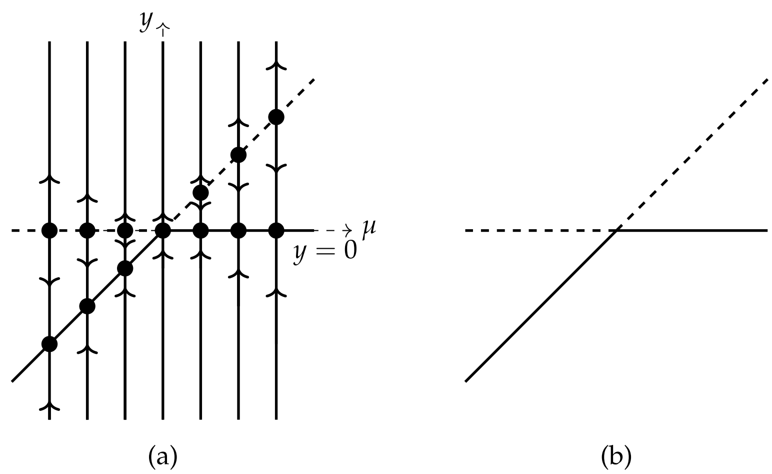



Figure 3.8: (a) Collection of phase lines for $y' = y^2 - \mu y$. (b) Bifurcation diagram for $y' = y^2 - \mu y$. This is an example of a transcritical bifurcation.

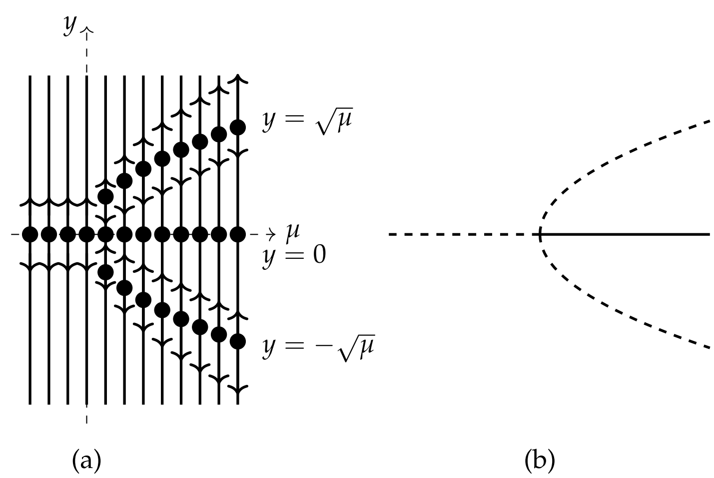



Figure 3.9: (a) The phase lines for $y' = y^3 - \mu y$. The left one corresponds to $\mu < 0$ and the right phase line is for $\mu > 0$. (b)Bifurcation diagram for $y' = y^3 - \mu y$. This is an example of a pitchfork bifurcation.

In Figure 3.9 we show the phase lines for these two cases. The corresponding bifurcation diagram is then sketched on the right side of Figure 3.9. For obvious reasons this has been labeled a *pitchfork bifurcation*.

When two of the prongs of the pitchfork are unstable branches, the bifurcation is called a subcritical pitchfork bifurcation. When two prongs are stable branches, the bifurcation is a supercritical pitchfork bifurcation.

Since $f'(y) = 3y^2 - \mu$, the stability analysis gives that $f'(0) = -\mu$. So, $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. For $\mu > 0$, we have that $f'(\pm\sqrt{\mu}) = 2\mu$. Therefore, $y = \pm\sqrt{\mu}$, $\mu > 0$, is unstable. Thus, we have a subcritical pitchfork bifurcation.

## 3.5 *The Stability of Fixed Points in Nonlinear Systems*

We next investigate the stability of the equilibrium solutions of the nonlinear pendulum which we first encountered in Section **??**. Along the way we will develop some basic methods for studying the stability of equilibria in nonlinear systems in general.

Recall that the derivation of the pendulum equation was based upon a simple point mass $m$ hanging on a string of length $L$ from some support as shown in Figure 3.10. One pulls the mass back to some starting angle, $\theta_0$, and releases it. The goal is to find the angular position as a function of time, $\theta(t)$.

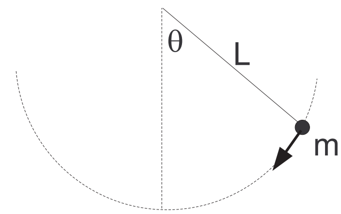


Figure 3.10: A simple pendulum consists of a point mass $m$ attached to a string of length $L$. It is released from an angle $\theta_0$.

In Chapter 2 we derived the nonlinear pendulum equation,

$$L\ddot{\theta} + g\sin\theta = 0. \tag{3.11}$$

There are several variations of Equation (3.11) which we have used in this text. The first one is the linear pendulum, which was obtained using a small angle approximation,

$$L\ddot{\theta} + g\theta = 0. \tag{3.12}$$

We also made the system more realistic by adding damping and forcing. A variety of these oscillation problems are summarized in the table below.

**Equations for Pendulum Motion**

1. Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0.$
2. Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0.$
3. Linear Pendulum: $L\ddot{\theta} + g\theta = 0.$
4. Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0.$
5. Forced Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t.$
6. Forced Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t.$

There are two simple systems that we will consider, the damped linear pendulum, in the form

$$x'' + bx' + \omega^2 x = 0$$

and the the damped nonlinear pendulum, in the form

$$x'' + bx' + \omega^2 \sin x = 0.$$

These are second order differential equations and can be cast as a system of two first order differential equations using the methods of Chapter 6.

The linear equation can be written as

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 x. \end{aligned} \tag{3.13}$$

This system has only one equilibrium solution, $x = 0$, $y = 0$.

The damped nonlinear pendulum takes the form

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.14}$$

This system also has the equilibrium solution $x = 0$, $y = 0$. However, there are actually an infinite number of solutions. The equilibria are determined from

$$y = 0 \text{ and } -by - \omega^2 \sin x = 0. \tag{3.15}$$

These equations imply that $y = 0$ and $\sin x = 0$. There are an infinite number of solutions to the latter equation: $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So, this system has an infinite number of equilibria, $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

The next step is to determine the stability of the equilibrium solutions these systems. This can be accomplished just as we had done for first order equations. To do this we need a more general theory for nonlinear systems. So, we will develop the needed machinery.

We begin with the $n-$dimensional system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n. \tag{3.16}$$

Linear stability analysis of systems.

Here $\mathbf{f} : \mathrm{R}^n \to \mathrm{R}^n$ is a mapping from $\mathrm{R}^n$ to $\mathrm{R}^n$. We define the equilibrium solutions, or fixed points, of this system as the points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

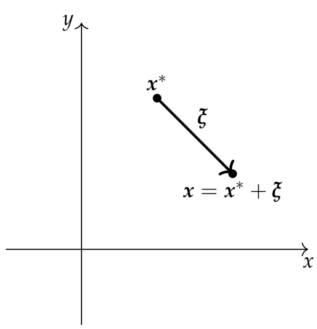



Figure 3.11: A general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$,

The stability in the neighborhood of equilibria will now be determined. We are interested in what happens to solutions of the system with initial conditions starting near a fixed point. We will represent a general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$. We note that the length of $\boldsymbol{\xi}$ gives an indication of how close we are to the fixed point. So, we consider that initially, $|\boldsymbol{\xi}| \ll 1$.

As the system evolves, $\boldsymbol{\xi}$ will change. The change of $\boldsymbol{\xi}$ in time is in turn governed by a system of equations. We can approximate this evolution as follows. First, we note that

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

We can expand the right side about the fixed point using a multidimensional version of Taylor's Theorem. Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

The Jacobian matrix.

Here $D\mathbf{f}(\mathbf{x})$ is the *Jacobian matrix*, defined as

$$D\mathbf{f}(\mathbf{x}^*) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Noting that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we then have that system (3.16) becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi}. \tag{3.17}$$

Linearization of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

It is this equation which describes the behavior of the system near the fixed point. As with first order equations, we say that system (3.16) has been linearized or that Equation (3.17) is the linearization of system (3.16).

The stability of the equilibrium point of the nonlinear system is now reduced to analyzing the behavior of the linearized system given by Equation (3.17). We can use the methods from the last two chapters to investigate the eigenvalues of the Jacobian matrix evaluated at each equilibrium point. We will demonstrate this procedure with several examples.

**Example 3.10.** Determine the equilibrium points and their stability for the system

$$\begin{aligned} x' &= -2x - 3xy, \\ y' &= 3y - y^2. \end{aligned} \tag{3.18}$$

We first determine the fixed points. Setting the right-hand side equal to zero and factoring, we have

$$\begin{aligned} -x(2+3y) &= 0, \\ y(3-y) &= 0. \end{aligned} \tag{3.19}$$

From the second equation, we see that either $y = 0$ or $y = 3$. The first equation then gives $x = 0$ in either case. So, there are two fixed points: $(0,0)$ and $(0,3)$.

Next, we linearize the system of differential equations about each fixed point. First, we note that the Jacobian matrix is given by

$$D\mathbf{f}(x,y) = \begin{pmatrix} -2-3y & -3x \\ 0 & 3-2y \end{pmatrix}. \tag{3.20}$$

1. Case I Equilibrium point $(0,0)$.

   In this case we find that

   $$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3.21}$$

   Therefore, the linearized equation becomes

   $$\xi' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{\xi}. \tag{3.22}$$

   This is equivalently written out as the system

   $$\begin{aligned} \xi_1' &= -2\xi_1, \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{3.23}$$

   This is the linearized system about the origin. Note the similarity with the original system.

   We should emphasize that the linearized equations are constant coefficient equations and we can use matrix methods to determine the nature of the equilibrium point. The eigenvalues of this system are obviously $\lambda = -2, 3$. Therefore, we have that the origin is a saddle point.

2. Case II Equilibrium point $(0,3)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}(0,3) = \begin{pmatrix} -11 & 0 \\ 0 & -3 \end{pmatrix}. \tag{3.24}$$

   The eigenvalues are $\lambda = -11, -3$. So, this fixed point is a stable node.

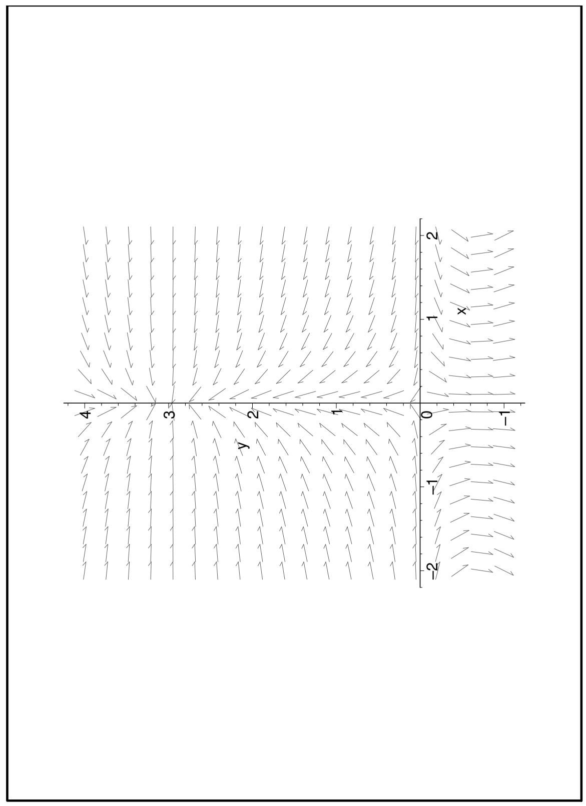



Figure 3.12: Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

This analysis has given us a saddle and a stable node. We know what the behavior is like near each fixed point, but we have to resort to other means to say anything about the behavior far from these points. The phase portrait for this system is given in Figure 3.12. You should be able to locate the saddle point and the node in the figure. Notice how solutions behave in regions far from these points.

We can expect to be able to perform a linearization under general conditions. These are given in the *Hartman-Großman Theorem*:

**Theorem 3.1.** *A continuous map exists between the linear and nonlinear systems when $D\mathbf{f}(\mathbf{x}^*)$ does not have any eigenvalues with zero real part.*

Generally, there are several types of behavior that one can see in nonlinear systems. One can see sinks or sources, hyperbolic (saddle) points, elliptic points (centers) or foci. We have defined some of these for planar systems. In general, if at least two eigenvalues have real parts with opposite signs, then the fixed point is a *hyperbolic point*. If the real part of a nonzero eigenvalue is zero, then we have a center, or *elliptic point.*

For linear systems in the plane, this classification was done in Chapter 6. The Jacobian matrix evaluated at the equilibrium points is simply the $2 \times 2$ coefficient matrix we had called $A$.

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.25}$$

Here we are using $J = D\mathbf{f}(\mathbf{x}^*)$.

The eigenvalue equation is given by

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

However, $a + d$ is the trace, $\mathrm{tr}(J)$ and $\det(J) = ad - bc$. Therefore, we can write the eigenvalue equation as

$$\lambda^2 - \mathrm{tr}(J)\lambda + \det(J) = 0.$$

The solution of this equation is found using the quadratic formula,

$$\lambda = \frac{1}{2}\left[-\mathrm{tr}(J) \pm \sqrt{\mathrm{tr}^2(J) - 4\det(J)}\right].$$

We had seen in previous chapter that equilibrium points in planar systems can be classified as nodes, saddles, centers, or spirals (foci). The type of behavior can be determined from solutions of the eigenvalue equation. Since the nature of the eigenvalues depends on the trace and determinant of the Jacobian matrix at the equilibrium point, we can relate the types of equilibria to points in the det-tr plane. This is shown in Figure 3.13, which is similar to Figure 2.25.

In Figure 3.13 the parabola $\mathrm{tr}^2(J) = 4\det(J)$ divides the det-tr plane. Points on this curve give a vanishing discriminant in the computation of the eigenvalues. In these cases one finds repeated roots, or eigenvalues. Along this curve one can find stable and unstable degenerate nodes. Also along this line are stable and unstable proper nodes, called star nodes. These arise from systems of the form $x' = ax$, $y' = ay$.

In the case that $\det(J) < 0$, we have that the discriminant

$$\Delta \equiv \mathrm{tr}^2(J) - 4\det(J)$$

is positive. Not only that, $\Delta > \mathrm{tr}^2(J)$. Thus, we obtain two real and distinct eigenvalues with opposite signs. These lead to saddle points.

In the case that $\det(J) > 0$, we can have either $\Delta > 0$ or $\Delta < 0$. The discriminant is negative for points inside the parabolic curve. It is in this region that one finds centers and spirals, corresponding to complex eigenvalues. When $\mathrm{tr}(J) > 0$, there are unstable spirals. There are stable spirals when $\mathrm{tr}(J) < 0$. For the case that $\mathrm{tr}(J) = 0$, the eigenvalues are pure imaginary, giving centers.

There are several other types of behavior depicted in the figure, but we will now turn to studying a few of examples.

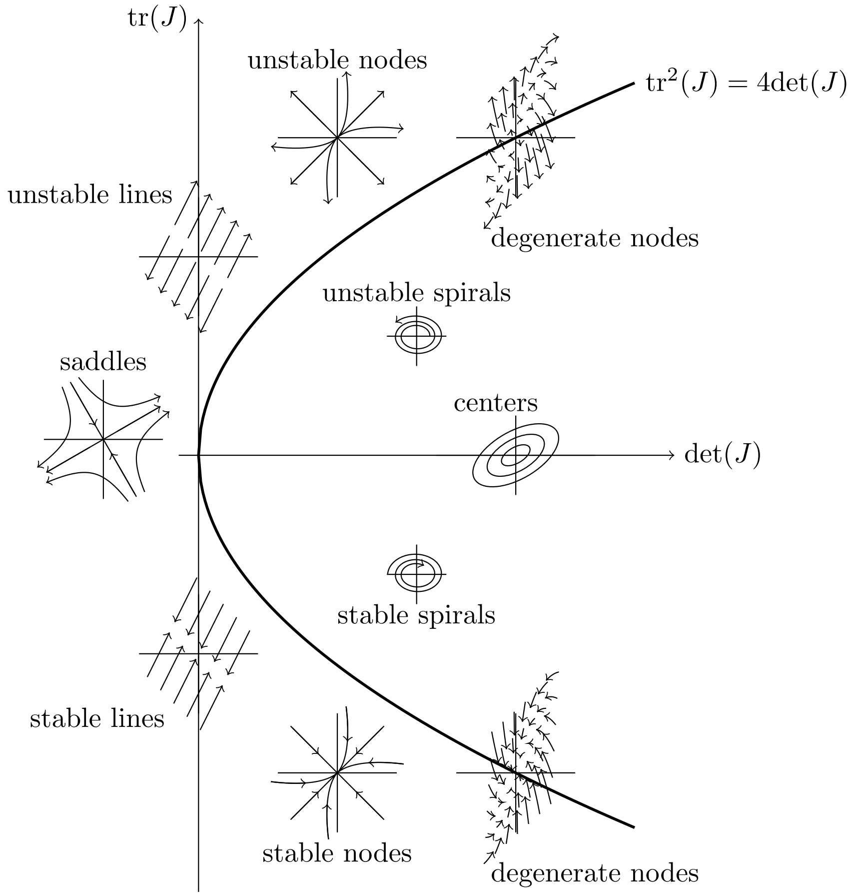



Figure 3.13: Diagram indicating the behavior of equilibrium points in the det − tr plane. The parabolic curve

$$\mathrm{tr}^2(J) = 4\det(J)$$

indicates where the discriminant vanishes.

**Example 3.11.** Find and classify all of the equilibrium solutions of the nonlinear system

$$\begin{aligned} x' &= 2x - y + 2xy + 3(x^2 - y^2), \\ y' &= x - 3y + xy - 3(x^2 - y^2). \end{aligned} \tag{3.26}$$

In Figure 3.14 we show the direction field for this system. Try to locate and classify the equilibrium points visually. After the stability analysis, you should return to this figure and determine if you identified the equilibrium points correctly.

We will first determine the equilibrium points. Setting the right-hand side of each differential equation to zero, we have

$$\begin{aligned} 2x - y + 2xy + 3(x^2 - y^2) &= 0, \\ x - 3y + xy - 3(x^2 - y^2) &= 0. \end{aligned} \tag{3.27}$$

This system of algebraic equations can be solved exactly. Adding the equations, we have

$$3x - 4y + 3xy = 0.$$

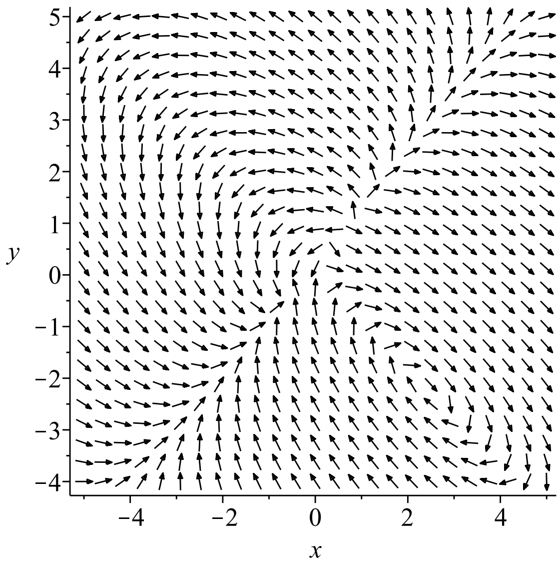



Figure 3.14: Phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$
$$y' = x - 3y + xy - 3(x^2 - y^2).$$

Solving for $x$,

$$x = \frac{4y}{3(1+y)},$$

and substituting the result for $x$ into the first algebraic equation, we find an equation for $y$ :

$$\frac{y(1-y)(9y^2 + 22y + 5)}{3(1+y)^2} = 0.$$

The solutions to this equation are

$$y = 0, 1, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}.$$

The corresponding values for $x$ are

$$x = 0, \frac{2}{3}, 1 \mp \frac{\sqrt{19}}{3}.$$

Now that we have located the equilibria, we can classify them. The Jacobian matrix is given by

$$D\mathbf{f}(x, y) = \begin{pmatrix} 6x + 2y + 2 & 2x - 6y - 1 \\ -6x + y + 1 & x + 6y - 3 \end{pmatrix}. \tag{3.28}$$

Now, we evaluate the Jacobian at each equilibrium point and find the eigenvalues.

1. Case I. Equilibrium point $(0, 0)$.
   In this case we find that

$$D\mathbf{f}(0, 0) = \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix}. \tag{3.29}$$

   The eigenvalues of this matrix are $\lambda = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$. Therefore, the origin is a saddle point.

2. Case II. Equilibrium point $(\frac{2}{3}, 1)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(\frac{2}{3}, 1\right) = \begin{pmatrix} 8 & -\frac{17}{3} \\ -2 & \frac{11}{3} \end{pmatrix}. \tag{3.30}$$

   The eigenvalues are $\lambda = \frac{35}{6} \pm \frac{\sqrt{577}}{6} \approx 9.84, 1.83$. This fixed point is an unstable node.

3. Case III. Equilibrium point $(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19})$.

   The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}\right) = \begin{pmatrix} \frac{50}{9} \mp \frac{14}{9}\sqrt{19} & \frac{25}{3} \mp 2\sqrt{19} \\ -\frac{56}{9} \pm \frac{20}{9}\sqrt{19} & -\frac{28}{3} \pm \sqrt{19} \end{pmatrix}. \tag{3.31}$$

   There are two equilibrium points under this case. The first is given by

$$(1 - \frac{\sqrt{19}}{3}, -\frac{11}{9} + \frac{2}{9}\sqrt{19}) \approx (0.453, -0.254).$$

   The eigenvalues for this point are

$$\lambda = -\frac{17}{9} - \frac{5}{18}\sqrt{19} \pm \frac{1}{18}\sqrt{3868\sqrt{19} - 16153}.$$

   These are approximately $-4.58$ and $-1.62$ So, this equilibrium point is a stable node.

   The other equilibrium is $(1 + \frac{\sqrt{19}}{3}, -\frac{11}{9} - \frac{2}{9}\sqrt{19}) \approx (2.45, -2.19)$. The corresponding eigenvalues are complex with negative real parts,

$$\lambda = -\frac{17}{9} + \frac{5}{18}\sqrt{19} \pm \frac{i}{18}\sqrt{16153 + 3868\sqrt{19}},$$

   or $\lambda \approx -0.678 \pm 10.1i$. This point is a stable spiral.

Plots of the phase plane are given in Figures 3.12 and 3.14. The reader can look at the direction field and verify these results for the behavior of equilibrium solutions. A zoomed in view is shown in Figure 3.15 with several orbits indicated.

**Example 3.12. Damped Nonlinear Pendulum Equilibria**

We are now ready to establish the behavior of the fixed points of the damped nonlinear pendulum system in Equation (3.14). Recall that the system for the damped nonlinear pendulum was given by

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.32}$$

For a damped system, we will need $b > 0$. We had found that there are an infinite number of equilibrium points at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

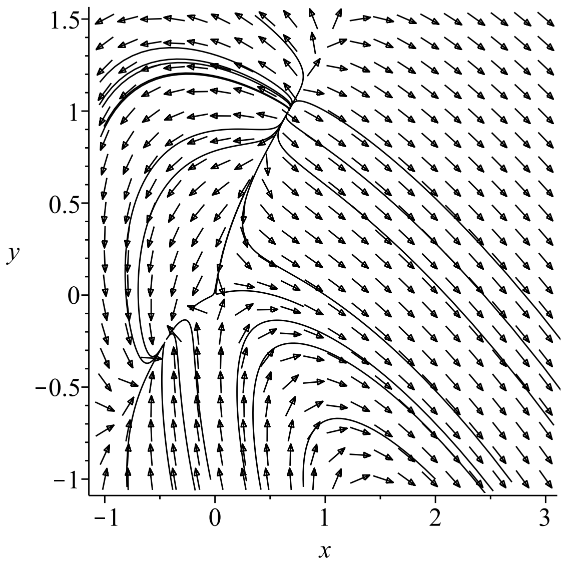



Figure 3.15: A closer look at the phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$
$$y' = x - 3y + xy - 3(x^2 - y^2)$$

with a few trajectories shown.

The Jacobian matrix for this systems is

$$D\mathbf{f}(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}. \tag{3.33}$$

Evaluating this matrix at the fixed points, we find that

$$D\mathbf{f}(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & -b \end{pmatrix}. \tag{3.34}$$

The eigenvalue equation is given by

$$\lambda^2 + b\lambda + (-1)^n \omega^2 = 0.$$

There are two cases to consider: $n$ even and $n$ odd. For the first case, we find the eigenvalues

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even $n$ values. If there is no damping, then we obtain centers ($\lambda = \pm i\omega$).

In the second case, $n$ odd, we find

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have hyperbolic points, or saddles. If there is no damping, the eigenvalues reduce to $\lambda = \pm\omega$.

In Figure (3.16) we show the phase plane for the undamped nonlinear pendulum with $\omega = 1.25$. We see that we have a mixture of centers

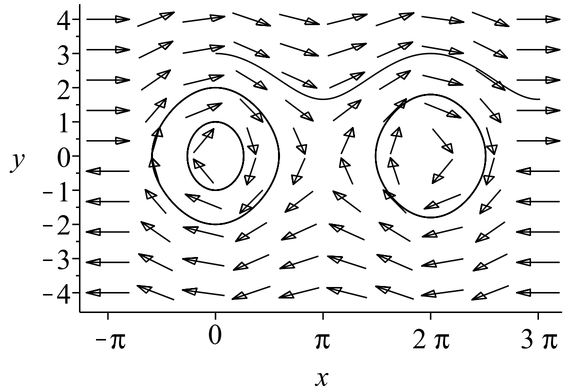



Figure 3.16: Phase plane for the undamped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

and saddles. There are orbits for which there is periodic motion. In the case that $\theta = \pi$ we have an inverted pendulum. This is an unstable position and this is reflected in the presence of saddle points, especially if the pendulum is constructed using a massless rod.

There are also unbounded orbits, going through all possible angles. These correspond to the mass spinning around the pivot in one direction forever due to initially having large enough energies.

We have indicated in the figure solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$. These show the various types of motions that we have described.

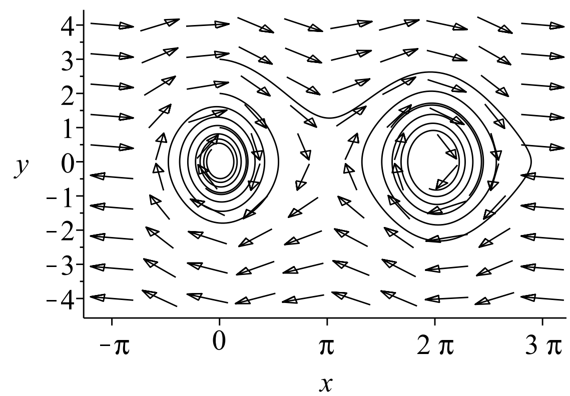



Figure 3.17: Phase plane for the damped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

When there is damping, we see that we can have a variety of other behaviors as seen in Figure (3.17). In this example we have set $b = 0.08$ and $\omega = 1.25$. We see that energy loss results in the mass settling around one of the stable fixed points. This leads to an understanding as to why there are an infinite number of equilibria, even though physically the mass traces out a bound set of Cartesian points. We have indicated in the Figure (3.17) solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

In Figure 3.18 we show a region of the phase plane which corresponds to oscillations about $x = 0$. For small angles the pendulum oscillates following somewhat elliptical orbits. As the angles get larger, due to greater initial

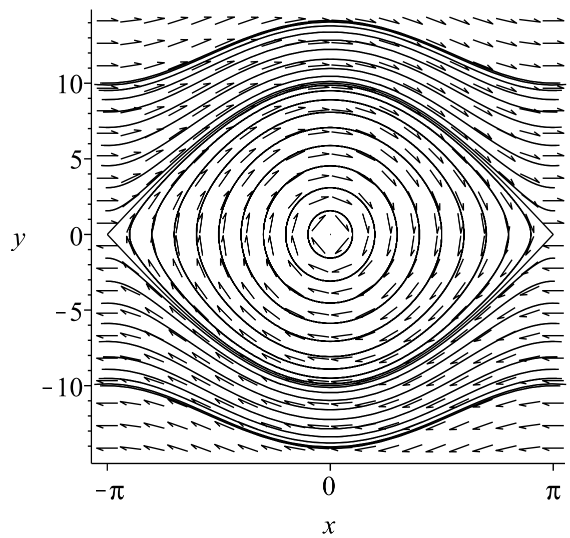



Figure 3.18: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$. The orbits surround a center at $(0,0)$. At the edges there are saddle points, $(\pm\pi, 0)$.

energies, these orbits begin to change from ellipses to other periodic orbits. There is a limiting orbit, beyond which one has unbounded motion. The limiting orbit connects the saddle points on either side of the center. The curve is called a separatrix and being that these trajectories connect two saddles, they are often referred to as heteroclinic orbits.

Heteroclinc orbits and separatrices.

In Figures 3.19-3.19 we show more orbits, including both bound and unbound motion beyond the interval $x \in [-\pi, \pi]$. For both plots we have chosen $\omega = 5$ and the same set of initial conditions, $x(0) = \pi k/10$, $k = -20, \ldots, 20$. for $y(0) = 0, \pm 10$. The time interval is taken for $t \in [-3, 3]$. The only difference is that in the damped case we have $b = 0.5$. In these plots one can see what happens to the heteroclinic orbits and nearby unbounded orbits under damping.

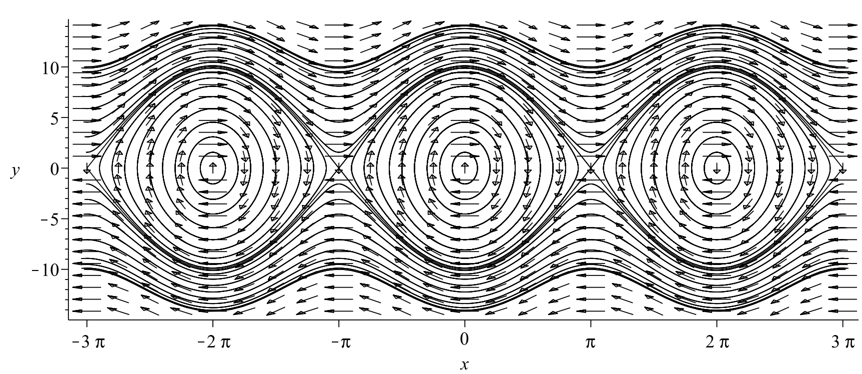



Figure 3.19: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$.

Before leaving this problem, we should note that the orbits in the phase plane for the undamped nonlinear pendulum can be obtained graphically. Recall from Equation (3.70), the total mechanical energy for the nonlinear

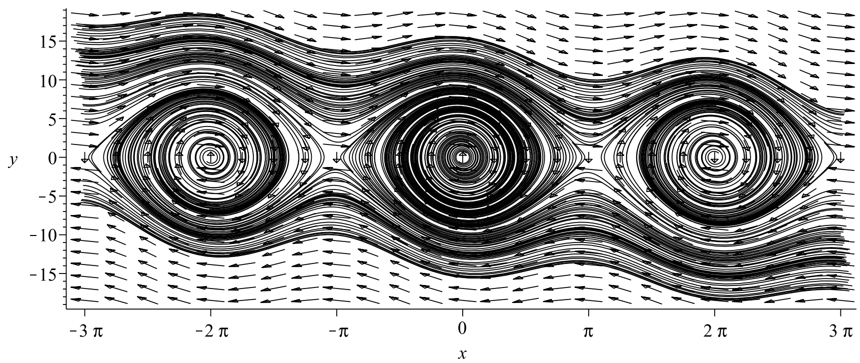



Figure 3.20: Several orbits in the phase plane for the damped nonlinear pendulum with $\omega = 5.0$ and $b = 0.5$.

pendulum is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta).$$

From this equation we obtained Equation (3.71),

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

Letting $y = \dot{\theta}$, $x = \theta$, and defining $z = -\omega^2\cos\theta_0$, this equation can be written as

$$\frac{1}{2}y^2 - \omega^2\cos x = z. \tag{3.35}$$

For each energy ($z$), this gives a constant energy curve. Plotting the family of energy curves we obtain the phase portrait shown in Figure 3.21.

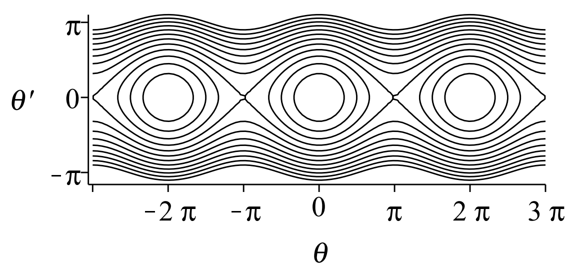



Figure 3.21: A family of energy curves in the phase plane for $\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = z$. Here we took $\omega = 1.0$ and $z \in [-5, 15]$.

## 3.6 Nonlinear Population Models

We have already encountered several models of population dynamics in this and previous chapters. Of course, one could dream up several other examples. While such models might seem far from applications in physics, it turns out that these models lead to systems od differential equations which also appear in physical systems such as the coupling of waves in lasers, in plasma physics, and in chemical reactions.

Two well-known nonlinear population models are the predator-prey and competing species models. In the predator-prey model, one typically has one species, the predator, feeding on the other, the prey. We will look at the standard Lotka-Volterra model in this section. The competing species model looks similar, except there are a few sign changes, since one species is not feeding on the other. Also, we can build in logistic terms into our model. We will save this latter type of model for the homework.

The Lotka-Volterra model is named after Alfred James Lotka (1880-1949) and Vito Volterra (1860-1940).

The Lotka-Volterra model takes the form

The Lotka-Volterra model of population dynamics.

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= -dy + cxy, \end{aligned} \tag{3.36}$$

where $a$, $b$, $c$, and $d$ are positive constants. In this model, we can think of $x$ as the population of rabbits (prey) and $y$ is the population of foxes (predators). Choosing all constants to be positive, we can describe the terms.

- $ax$: When left alone, the rabbit population will grow. Thus $a$ is the natural growth rate without predators.
- $-dy$: When there are no rabbits, the fox population should decay. Thus, the coefficient needs to be negative.
- $-bxy$: We add a nonlinear term corresponding to the depletion of the rabbits when the foxes are around.
- $cxy$: The more rabbits there are, the more food for the foxes. So, we add a nonlinear term giving rise to an increase in fox population.

**Example 3.13.** Determine the equilibrium points and their stability for the Lotka-Volterra system.

The analysis of the Lotka-Volterra model begins with determining the fixed points. So, we have from Equation (3.36)

$$\begin{aligned} x(a - by) &= 0, \\ y(-d + cx) &= 0. \end{aligned} \tag{3.37}$$

Therefore, the origin, $(0,0)$, and $(\frac{d}{c}, \frac{a}{b})$ are the fixed points.

Next, we determine their stability, by linearization about the fixed points. We can use the Jacobian matrix, or we could just expand the right-hand side of each equation in (3.36) about the equilibrium points as shown in he next example. The Jacobian matrix for this system is

$$Df(x,y) = \begin{pmatrix} a - by & -bx \\ cy & -d + cx \end{pmatrix}.$$

Evaluating at each fixed point, we have

$$Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.38}$$

$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}. \tag{3.39}$$

The eigenvalues of (3.38) are $\lambda = a, -d$. So, the origin is a saddle point.

The eigenvalues of (3.39) satisfy $\lambda^2 + ad = 0$. So, the other point is a center. In Figure 3.22 we show a sample direction field for the Lotka-Volterra system.

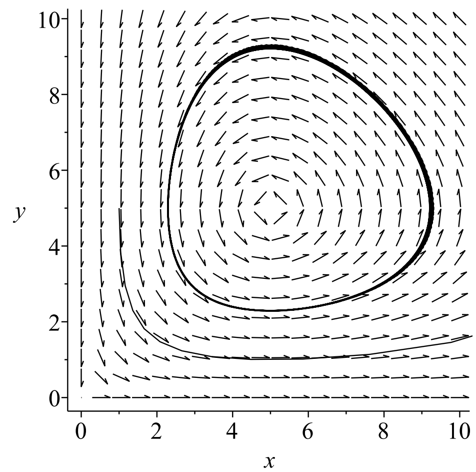



Figure 3.22: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$. Solution curves are shown for initial conditions $(x_0, y_0) = (8,3), (1,5)$.

Another way to carry out the linearization of the system of differential equations is to expand the equations about the fixed points. For fixed points $(x^*, y^*)$, we let

$$(x, y) = (x^* + u, y^* + v).$$

Inserting this translation of the origin into the equations of the system, and dropping nonlinear terms in $u$ and $v$, results in the linearized system. This method is equivalent to analyzing the Jacobian matrix for each fixed point.

Direct linearization of a system is carried out by introducing $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$, or $(x, y) = (x^* + u, y^* + v)$ into the system and dropping nonlinear terms in $u$ and $v$.

**Example 3.14.** Expand the Lotka-Volterra system about the equilibrium points.

For the origin $(0,0)$ the linearization about the origin amounts to simply dropping the nonlinear terms. In this case we have

$$\begin{aligned} \dot{u} &= au, \\ \dot{v} &= -dv. \end{aligned} \tag{3.40}$$

The coefficient matrix for this system is the same as $Df(0,0)$.

For the second fixed point, we let

$$(x, y) = \left(\frac{d}{c} + u, \frac{a}{b} + v\right).$$

Inserting this transformation into the system gives

$$\begin{aligned} \dot{u} &= a\left(\frac{d}{c} + u\right) - b\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right), \\ \dot{v} &= -d\left(\frac{a}{b} + v\right) + c\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right). \end{aligned} \tag{3.41}$$

Expanding, we obtain

$$\begin{aligned}
\dot{u} &= \frac{ad}{c} + au - b\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right), \\
\dot{v} &= -\frac{ad}{b} - dv + c\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right).
\end{aligned} \tag{3.42}$$

In both equations the constant terms cancel and linearization is simply getting rid of the $uv$ terms. This leaves the linearized system

$$\begin{aligned}
\dot{u} &= au - b\left(\frac{d}{c}v + \frac{a}{b}u\right), \\
\dot{v} &= -dv + c\left(+\frac{d}{c}v + \frac{a}{b}u\right),
\end{aligned} \tag{3.43}$$

or

$$\begin{aligned}
\dot{u} &= -\frac{bd}{c}v, \\
\dot{v} &= \frac{ac}{b}u.
\end{aligned} \tag{3.44}$$

The coefficient matrix for this linearized system is the same as $Df\left(\frac{d}{c}, \frac{a}{b}\right)$. In fact, for nearby orbits, they are almost circular orbits. From this linearized system, we have $\ddot{u} + adu = 0$.

We can take $u = A\cos(\sqrt{ad}t + \phi)$, where $A$ and $\phi$ can be determined from the initial conditions. Then,

$$\begin{aligned}
v &= -\frac{c}{bd}\dot{u} \\
&= \frac{c}{bd}A\sqrt{ad}\sin(\sqrt{ad}t + \phi) \\
&= \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi).
\end{aligned} \tag{3.45}$$

Therefore, the solutions near the center are given by

$$(x, y) = \left(\frac{d}{c} + A\cos(\sqrt{ad}t + \phi), \frac{a}{b} + \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi)\right).$$

For $a = d = 1$, $b = c = 0.2$, and initial values of $(x_0, y_0) = (5.5, 5)$, these solutions become

$$x(t) = 5.0 + 0.5\cos t, \quad y(t) = 5.0 + 0.5\sin t.$$

Plots of these solutions are shown in Figure (3.23).

It is also possible to find a first integral of the Lotka-Volterra system whose level curves give the phase portrait of the system. As we had done in Chapter 2, we can write

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} \\
&= \frac{-dy + cxy}{ax - bxy} \\
&= \frac{y(-d + cx)}{x(a - by)}.
\end{aligned} \tag{3.46}$$

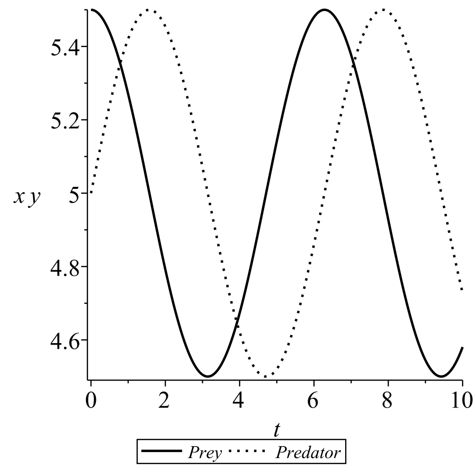



Figure 3.23: The linearized solutions of Lotka-Volterra system $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ for the initial conditions $(x_0, y_0) = (5.5, 5)$.

This is an equation of the form seen in Problem 2.**??**. This equation is now a separable differential equation. The solution this differential equation is given in implicit form as

$$a \ln y + d \ln x - cx - by = C,$$

where $C$ is an arbitrary constant. This expression is known as the first integral of the Lotka-Volterra system. These level curves are shown in Figure 3.24.

The first integral of the Lotka-Volterra system.

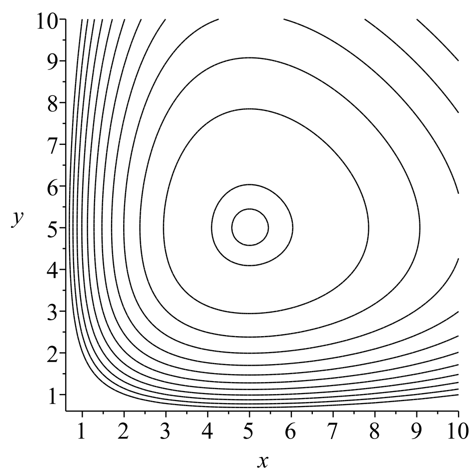



Figure 3.24: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ based upon the first integral of the system.

## 3.7 *Limit Cycles*

So far we have just been concerned with equilibrium solutions and their behavior. However, asymptotically stable fixed points are not the only attractors. There are other types of solutions, known as limit cycles, towards which a solution may tend. In this section we will look at some examples of these periodic solutions.

Such solutions are common in nature. Rayleigh investigated the problem

$$x'' + c\left(\frac{1}{3}(x')^2 - 1\right)x' + x = 0 \tag{3.47}$$

in the study of the vibrations of a violin string. Balthasar van der Pol (1889-1959) studied an electrical circuit, modeling this behavior. Others have looked into biological systems, such as neural systems, chemical reactions, such as Michaelis-Menten kinetics, and other chemical systems leading to chemical oscillations. One of the most important models in the historical study of dynamical systems is that of planetary motion and investigating the stability of planetary orbits. As is well known, these orbits are periodic.

Limit cycles are isolated periodic solutions towards which neighboring states might tend when stable. A key example exhibiting a limit cycle is given in the next example.

**Example 3.15.** Find the limit cycle in the system

$$\begin{aligned} x' &= \mu x - y - x(x^2+y^2) \\ y' &= x + \mu y - y(x^2+y^2). \end{aligned} \tag{3.48}$$

It is clear that the origin is a fixed point. The Jacobian matrix is given as

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}. \tag{3.49}$$

The eigenvalues are found to be $\lambda = \mu \pm i$. For $\mu = 0$ we have a center. For $\mu < 0$ we have a stable spiral and for $\mu > 0$ we have an unstable spiral. However, this spiral does not wander off to infinity. We see in Figure 3.25 that the equilibrium point is a spiral. However, in Figure 3.26 it is clear that the solution does not spiral out to infinity. It is bounded by a circle.

One can actually find the radius of this circle. This requires rewriting the system in polar form. Recall from Chapter 2 that we can change derivatives of Cartesian coordinates to derivatives of polar coordinates by using the relations

$$rr' = xx' + yy', \tag{3.50}$$

$$r^2\theta' = xy' - yx'. \tag{3.51}$$

Inserting the system (3.48) into these expressions, we have

$$rr' = \mu r^2 - r^4, \qquad r^2\theta' = r^2.$$

This leads to the system

$$\begin{aligned} r' &= \mu r - r^3, \\ \theta' &= 1. \end{aligned} \tag{3.52}$$

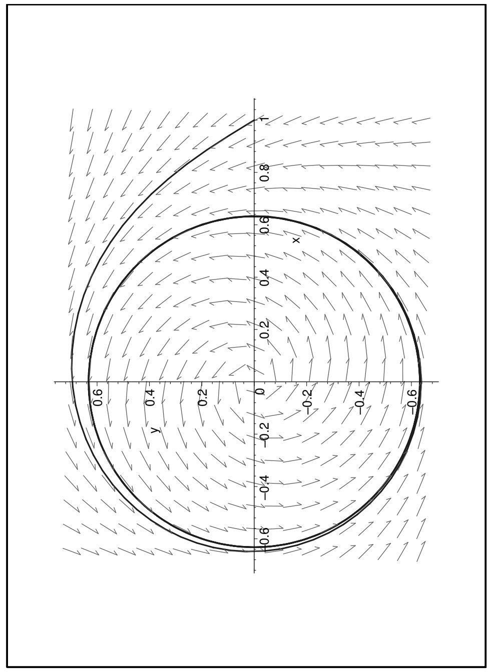



Figure 3.25: Phase plane for system (3.48) with $\mu = 0.4$.

Of course, for a circle the radius is constant, $r = \text{const}$. Therefore, in order to find the limit cycle, we need to look at the equilibrium solutions of Equation (3.52). This amounts to finding the constant solutions of $\mu r - r^3 = 0$. The equilibrium solutions are $r = 0, \pm\sqrt{\mu}$. The limit cycle corresponds to the positive radius solution, $r = \sqrt{\mu}$.

In Figures 3.25-3.26 we take $\mu = 0.4$. In this case we expect a circle with $r = \sqrt{0.4} \approx 0.63$. From the $\theta$ equation, we have that $\theta' > 0$. This means that we follow the limit cycle in a counterclockwise direction as time increases.

Limit cycles are not always circles. In Figures 3.27-3.28 we show the behavior of the Rayleigh system (3.47) for $c = 0.4$ and $c = 2.0$. In this case we see that solutions tend towards a noncircular limit cycle in a clockwise direction.

A slight change of the Rayleigh system leads to the van der Pol equation:

$$x'' + c(x^2 - 1)x' + x = 0 \tag{3.53}$$

The van der Pol system.

The limit cycle for $c = 2.0$ is shown in Figure 3.29.

Can one determine ahead of time if a given nonlinear system will have a limit cycle? In order to answer this question, we will introduce some definitions.

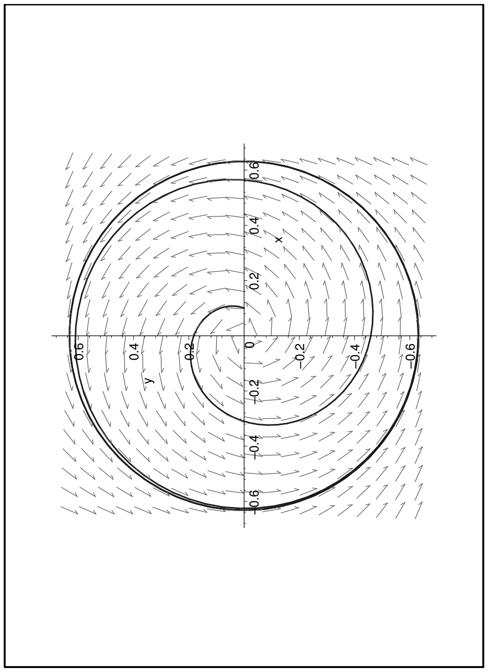



Figure 3.26: Phase plane for system (3.48) with $\mu = 0.4$ showing that the inner spiral is bounded by a limit cycle.

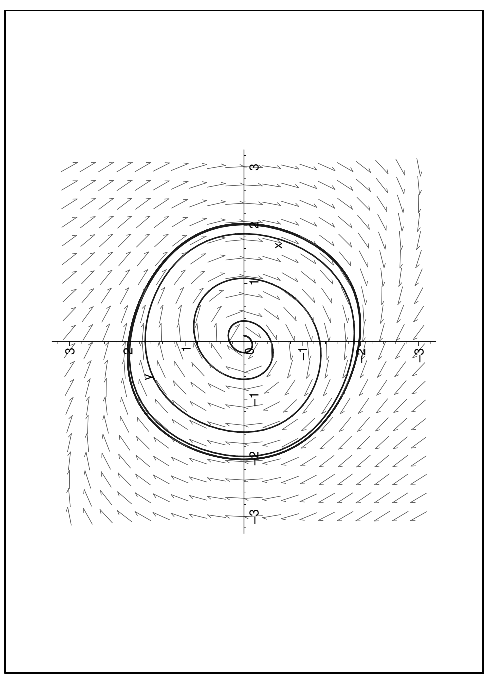



Figure 3.27: Phase plane for the Rayleigh system (3.47) with $c = 0.4$.

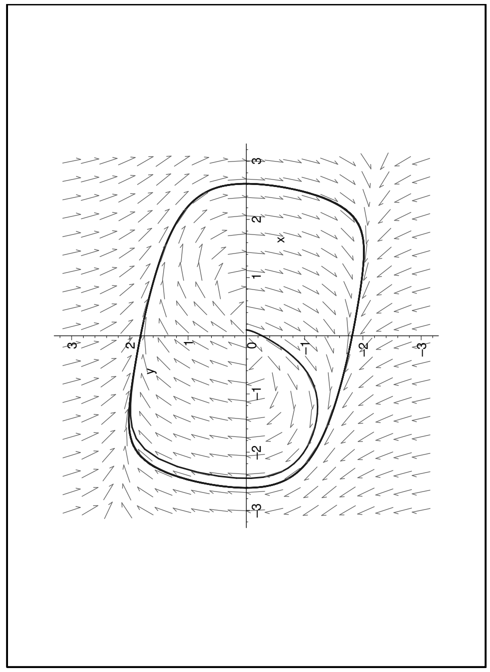



Figure 3.28: Phase plane for the van der Pol system (3.53) with $c = 2.0$.

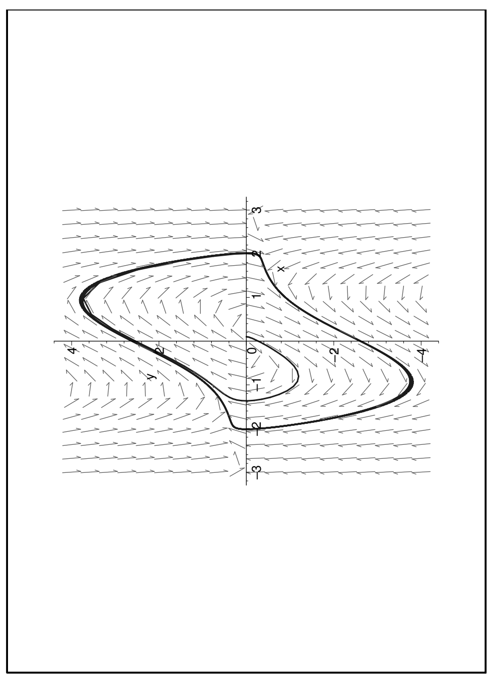



Figure 3.29: Phase plane for the van der Pol system (3.53) with $c = 0.4$.

We first describe different trajectories and families of trajectories. A *flow* on $R^2$ is a function $\phi$ that satisfies the following

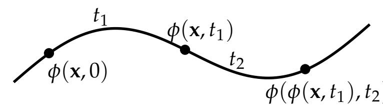



Figure 3.30: A sketch depicting the idea of trajectory, or orbit, passing through $\mathbf{x}$.

Flows.

1. $\phi(\mathbf{x}, t)$ is continuous in both arguments.

2. $\phi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in R^2$.

3. $\phi(\phi(\mathbf{x}, t_1), t_2) = \phi(\mathbf{x}, t_1 + t_2)$.

Orbits and trajectories.

The *orbit*, or *trajectory*, through $\mathbf{x}$ is defined as $\gamma = \{\phi(\mathbf{x}, t) | t \in I\}$. In Figure 3.30 we demonstrate these properties. For $t = 0$, $\phi(\mathbf{x}, 0) = \mathbf{x}$. Increasing $t$, one follows the trajectory until one reaches the point $\phi(\mathbf{x}, t_1)$. Continuing $t_2$ further, one is then at $\phi(\phi(\mathbf{x}, t_1), t_2)$. By the third property, this is the same as going from $\mathbf{x}$ to $\phi(\mathbf{x}, t_1 + t_2)$ for $t = t_1 + t_2$.

Having defined the orbits, we need to define the asymptotic behavior of the orbit for both positive and negative large times. We define the *positive semiorbit* through $\mathbf{x}$ as $\gamma^+ = \{\phi(\mathbf{x}, t) | t > 0\}$. The *negative semiorbit* through $\mathbf{x}$ is defined as $\gamma^- = \{\phi(\mathbf{x}, t) | t < 0\}$. Thus, we have $\gamma = \gamma^+ \cup \gamma^-$.

Limit sets and limit points.

The *positive limit set*, or *$\omega$-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^+ = \{\mathbf{y} | \text{ there exists a sequence of } t_n \to \infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}.$$

The $\mathbf{y}$'s are referred to as *$\omega$-limit points*. This is shown in Figure 3.31.

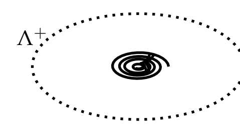



Figure 3.31: A sketch depicting an $\omega$-limit set. Note that the orbits tend towards the set as $t$ increases.

Similarly, we define the *negative limit set*, or the *alpha-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^- = \{\mathbf{y} | \text{ there exists a sequences of } t_n \to -\infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}$$

and the corresponding $\mathbf{y}$'s are *$\alpha$-limit points*. This is shown in Figure 3.32.

Cycles and periodic orbits.

There are several types of orbits that a system might possess. A *cycle* or *periodic orbit* is any closed orbit which is not an equilibrium point. A periodic orbit is stable if for every neighborhood of the orbit such that all nearby orbits stay inside the neighborhood. Otherwise, it is unstable. The orbit is asymptotically stable if all nearby orbits converge to the periodic orbit.

A limit cycle is a cycle which is the $\alpha$ or $\omega$-limit set of some trajectory other than the limit cycle. A limit cycle $\Gamma$ is stable if $\Lambda^+ = \Gamma$ for all $\mathbf{x}$ in some neighborhood of $\Gamma$. A limit cycle $\Gamma$ is unstable if $\Lambda^- = \Gamma$ for all $\mathbf{x}$ in some neighborhood of $\Gamma$. Finally, a limit cycle is semistable if it is attracting on one side and repelling on the other side. In the previous examples, we saw limit cycles that were stable. Figures 3.31 and 3.32 depict stable and unstable limit cycles, respectively.

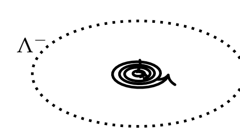



Figure 3.32: A sketch depicting an $\alpha$-limit set. Note that the orbits tend away from the set as $t$ increases.

We now state a theorem which describes the type of orbits we might find in our system.

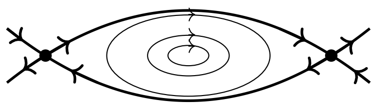


Figure 3.33: A heteroclinic orbit connecting two critical points.

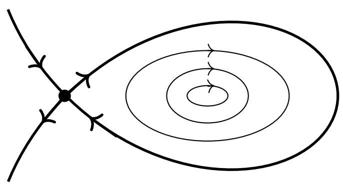


Figure 3.34: A homoclinic orbit returning to the point it left.

**Theorem 3.2. Poincaré-Bendixon Theorem** *Let $\gamma^+$ be contained in a bounded region in which there are finitely many critical points. Then $\Lambda^+$ is either*

1. *a single critical point;*
2. *a single closed orbit;*
3. *a set of critical points joined by heteroclinic orbits. [Compare Figures 3.33 and 3.34.]*

We are interested in determining when limit cycles may, or may not, exist. A consequence of the Poincaré-Bendixon Theorem is given by the following corollary.

**Corollary** Let $D$ be a bounded closed set containing no critical points and suppose that $\gamma^+ \subset D$. Then there exists a limit cycle contained in $D$.

More specific criteria allow us to determine if there is a limit cycle in a given region. These are given by Dulac's Criteria and Bendixon's Criteria.

**Dulac's Criteria** Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

and a continuously differentiable function $\psi$ defined on an annular region $D$ contained in some open set. If

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g)$$

does not change sign in $D$, then there is at most one limit cycle contained entirely in $D$.

**Bendixon's Criteria** Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

defined on a simply connected domain $D$ such that

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g) \neq 0$$

in $D$. Then, there are no limit cycles entirely in $D$.

*Proof.* These are easily proved using Green's Theorem in the Plane. (See your calculus text.) We prove Bendixon's Criteria. Let $\mathbf{f} = (f,g)$. Assume that $\Gamma$ is a closed orbit lying in $D$. Let $S$ be the interior of $\Gamma$. Then

$$\begin{aligned} \int_S \nabla \cdot \mathbf{f}\, dxdy &= \oint_\Gamma (f\,dy - g\,dx) \\ &= \int_0^T (f\dot{y} - g\dot{x})dt \\ &= \int_0^T (fg - gf)dt = 0. \end{aligned} \tag{3.54}$$

So, if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in $S$, then from the continuity of $\nabla \cdot \mathbf{f}$ in $S$ we have that the right side above is either positive or negative. Thus, we have a contradiction and there is no closed orbit lying in $D$ □

**Example 3.16.** Consider the earlier example in (3.48) with $\mu = 1$.

$$\begin{aligned} x' &= x - y - x(x^2 + y^2) \\ y' &= x + y - y(x^2 + y^2). \end{aligned} \tag{3.55}$$

We already know that a limit cycle exists at $x^2 + y^2 = 1$. A simple computation gives that

$$\nabla \cdot \mathbf{f} = 2 - 4x^2 - 4y^2.$$

For an arbitrary annulus $a < x^2 + y^2 < b$, we have

$$2 - 4b < \nabla \cdot \mathbf{f} < 2 - 4a.$$

For $a = 3/4$ and $b = 5/4$, $-3 < \nabla \cdot \mathbf{f} < -1$. Thus, $\nabla \cdot \mathbf{f} < 0$ in the annulus $3/4 < x^2 + y^2 < 5/4$. Therefore, by Dulac's Criteria there is at most one limit cycle in this annulus.

**Example 3.17.** Consider the system

$$\begin{aligned} x' &= y \\ y' &= -ax - by + cx^2 + dy^2. \end{aligned} \tag{3.56}$$

Let $\psi(x, y) = e^{-2dx}$. Then,

$$\frac{\partial}{\partial x}(\psi y) + \frac{\partial}{\partial y}(\psi(-ax - by + cx^2 + dy^2)) = -be^{-2dx} \neq 0.$$

We conclude by Bendixon's Criteria that there are no limit cycles for this system.

## 3.8 Nonautonomous Nonlinear Systems

In this section we discuss nonautonomous systems. Recall that an autonomous system is one in which there is no explicit time dependence. A simple example is the forced nonlinear pendulum given by the nonhomogeneous equation

$$\ddot{x} + \omega^2 \sin x = f(t). \tag{3.57}$$

We can set this up as a system of two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(t). \end{aligned} \tag{3.58}$$

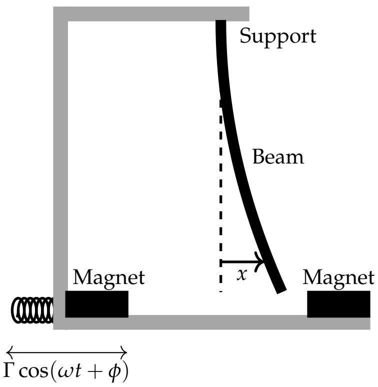



Figure 3.35: One model of the Duffing equation describes a periodically forced beam which interacts with two magnets.

This system is not in a form for which we could use the earlier methods. Namely, it is a nonautonomous system. However, we introduce a new variable $z(t) = t$ and turn it into an autonomous system in one more dimension. The new system takes the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(z). \\ \dot{z} &= 1. \end{aligned} \tag{3.59}$$

The system is now a three dimensional autonomous, possibly nonlinear, system and can be explored using methods from Chapters 2 and 3.

A more interesting model is provided by the Duffing Equation. This equation, named after Georg Wilhelm Christian Caspar Duffing (1861-1944), models hard spring and soft spring oscillations. It also models a periodically forced beam as shown in Figure 3.35. It is of interest because it is a simple system which exhibits chaotic dynamics and will motivate us towards using new visualization methods for nonautonomous systems.

The most general form of Duffing's equation is given by the damped, forced system

$$\ddot{x} + k\dot{x} + (\beta x^3 \pm \omega_0^2 x) = \Gamma \cos(\omega t + \phi). \tag{3.60}$$

This equation models hard spring, $(\beta > 0)$, and soft spring, $(\beta < 0)$, oscillations. However, we will use the simpler version of the Duffing equation:

$$\ddot{x} + k\dot{x} + x^3 - x = \Gamma \cos \omega t. \tag{3.61}$$

An equation of this form can be obtained by setting $\phi = 0$ and rescaling $x$ and $t$ in the original equation. We will explore the behavior of the system as we vary the remaining parameters. In Figures 3.36-3.39 we show some typical solution plots superimposed on the direction field.

The undamped, unforced Duffing equation.

We start with the undamped ($k = 0$) and unforced ($\Gamma = 0$) Duffing equation,

$$\ddot{x} + x^3 - x = 0.$$

We can write this second order equation as the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2). \end{aligned} \tag{3.62}$$

We see that there are three equilibrium points at $(0, 0)$ and $(\pm 1, 0)$. In Figure 3.36 we plot several orbits for $k = 0$, and $\Gamma = 0$. We see that the three equilibrium points consist of two centers and a saddle.

The unforced Duffing equation.

We now turn on the damping. The system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky + x(1 - x^2). \end{aligned} \tag{3.63}$$

In Figures 3.37 and 3.38 we show what happens when $k = 0.1$. These plots are reminiscent of the plots for the nonlinear pendulum; however, there are fewer equilibria. Note that the centers become stable spirals for $k > 0$.

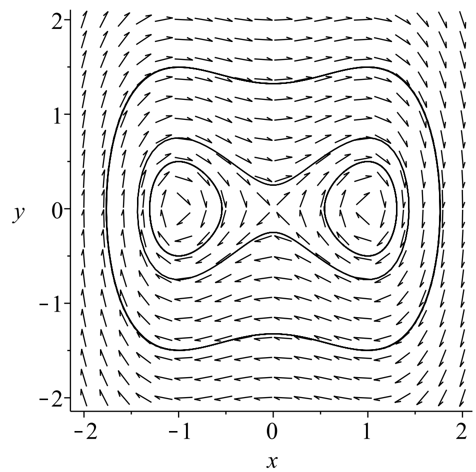



Figure 3.36: Phase plane for the undamped, unforced Duffing equation ($k = 0, \Gamma = 0$).

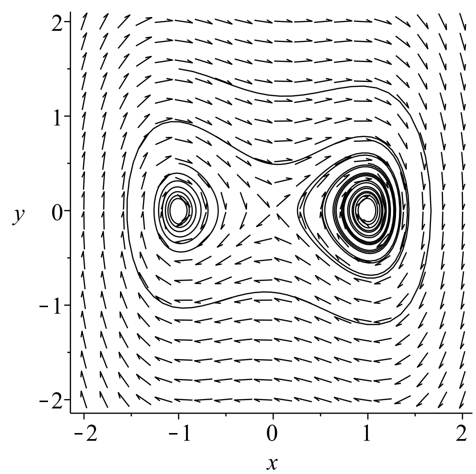



Figure 3.37: Phase plane for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

Next we turn on the forcing to obtain a damped, forced Duffing equation. The system is now nonautonomous.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-x^2) + \Gamma \cos \omega t. \end{aligned} \tag{3.64}$$

The damped, forced Duffing equation.

In Figure 3.39 we only show one orbit with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. The solution intersects itself and look a bit messy. We can imagine what we would get if we added any more orbits. For completeness, we show in Figure 3.40 an example with four different orbits.

In cases for which one has periodic orbits such as the Duffing equation, Poincaré introduced the notion of *surfaces of section*. One embeds the orbit in a higher dimensional space so that there are no self intersections, like we saw in Figures 3.39 and 3.40. In Figure 3.42 we show an example where a simple orbit is shown as it periodically pierces a given surface.

In order to simplify the resulting pictures, one only plots the points at which the orbit pierces the surface as sketched in Figure 3.41. In practice, there is a natural frequency, such as $\omega$ in the forced Duffing equation. Then,

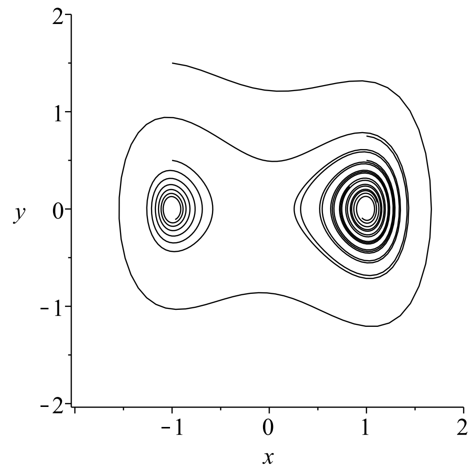



Figure 3.38: Display of two orbits for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

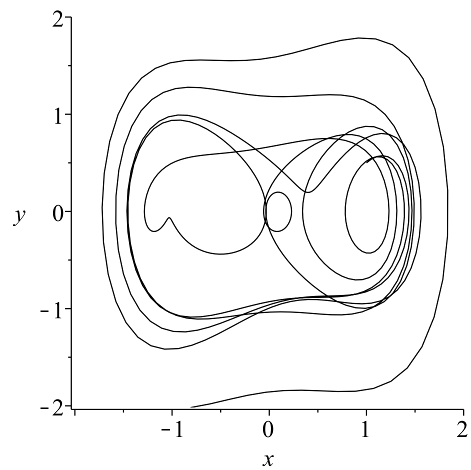



Figure 3.39: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case we show only one orbit which was generated from the initial condition $(x_0 = 1.0, \quad y_0 = 0.5)$.

one plots points at times that are multiples of the period, $T = \frac{2\pi}{\omega}$. In Figure 3.43 we show what the plot for one orbit would look like for the damped, unforced Duffing equation.

The more interesting case, is when there is forcing and damping. In this case the surface of section plot is given in Figure 3.44. While this is not as busy as the solution plot in Figure 3.39, it still provides some interesting behavior. What one finds is what is called a strange attractor. Plotting many orbits, we find that after a long time, all of the orbits are attracted to a small region in the plane, much like a stable node attracts nearby orbits. However, this set consists of more than one point. Also, the flow on the attractor is chaotic in nature. Thus, points wander in an irregular way throughout the attractor. This is one of the interesting topics in chaos theory and this whole theory of dynamical systems has only been touched in this text leaving the reader to wander of into further depth into this fascinating field.

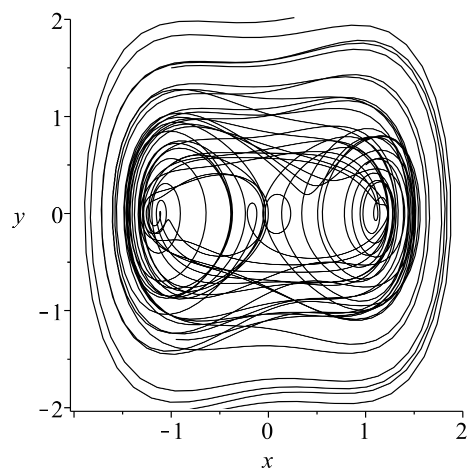



Figure 3.40: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case four initial conditions were used to generate four orbits.

The surface of section plots at the end of the last section were obtained using code from S. Lynch's book *Dynamical Systems with Applications Using Maple*. For reference, the plots in Figures 3.36 and 3.37 were generated in Maple using the following commands:

```
> with(DEtools):
> Gamma:=0.5:omega:=1.25:k:=0.1:
> DEplot([diff(x(t),t)=y(t), diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t)], [x(t),y(t)],t=0..500,[[x(0)=1,y(0)=0.5],
  [x(0)=-1,y(0)=0.5], [x(0)=1,y(0)=0.75], [x(0)=-1,y(0)=1.5]],
  x=-2..2,y=-2..2, stepsize=0.1, linecolor=blue, thickness=1,
  color=black);
```

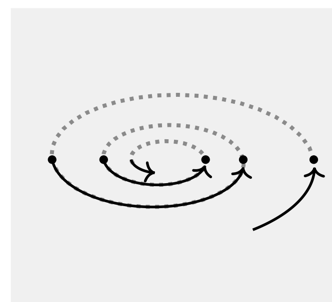


Figure 3.41: As an orbit pierces the surface of section, one plots the point of intersection in that plane to produce the surface of section plot.

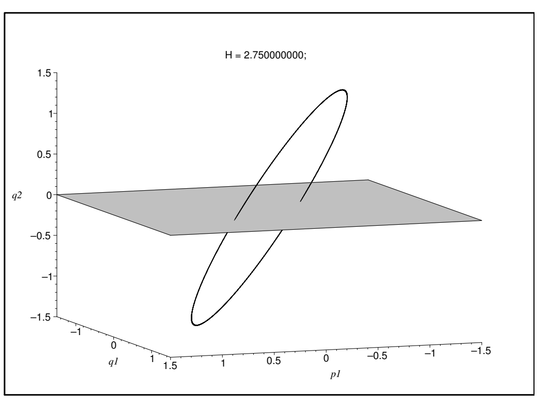



Figure 3.42: Poincaré's surface of section. One notes each time the orbit pierces the surface.

Figure 3.43: Poincaré's surface of section plot for the damped, unforced Duffing equation.

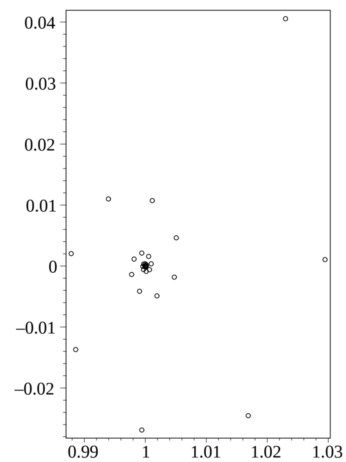
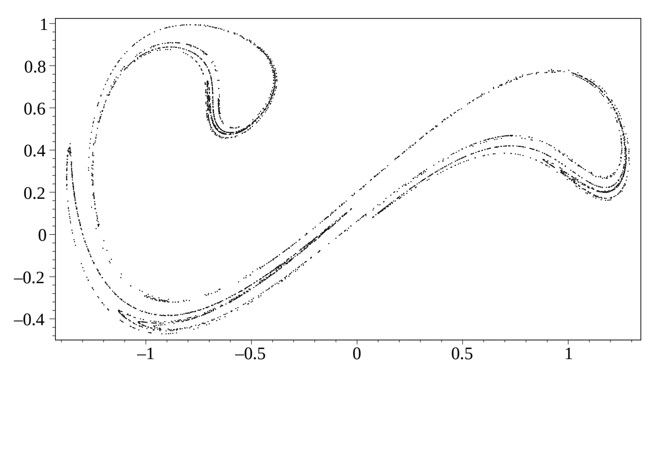



Figure 3.44: Poincaré's surface of section plot for the damped, forced Duffing equation. This leads to what is known as a strange attractor.



## 3.9 *The Period of the Nonlinear Pendulum**

RECALL THAT THE PERIOD OF THE SIMPLE PENDULUM is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.65}$$

for

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{3.66}$$

This was based upon the solving the linear pendulum equation (3.12). This equation was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle?

We recall that the Taylor series approximation of $\sin\theta$ about $\theta = 0$ :

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{3.67}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \left|\frac{\sin\theta - \theta}{\sin\theta}\right|.$$

A plot of the relative error is given in Figure 3.45. Thus for $\theta \approx 0.4$ radians (or, $23^o$) we have that the relative error is about 2.6%.

Relative error in $\sin\theta$ approximation.

We would like to do better than this. So, we now turn to the nonlinear pendulum equation (3.11) in the simpler form

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{3.68}$$

Solution of nonlinear pendulum equation.

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For the nonlinear pendulum problem, we multiply Equation (3.68) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

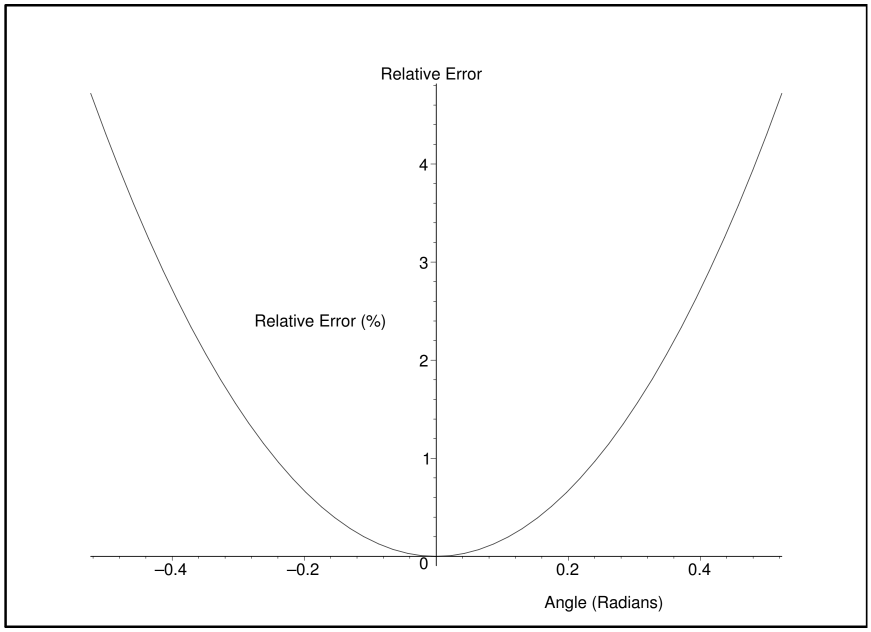
and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{3.69}$$

Figure 3.45: The relative error in percent when approximating $\sin\theta$ by $\theta$.




Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2 \cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2 \cos\theta)}}.$$

Of course, we need to be able to do the integral. When one finds a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

In fact, the above integral can be transformed into what is known as an elliptic integral of the first kind. We will rewrite this result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (3.69). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the mass on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where $h$ is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1-\cos\theta)$. So,

$$U = mgL(1-\cos\theta).$$

So, the total mechanical energy is

Total mechanical energy for the nonlinear pendulum.

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta). \tag{3.70}$$

We note that a little rearranging shows that we can relate this result to Equation (3.69). Dividing by $m$ and $L^2$ and using the definition of $\omega^2 = g/L$, we have

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2.$$

Therefore, we have determined the integration constant in terms of the total mechanical energy,

$$c = \frac{1}{mL^2}E - \omega^2.$$

We can use Equation (3.70) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total mechanical energy is pure potential energy. Letting $\theta_0$ denote the angle at the highest angular position, we have that

$$E = mgL(1-\cos\theta_0) = mL^2\omega^2(1-\cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0. \tag{3.71}$$

We can solve for $\dot{\theta}$ and integrate the differential equation to obtain

$$t = \int dt = \int \frac{d\theta}{\omega\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta),$$

we can rewrite the argument in the radical as

$$\cos\theta - \cos\theta_0 = 2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right].$$

Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{3.72}$$

This result can now be transformed into an elliptic integral.[2] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

[2] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate,

$$(x^2+y^2)^2 = x^2 - y^2.$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then, Equation (3.72) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.73}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is called the complete elliptic integral of the first kind.

We note that the incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The complete and incomplete elliptic integrals of the first kind.

Then, the complete elliptic integral of the first kind is given by $K(k) = F(\frac{\pi}{2},k)$, or

$$K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Therefore, the period of the nonlinear pendulum is given by

$$T = \frac{4}{\omega}K\left(\sin\frac{\theta_0}{2}\right). \tag{3.74}$$

There are table of values for elliptic integrals. However, one can use a computer algebra system to compute values of such integrals. We will look for small angle approximations.

For small angles ($\theta_0 \ll \frac{\pi}{2}$), we have that $k$ is small. So, we can develop a series expansion for the period, $T$, for small $k$. This is simply done by using the binomial expansion,

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

Inserting this expansion into the integrand for the complete elliptic integral and integrating term by term, we find that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right]. \tag{3.75}$$

The first term of the expansion gives the well known period of the simple pendulum for small angles. The next terms in the expression give further corrections to the linear result which are useful for larger amplitudes of oscillation. In Figure 3.46 we show the relative errors incurred when keeping the $k^2$ (quadratic) and $k^4$ (quartic) terms as compared to the exact value of the period.

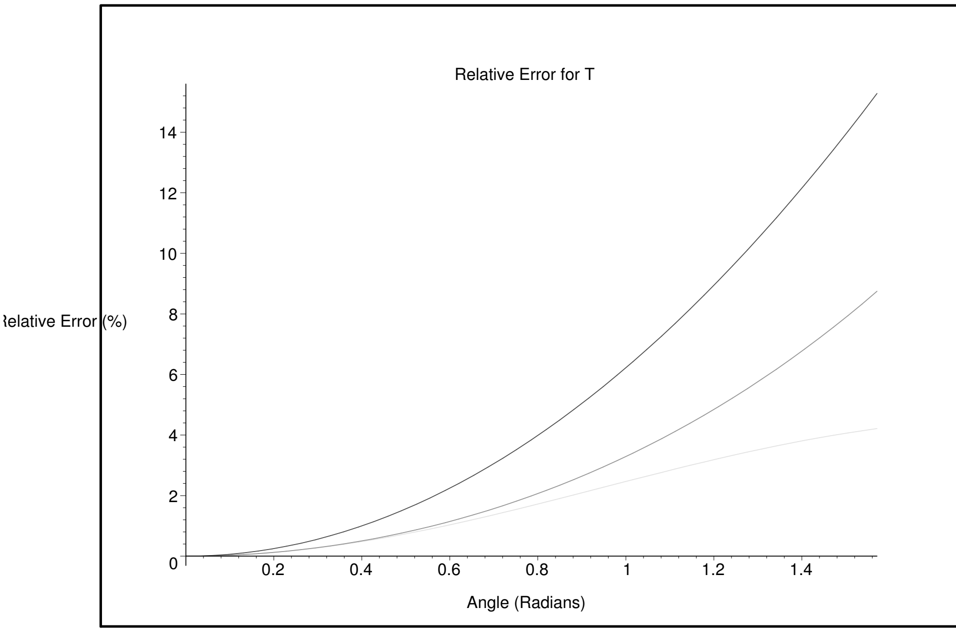



Figure 3.46: The relative error in percent when approximating the exact period of a nonlinear pendulum with one (solid), two (dashed), or three (dotted) terms in Equation (3.75).

## 3.10 *Exact Solutions Using Elliptic Functions**

The solution in Equation (3.73) of the nonlinear pendulum equation led to the introduction of elliptic integrals. The incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.76}$$

The complete integral of the first kind is given by $K(k) = F(\frac{\pi}{2}, k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Elliptic integrals of the second kind are defined as

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{\sin\phi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{3.77}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{1} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{3.78}$$

Recall, a first integration of the nonlinear pendulum equation from Equation (3.70),

$$\left(\frac{d\theta}{dt}\right)^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega^2\left[\sin^2\frac{\theta}{2} - \sin^2\frac{\theta_0}{2}\right].$$

Letting

$$kz = \sin\frac{\theta}{2} \text{ and } k = \sin\frac{\theta_0}{2},$$

the differential equation becomes

$$\frac{dz}{d\tau} = \pm\omega\sqrt{1-z^2}\sqrt{1-k^2z^2}.$$

Applying separation of variables, we find

$$\begin{aligned} \pm\omega(t-t_0) &= \frac{1}{\omega}\int_1^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} & (3.79)\\ &= \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} - \int_0^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} & (3.80)\\ &= K(k) - F(\sin^{-1}(k^{-1}\sin\theta), k). & (3.81) \end{aligned}$$

The solution, $\theta(t)$, is then found by solving for $z$ and using $kz = \sin\frac{\theta}{2}$ to solve for $\theta$. This requires that we know how to invert the elliptic integral, $F(z,k)$.

Elliptic functions result from the inversion of elliptic integrals. Consider

$$\begin{aligned} u(\sin\phi, k) = F(\phi, k) &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. & (3.82)\\ &= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. & (3.83) \end{aligned}$$

Note:$F(\phi, 0) = \phi$ and $F(\phi, 1) = \ln(\sec\phi + \tan\phi)$. In these cases $F$ is obviously monotone increasing and thus there must be an inverse.

The inverse of Equation (3.76) is defined as $\phi = F^{-1}(u,k) = \text{am}(u,k)$, where $u = \sin\phi$. The function $\text{am}(u,k)$ is called the Jacobi amplitude function and $k$ is the elliptic modulus. [In some references and software like MATLAB packages, $m = k^2$ is used as the parameter.] Three of the Jacobi elliptic functions, shown in Figure 3.47, can be defined in terms of the amplitude function by

$$\text{sn}(u,k) = \sin\text{am}(u,k) = \sin\phi,$$

$$\text{cn}(u,k) = \cos\text{am}(u,k) = \cos\phi,$$

Jacobi elliptic functions.

and the delta amplitude

$$\text{dn}(u,k) = \sqrt{1-k^2\sin^2\phi}.$$

They are related through the identities

$$\text{cn}^2(u,k) + \text{sn}^2(u,k) = 1, \tag{3.84}$$

$$\text{dn}^2(u,k) + k^2\,\text{sn}^2(u,k) = 1. \tag{3.85}$$

The elliptic functions can be extended to the complex plane. In this case the functions are doubly periodic. However, we will not need to consider this in the current text.

Also, we see that these functions are periodic. The period is given in terms of the complete elliptic integral of the first kind, $K(k)$. Consider

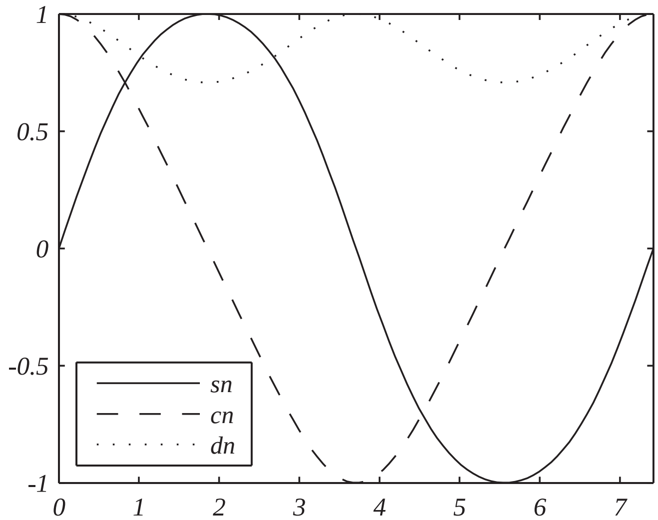



Figure 3.47: Plots of the Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ for $m = k^2 = 0.5$. Here $K(k) = 1.8541$.

$$\begin{aligned}
F(\phi+2\pi,k) &= \int_0^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \\
&= \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_{\phi}^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= F(\phi,k) + \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= F(\phi,k) + 4K(k). \qquad (3.86)
\end{aligned}$$

Since $F(\phi+2\pi,k) = u + 4K$, we have

$$\mathrm{sn}(u+4K) = \sin(\mathrm{am}(u+4K)) = \sin(\mathrm{am}(u)+2\pi) = \sin \mathrm{am}(u) = \mathrm{sn}\, u.$$

In general, we have

$$\begin{aligned}
\mathrm{sn}(u+2K,k) &= -\,\mathrm{sn}(u,k) \qquad (3.87)\\
\mathrm{cn}(u+2K,k) &= -\,\mathrm{cn}(u,k) \qquad (3.88)\\
\mathrm{dn}(u+2K,k) &= \mathrm{dn}(u,k). \qquad (3.89)
\end{aligned}$$

The plots of $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $dn(u)$, are shown in Figures 3.48-3.50.

Namely,

$$\mathrm{sn}(u+K,k) = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}, \quad \mathrm{sn}(u+2K,k) = -\,\mathrm{sn}\, u,$$

$$\mathrm{cn}(u+K,k) = -\sqrt{1-k^2}\frac{\mathrm{sn}\, u}{\mathrm{dn}\, u}, \quad \mathrm{dn}(u+2K,k) = -\,\mathrm{cn}\, u,$$

$$\mathrm{dn}(u+K,k) = \frac{\sqrt{1-k^2}}{\mathrm{dn}\, u}, \quad \mathrm{dn}(u+2K,k) = \mathrm{dn}\, u.$$

Therefore, dn and cn have a period of $4K$ and dn has a period of $2K$.

Special values found in Figure 3.47 are seen as

$$\mathrm{sn}(K,k) = 1,$$

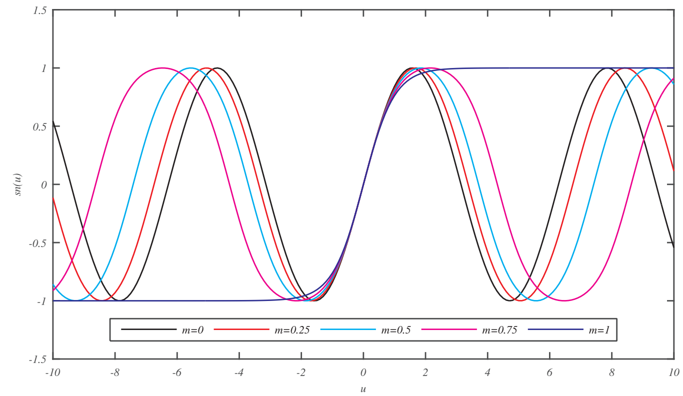



Figure 3.48: Plots of $\operatorname{sn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

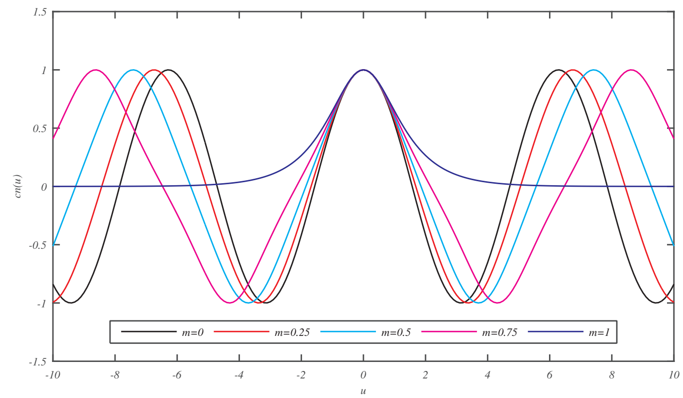



Figure 3.49: Plots of $\operatorname{cn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

$$\operatorname{cn}(K,k) = 0,$$
$$\operatorname{dn}(K,k) = \sqrt{1-k^2} = k',$$

where $k'$ is called the complementary modulus.

Important to this section are the derivatives of these elliptic functions,

$$\frac{\partial}{\partial u}\operatorname{sn}(u,k) = \operatorname{cn}(u,k)\operatorname{dn}(u,k),$$
$$\frac{\partial}{\partial u}\operatorname{cn}(u,k) = -\operatorname{sn}(u,k)\operatorname{dn}(u,k),$$

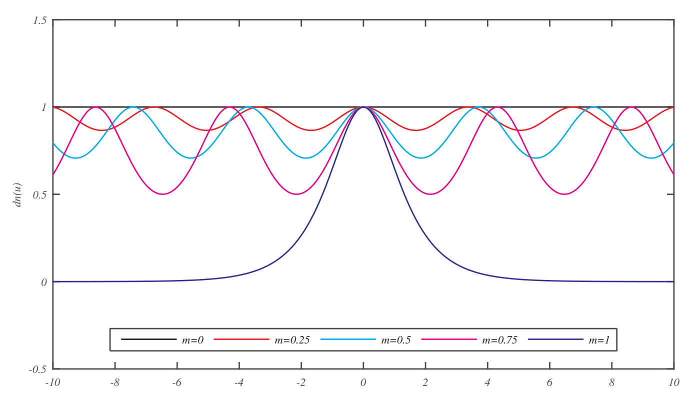



Figure 3.50: Plots of $\operatorname{dn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

$$\frac{\partial}{\partial u}\operatorname{dn}(u,k) = -k^2 \operatorname{sn}(u,k)\operatorname{cn}(u,k),$$

and the amplitude function

$$\frac{\partial}{\partial u}\operatorname{am}(u,k) = \operatorname{dn}(u,k).$$

Sometimes the Jacobi elliptic functions are displayed without reference to the elliptic modulus, such as $\operatorname{sn}(u) = \operatorname{sn}(u,k)$. When $k$ is understood, we can do the same.

**Example 3.18.** Show that $\operatorname{sn}(u)$ satisfies the differential equation

$$y'' + (1+k^2)y = 2k^2y^3.$$

From the above derivatives, we have that

$$\begin{aligned}\frac{d^2}{du^2}\operatorname{sn}(u) &= \frac{d}{du}(\operatorname{cn}(u)\operatorname{dn}(u)) \\ &= -\operatorname{sn}(u)\operatorname{dn}^2(u) - k^2\operatorname{sn}(u)\operatorname{cn}^2(u). \qquad (3.90)\end{aligned}$$

Letting $y(u) = \operatorname{sn}(u)$ and using the identities (3.84)-(3.85), we have that

$$y'' = -y(1-k^2y^2) - k^2y(1-y^2) = -(1+k^2)y + 2k^2y^3.$$

This is the desired result.

**Example 3.19.** Show that $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$ is a solution of the equation $\ddot{\theta} + \sin\theta = 0$.

Differentiating $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$, we have

$$\begin{aligned}\frac{d^2}{dt^2}\left(2\sin^{-1}(k\operatorname{sn} t)\right) &= \frac{d}{dt}\left(2\frac{k\operatorname{cn} t\operatorname{dn} t}{\sqrt{1-k^2\operatorname{sn}^2 t}}\right) \\ &= \frac{d}{dt}\left(2k\operatorname{cn} t\right) \\ &= -2k\operatorname{sn} t\operatorname{dn} t. \qquad (3.91)\end{aligned}$$

However, we can evaluate $\sin\theta$ for a range of $\theta$. Thus, we have

$$\begin{aligned}\sin\theta &= \sin(2\sin^{-1}(k\operatorname{sn} t)) \\ &= 2\sin(\sin^{-1}(k\operatorname{sn} t))\cos(\sin^{-1}(k\operatorname{sn} t)) \\ &= 2k\operatorname{sn} t\sqrt{1-k^2\operatorname{sn}^2 t} \\ &= 2k\operatorname{sn} t\operatorname{dn} t. \qquad (3.92)\end{aligned}$$

Comparing these results, we have shown that $\ddot{\theta} + \sin\theta = 0$.

The solution to the last example can be used to obtain the exact solution to the nonlinear pendulum problem, $\ddot{\theta} + \omega^2\sin\theta = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$. The general solution is given by $\theta(t) = 2\sin^{-1}(k\operatorname{sn}(\omega t + \phi))$ where $\phi$ has to be determined from the initial conditions. We note that

$$\begin{aligned}\frac{d\operatorname{sn}(u+K)}{du} &= \operatorname{cn}(u+K)\operatorname{dn}(u+K) \\ &= \left(-\sqrt{1-k^2}\frac{\operatorname{sn} u}{\operatorname{dn} u}\right)\left(\frac{\sqrt{1-k^2}}{\operatorname{dn} u}\right) \\ &= -(1-k^2)\frac{\operatorname{sn} u}{\operatorname{dn}^2 u}. \qquad (3.93)\end{aligned}$$

Evaluating at $u = 0$, we have $\mathrm{sn}'(K) = 0$.

Therefore, if we pick $\phi = K$, then $\dot{\theta}(0) = 0$ and the solution is

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\omega t + K)).$$

Furthermore, the other initial value is found to be

$$\theta(0) = 2\sin^{-1}(k\,\mathrm{sn}\,K) = 2\sin^{-1}k.$$

Thus, $k = \sin\frac{\theta_0}{2}$, as we had seen in the earlier derivation of the elliptic integral solution. The solution is given as

$$\theta(t) = 2\sin^{-1}(\sin\frac{\theta_0}{2}\,\mathrm{sn}(\omega t + K)).$$

In Figures 3.51-3.52 we show comparisons of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and initial angles $\theta_0 = 10^o$ and $\theta_0 = 50^o$.

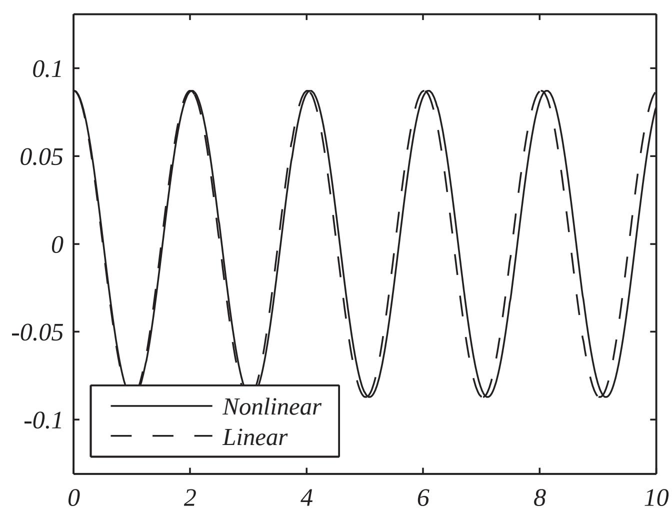



Figure 3.51: Comparison of exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 10^o$.

## *Problems*

**1.** Solve the general logistic problem,

$$\frac{dy}{dt} = ky - cy^2, \quad y(0) = y_0 \tag{3.94}$$

using separation of variables.

**2.** Find the equilibrium solutions and determine their stability for the following systems. For each case draw representative solutions and phase lines.

a. $y' = y^2 - 6y - 16.$

b. $y' = \cos y.$

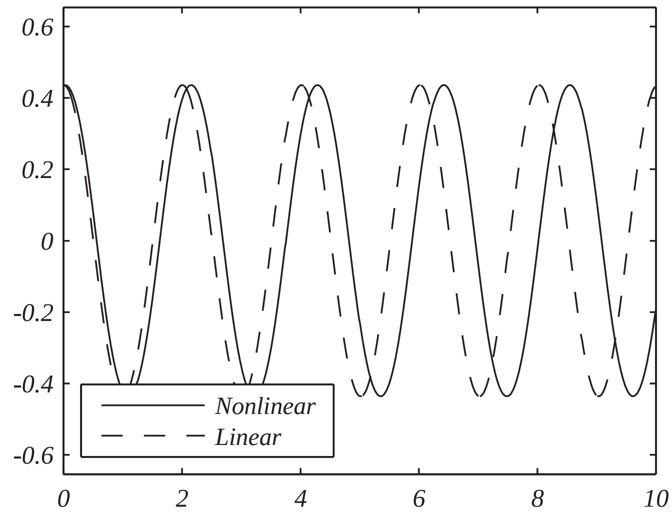



Figure 3.52: Comparison of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 50^o$.

c. $y' = y(y-2)(y+3)$.
d. $y' = y^2(y+1)(y-4)$.

**3.** For $y' = y - y^2$, find the general solution corresponding to $y(0) = y_0$. Provide specific solutions for the following initial conditions and sketch them: a. $y(0) = 0.25$, b. $y(0) = 1.5$, and c. $y(0) = -0.5$.

**4.** For each problem determine equilibrium points, bifurcation points and construct a bifurcation diagram. Discuss the different behaviors in each system.

a. $y' = y - \mu y^2$
b. $y' = y(\mu - y)(\mu - 2y)$
c. $x' = \mu - x^3$
d. $x' = x - \frac{\mu x}{1+x^2}$

**5.** Consider the family of differential equations $x' = x^3 + \delta x^2 - \mu x$.

a. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta = 0$.
b. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta > 0$.

Hint: Pick a few values of $\delta$ and $\mu$ in order to get a feel for how this system behaves.

**6.** System 3.52 can be solved exactly. Integrate the $r$-equation using separation of variables. For initial conditions a) $r(0) = 0.25$, $\theta(0) = 0$, and b) $r(0) = 1.5$, $\theta(0) = 0$, and $\mu = 1.0$, find and plot the solutions in the $xy$-plane showing the approach to a limit cycle.

**7.** Consider the system

$$\begin{aligned} x' &= -y + x\left[\mu - x^2 - y^2\right], \\ y' &= x + y\left[\mu - x^2 - y^2\right]. \end{aligned}$$

Rewrite this system in polar form. Look at the behavior of the $r$ equation and construct a bifurcation diagram in $\mu r$ space. What might this diagram look like in the three dimensional $\mu xy$ space? (Think about the symmetry in this problem.) This leads to what is called a *Hopf bifurcation.*

**8.** Find the fixed points of the following systems. Linearize the system about each fixed point and determine the nature and stability in the neighborhood of each fixed point, when possible. Verify your findings by plotting phase portraits using a computer.

a.

$$\begin{aligned} x' &= x(100 - x - 2y), \\ y' &= y(150 - x - 6y). \end{aligned}$$

b.

$$\begin{aligned} x' &= x + x^3, \\ y' &= y + y^3. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - x^2 + xy, \\ y' &= 2y - xy - 6y^2. \end{aligned}$$

d.

$$\begin{aligned} x' &= -2xy, \\ y' &= -x + y + xy - y^3. \end{aligned}$$

**9.** Plot phase portraits for the Lienard system

$$\begin{aligned} x' &= y - \mu(x^3 - x) \\ y' &= -x. \end{aligned}$$

for a small and a not so small value of $\mu$. Describe what happens as one varies $\mu$.

**10.** Consider the period of a nonlinear pendulum. Let the length be $L = 1.0$ m and $g = 9.8\ \mathrm{m/s^2}$. Sketch $T$ vs the initial angle $\theta_0$ and compare the linear and nonlinear values for the period. For what angles can you use the linear approximation confidently?

**11.** Another population model is one in which species compete for resources, such as a limited food supply. Such a model is given by

$$x' = ax - bx^2 - cxy,$$
$$y' = dy - ey^2 - fxy.$$

In this case, assume that all constants are positive.

a Describe the effects/purpose of each terms.

b Find the fixed points of the model.

c Linearize the system about each fixed point and determine the stability.

d From the above, describe the types of solution behavior you might expect, in terms of the model.

**12.** Consider a model of a food chain of three species. Assume that each population on its own can be modeled by logistic growth. Let the species be labeled by $x(t)$, $y(t)$, and $z(t)$. Assume that population $x$ is at the bottom of the chain. That population will be depleted by population $y$. Population $y$ is sustained by $x$'s, but eaten by $z$'s. A simple, but scaled, model for this system can be given by the system

$$\begin{aligned} x' &= x(1-x) - xy \\ y' &= y(1-y) + xy - yz \\ z' &= z(1-z) + yz. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Find the eigenvalues and eigenvectors.

d. Describe the solution behavior near each equilibrium point.

e. Which of these equilibria are important in the study of the population model and describe the interactions of the species in the neighborhood of these point(s).

**13.** Derive the first integral of the Lotka-Volterra system, $a \ln y + d \ln x - cx - by = C$.

**14.** Show that the system $x' = x - y - x^3$, $y' = x + y - y^3$, has a unique limit cycle by picking an appropriate $\psi(x, y)$ in Dulac's Criteria.

**15.** The Lorenz model is a simple model for atmospheric convection developed by Edward Lorenz in 1963. The system is given by the three equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Determine any bifurcation points and describe what happens near the bifurcation point(s). Consider $\sigma = 10$, $\beta = 8/3$, and vary $\rho$.

d. This system is know to exhibit chaotic behavior. Lorenz found a so-called strange attractor for parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. Using a computer, locate this strange attractor.

**16.** The Michaelis-Menten kinetics reaction is given by

$$\boxed{E + S \underset{k_1}{\overset{k_3}{\longleftrightarrow}} ES \underset{k_2}{\longrightarrow} E + P.}$$

The resulting system of equations for the chemical concentrations is

$$\begin{aligned}
\frac{d[S]}{dt} &= -k_1[E][S] + k_3[ES], \\
\frac{d[E]}{dt} &= -k_1[E][S] + (k_2 + k_3)[ES], \\
\frac{d[ES]}{dt} &= k_1[E][S] - (k_2 + k_3)[ES], \\
\frac{d[P]}{dt} &= k_2[ES].
\end{aligned} \tag{3.95}$$

In chemical kinetics one seeks to determine the rate of product formation ($v = d[P]/dt = k_3[ES]$). Assuming that $[ES]$ is a constant, find $v$ as a function of $[S]$ and the total enzyme concentration $[E_T] = [E] + [ES]$. As a nonlinear dynamical system, what are the equilibrium points?

**17.** In Equation (2.58) we saw a linear version of an epidemic model. The commonly used nonlinear SIR model is given by

$$\begin{aligned}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I,
\end{aligned} \tag{3.96}$$

where $S$ is the number of susceptible individuals, $I$ is the number of infected individuals, and $R$ are the number who have been removed from the the other groups, either by recovering or dying.

a. Let $N = S + I + R$ be the total population. Prove that $N =$ constant. Thus, one need only solve the first two equations and find $R = N - S - I$ afterwards.

b. Find and classify the equilibria. Describe the equilibria in terms of the population behavior.

c. Let $\beta = 0.05$ and $\gamma = 0.2$. Assume that in a population of 100 there is one infected person. Numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

d. The equations can be modified by adding constant birth and death rates. Assuming these are te same, one would have a new system.

$$\frac{dS}{dt} = -\beta SI + \mu(N - S)$$

$$\begin{aligned} \frac{dI}{dt} &= \beta SI - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{3.97}$$

How does this affect any equilibrium solutions?

e. Again, let $\beta = 0.05$ and $\gamma = 0.2$. Let $\mu = 0.1$ For a population of 100 with one infected person numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

**18.** An undamped, unforced Duffing equation, $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$, can be solved exactly in terms of elliptic functions. Using the results of Exercise 3.18, determine the solution of this equation and determine if there are any restrictions on the parameters.

**19.** Determine the circumference of an ellipse in terms of an elliptic integral.

**20.** Evaluate the following in terms of elliptic integrals and compute the values to four decimal places.

a. $\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$.

b. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}$.

c. $\int_0^{2} \frac{dx}{\sqrt{(9-x^2)(4-x^2)}}$.

d. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$.

e. $\int_1^{\infty} \frac{dx}{\sqrt{x^4-1}}$.

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