Chapter 3

Nonlinear Systems

"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful." - Jules Henri Poincaré (1854-1912)

3.1 Introduction

Some of the most interesting phenomena in the world are modeled by nonlinear systems. These systems can be modeled by differential equations when time is considered as a continuous variable or difference equations when time is treated in discrete steps. Applications involving differential equations can be found in many physical systems such as planetary systems, weather prediction, electrical circuits, and kinetics. Even in some simple dynamical systems a combination of damping and a driving force can lead to chaotic behavior. Namely, small changes in initial conditions could lead to very different outcomes. In this chapter we will explore a few different nonlinear systems and introduce some of the tools needed to investigate them. These tools are based on some of the material in Chapters 2 and 3 for linear systems of differential equations.

Nonlinear differential equations are either integrable, but difficult to solve, or they are not integrable and can only be solved numerically. We will see that we can sometimes approximate the solutions of nonlinear systems with linear systems in small regions of phase space and determine the qualitative behavior of the system without knowledge of the exact solution.

Nonlinear problems occur naturally. We will see problems from many of the same fields we explored in Section 2.3. We will concentrate mainly on continuous dynamical systems. We will begin with a simple population model and look at the behavior of equilibrium solutions of first order autonomous differential equations. We will then look at nonlinear systems in the plane, such as the nonlinear pendulum and other nonlinear oscillations. We will conclude by discussing a few other interesting physical examples stressing some of the key ideas of nonlinear dynamics.

3.2 The Logistic Equation

In this section we will explore a simple nonlinear population model. Typically, we want to model the growth of a given population, $y(t)$, and the differential equation governing the growth behavior of this population is developed in a manner similar to that used previously for mixing problems. Namely, we note that the rate of change of the population is given by an equation of the form

$$\frac{dy}{dt} = \text{Rate In} - \text{Rate Out}.$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. While there are other potential contributions to these rates we will consider the birth and death rates in the simplest examples.

A simple population model can be obtained if one assumes that these rates are linear in the population. Thus, we assume that the

$$\text{Rate In} = by \text{ and the } \text{Rate Out} = my.$$

Here we have denoted the birth rate as b and the mortality rate as m. This gives the rate of change of population as

$$\frac{dy}{dt} = by - my. \tag{3.1}$$

Generally, these rates could depend on the time. In the case that they are both constant rates, we can define $k = b - m$ and obtain the familiar exponential model of population growth:

Malthusian population growth.

$$\frac{dy}{dt} = ky.$$

This is easily solved and one obtains exponential growth $(k > 0)$ or decay $(k < 0)$. This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

When populations get large enough, there is competition for resources, such as space and food, which can lead to a higher mortality rate. Thus, the mortality rate may be a function of the population size, $m = m(y)$. The simplest model would be a linear dependence, $m = \tilde{m} + cy$. Then, the previous exponential model takes the form

$$\frac{dy}{dt} = ky - cy^2, \tag{3.2}$$

where $k = b - \tilde{m}$. This is known as the *logistic model* of population growth. Typically, c is small and the added nonlinear term does not really kick in until the population gets large enough.

The logistic model was first published in 1838 by Pierre François Verhulst (1804-1849) in the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where N is the population at time t, r is the growth rate, and K is what is called the carrying capacity. Note that in our model $r = k = Kc$.

Example 3.1. Show that Equation (3.2) can be written in the form

$$z' = kz(1-z)$$

which has only one parameter.

We carry this out be rescaling the population, $y(t) = \alpha z(t)$, where α is to be determined. Inserting this transformation, we have

$$\begin{aligned} y' &= ky - cy^2 \\ \alpha z' &= \alpha kz - c\alpha^2 z^2, \end{aligned}$$

or

$$z' = kz\left(1 - \alpha\frac{c}{k}z\right).$$

Thus, we obtain the result, $z' = kz(1-z)$, if we pick $\alpha = \frac{k}{c}$.

Before we obtain the exact solution, it is instructive to study the qualitative behavior of the solutions without actually writing down any explicit solutions. Such methods are useful for more difficult nonlinear equations as we will see later in this chapter.

We will demonstrate this analysis with a simple logistic equation example. We will first look for constant solutions, called equilibrium solutions, satisfying $y'(t) = 0$. Then, we will look at the behavior of solutions near the equilibrium solutions, or fixed points, and determine the stability of the equilibrium solutions. In the next section we will extend these ideas to other first order differential equations.

Example 3.2. Find and classify the equilibrium solutions of the logistic equation,

$$\frac{dy}{dt} = y - y^2. \tag{3.3}$$

First, we determine the equilibrium, or constant, solutions given by $y' = 0$. For this case, we have $y - y^2 = 0$. So, the equilibrium solutions are $y = 0$ and $y = 1$.

These solutions divide the ty-plane into three regions, $y < 0$, $0 < y < 1$, and $y > 1$. Solutions that originate in one of these regions at $t = t_0$ will remain in that region for all $t > t_0$ since solutions of this differential equation cannot intersect.

Note: If two solutions of the differential equation intersect then they have common values y_1 at time t_1. Using this information, we could set up an initial value problem for which the initial condition is $y(t_1) = y_1$. Since the two different solutions intersect at this point in the phase plane, we would have an initial value problem with two different solutions. This would violate the uniqueness theorem for initial value problems.

Next, we determine the behavior of solutions in the three regions. Noting that $y'(t)$ gives the slope of any solution in the plane, then we find that the solutions are monotonic in each region. Namely, in regions where $y'(t) > 0$, we have monotonically increasing functions and in regions where $y'(t) < 0$, we have monotonically decreasing functions. We determine the sign of $y'(t)$ from the right-hand side of the differential equation.

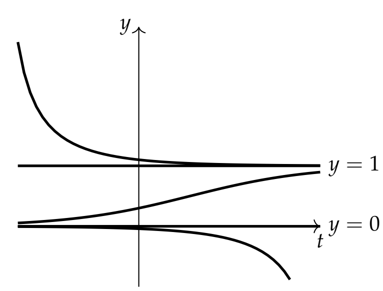


Figure 3.1: Representative solution behavior for $y' = y - y^2$.

For example, in this problem $y - y^2 > 0$ only for the middle region and $y - y^2 < 0$ for the other two regions. Thus, the slope is positive in the middle region, giving a rising solution as shown in Figure 3.1. Note that this solution does not cross the equilibrium solutions. Similar statements can be made about the solutions in the other regions.

Stable and unstable equilibria.

We further note that the solutions on either side of the equilibrium solution $y = 1$ tend to approach this equilibrium solution for large values of t. In fact, no matter how far these solutions are from $y = 1$, as long as $y(t) > 0$, the solutions will eventually approach this equilibrium solution as $t \to \infty$. We then say that the equilibrium solution, $y = 1$, is a *stable equilibrium*.

Similarly, we note that the solutions on either side of the equilibrium solution $y = 0$ tend away from $y = 0$ for large values of t. No matter how close a solution is to $y = 0$ at some given time, eventually these solutions will diverge as $t \to \infty$. We say that such equilibrium solutions are *unstable equilibria*.

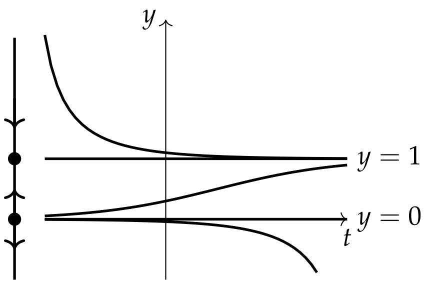


Figure 3.2: Representative solution behavior and the phase line for $y' = y - y^2$.

Phase lines.

If we are only interested in the behavior of the equilibrium solutions, we could just display a *phase line*. In Figure 3.2 we place a vertical line to the right of the ty-plane plot. On this line we first place dots at the corresponding equilibrium solutions and label the solutions. These points divide the phase line into three intervals.

In each interval we then place arrows pointing upward or downward indicating solutions with positive or negative slopes, respectively. For example, for the interval $y > 1$ there is a downward pointing arrow indicating that the slope is negative in that region.

Looking at the resulting phase line we can determine if a given equilibrium is stable (arrows pointing towards the point) or unstable (arrows pointing away from the point). In Figure 3.3 we draw the final phase line by itself. We see that $y = 1$ is a stable equilibrium point and $y = 0$ is an unstable equilibrium point.

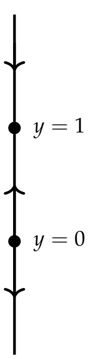


Figure 3.3: Phase line for $y' = y - y^2$.

3.2.1 *The Riccati Equation**

We have seen that one does not need an explicit solution of the logistic equation (3.2) in order to study the behavior of its solutions. However, the logistic equation is an example of a nonlinear first order equation that is solvable. It is also an example of a general Riccati equation, a first order differential equation quadratic in the unknown function.

The Riccati equation is named after the Italian mathematician Jacopo Francesco Riccati (1676-1754). When $a(t) = 0$, the equation becomes a Bernoulli equation.

The general form of the *Riccati equation* is

$$\frac{dy}{dt} = a(t) + b(t)y + c(t)y^2. \tag{3.4}$$

As long as $c(t) \neq 0$, this equation can be reduced to a second order linear differential equation through the transformation

$$y(t) = -\frac{1}{c(t)}\frac{x'(t)}{x(t)}.$$

We will demonstrate the use of this transformation in obtaining the solution of the logistic equation.

Example 3.3. Solve the logistic equation

$$\frac{dy}{dt} = ky - cy^2 \tag{3.5}$$

using the transformation

$$y = \frac{1}{c}\frac{x'}{x}.$$

differentiating this transformation with respect to t, we obtain

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{c}\left[\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right] \\ &= \frac{1}{c}\left[\frac{x''}{x} - (cy)^2\right] \\ &= \frac{1}{c}\frac{x''}{x} - cy^2. \end{aligned} \tag{3.6}$$

Inserting this result into the logistic equation (3.5), we have

$$\frac{1}{c}\frac{x''}{x} - cy^2 = k\frac{1}{c}\left(\frac{x'}{x}\right) - cy^2.$$

Simplifying, we see that the logistic equation has been reduced to a second order linear, differential equation,

$$x'' = kx'.$$

This equation is readily solved. One integration gives

$$x'(t) = Be^{kt}.$$

A second integration gives

$$x(t) = A + Be^{kt},$$

where A and B are two arbitrary constants.

Inserting this result into the Riccati transformation, we obtain

$$y(t) = \frac{1}{c}\frac{x'}{x} = \frac{kBe^{kt}}{c(A + Be^{kt})}.$$

It appears that we have two arbitrary constants. However, we started out with a first order differential equation and so we expect only one arbitrary constant. We can resolve this dilemma by dividing[1] the numerator and denominator by Be^{kt} and defining $C = \frac{A}{B}$. Then, we have the solution

$$y(t) = \frac{k/c}{1 + Ce^{-kt}}, \tag{3.7}$$

showing that there really is only one arbitrary constant in the solution.

[1] This general solution holds for $B \neq 0$. If $B = 0$, then we have $x(t) = A$ and, thus, $y(t)$ is the constant equilibrium solution.

Plots of the solution (3.7) of the logistic equation for different initial conditions gives the solutions seen in the last section. In particular, setting all of the constants to unity, we have the sigmoid function,

$$y(t) = \frac{1}{1+e^{-t}}.$$

This is the signature S-shaped curve of the logistic model as shown in Figure 3.4. We should note that this is not the only way to obtain the solution to the logistic equation, though this approach has provided us with an introduction to Riccati equations. A more direct approach would be to use separation of variables on the logistic equation, which is Problem 1.

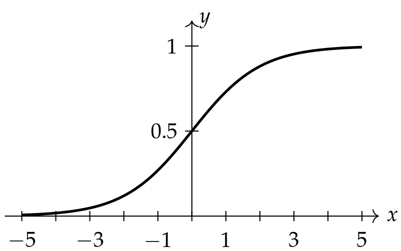


Figure 3.4: Plot of the sigmoid function.

3.3 *Autonomous First Order Equations*

IN THIS SECTION WE WILL STUDY THE STABILITY of nonlinear first order autonomous equations. We will then extend this study in the next section to looking at families of first order equations which are connected through a parameter.

Recall that a first order autonomous equation is given in the form

$$\frac{dy}{dt} = f(y). \tag{3.8}$$

We will assume that f and $\frac{\partial f}{\partial y}$ are continuous functions of y, so that we know that solutions of initial value problems exist and are unique.

A solution $y(t)$ of Equation (3.8) is called an *equilibrium solution*, or a *fixed point* solution, if it is a constant solution satisfying $y'(t) = 0$. Such solutions are the roots of the right-hand side of the differential equation, $f(y) = 0$.

Example 3.4. Find the equilibrium solutions of $y' = 1 - y^2$.

The equilibrium solutions are the roots of $f(y) = 1 - y^2 = 0$. The equilibria are found to be $y = \pm 1$.

Once we have determined the equilibrium solutions, we would like to classify them. Are they stable or unstable? As we had seen previously, we are interested in the behavior of solutions near the equilibria. This classification can be determined using a linearization of the given equation. This will provide an analytic criteria to establish the stability of equilibrium solutions without geometrically drawing the phase lines as we had done previously.

Linearization of first order equations.

Let y^* be an equilibrium solution of Equation (3.8). Then, any solution can be written in the form

$$y(t) = y^* + \xi(t),$$

where $\xi(t)$ measures how far the solution is from the equilibrium at any given time.

Inserting this form into Equation (3.8), we have

$$\frac{d\xi}{dt} = f(y^* + \xi).$$

We now consider small $\xi(t)$ in order to study solutions near the equilibrium solution. For such solutions, we can expand $f(y)$ about the equilibrium solution,

$$f(y^* + \xi) = f(y^*) + f'(y^*)\xi + \frac{1}{2!}f''(y^*)\xi^2 + \cdots.$$

Since y^* is an equilibrium solution, $f(y^*) = 0$, the first term in the Taylor series vanishes. If the first derivative does not vanish, then for solutions close to equilibrium, we can neglect higher order terms in the expansion. Then, $\xi(t)$ approximately satisfies the differential equation

$$\frac{d\xi}{dt} = f'(y^*)\xi. \tag{3.9}$$

This is called a linearization of the original nonlinear equation about the equilibrium point. This equation has exponential solutions for $f'(y^*) \neq 0$,

$$\xi(t) = \xi_0 e^{f'(y^*)t}.$$

Now we see how the stability criteria arise. If $f'(y^*) > 0$, $\xi(t)$ grows in time. Therefore, nearby solutions stray from the equilibrium solution for large times. On the other hand, if $f'(y^*) < 0$, $\xi(t)$ decays in time and nearby solutions approach the equilibrium solution for large t. Thus, we have the results:

$$\begin{aligned} f'(y^*) < 0, &\quad y^* \text{ is stable.} \\ f'(y^*) > 0, &\quad y^* \text{ is unstable.} \end{aligned} \tag{3.10}$$

The stability criteria for equilibrium solutions of a first order differential equation.

Example 3.5. Determine the stability of the equilibrium solutions of $y' = 1 - y^2$.

In the last example we found the equilibrium solutions, $y^* = \pm 1$. The stability criteria require computing

$$f'(y^*) = -2y^*.$$

For this problem we have $f'(\pm 1) = \mp 2$. Therefore, $y^* = 1$ is a stable equilibrium and $y^* = -1$ is an unstable equilibrium.

Example 3.6. Find and classify the equilibria for the logistic equation $y' = y - y^2$.

We had already investigated this problem using phase lines. There are two equilibria, $y = 0$ and $y = 1$.

We next apply the stability criteria. Noting that $f'(y) = 1 - 2y$, the first equilibrium solution gives $f'(0) = 1$. So, $y = 0$ is an unstable equilibrium. Since $f'(1) = -1 < 0$, we see that $y = 1$ is a stable equilibrium. These results are the same as we hade determined earlier using phase lines.

3.4 *Bifurcations for First Order Equations*

We now consider families of first order autonomous differential equations of the form

$$\frac{dy}{dt} = f(y; \mu).$$

Here μ is a parameter that we can change and then observe the resulting behaviors of the solutions of the differential equation. When a small change in the parameter leads to changes in the behavior of the solution, then the system is said to undergo a *bifurcation*. The value of the parameter, μ, at which the bifurcation occurs is called a *bifurcation point*.

Bifurcations and bifurcation points.

We will consider several generic examples, leading to special classes of bifurcations of first order autonomous differential equations. We will study the stability of equilibrium solutions using both phase lines and the stability criteria developed in the last section

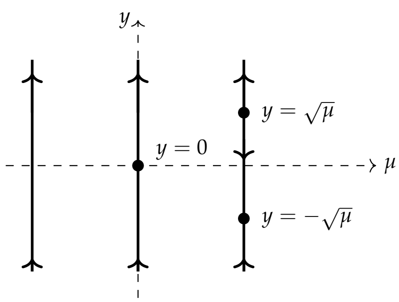


Figure 3.5: Phase lines for $y' = y^2 - \mu$. On the right $\mu > 0$ and on the left $\mu < 0$.

Example 3.7. $y' = y^2 - \mu$.

First note that equilibrium solutions occur for $y^2 = \mu$. In this problem, there are three cases to consider.

1. $\mu > 0$.

   In this case there are two real solutions of $y^2 = \mu$, $y = \pm\sqrt{\mu}$. Note that $y^2 - \mu < 0$ for $|y| < \sqrt{\mu}$. So, we have the right phase line in Figure 3.5.

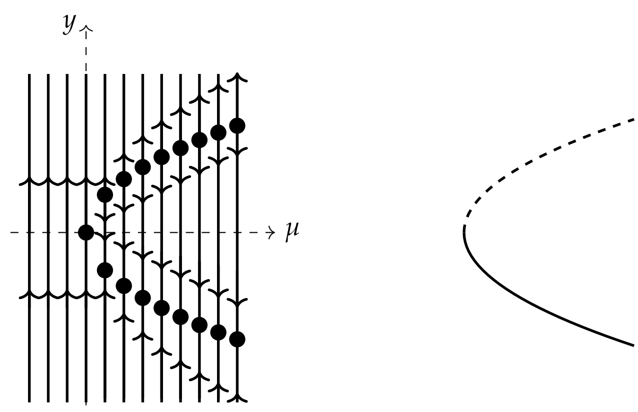
2. $\mu = 0$.

   There is only one equilibrium point at $y = 0$. The equation becomes $y' = y^2$. It is obvious that the right side of this equation is never negative. So, the phase line, which is shown as the middle line in Figure 3.5, has upward pointing arrows.

3. $\mu < 0$.

   In this case there are no equilibrium solutions. Since $y^2 - \mu > 0$, the slopes for all solutions are positive as indicated by the last phase line in Figure 3.5.

We can also confirm the behaviors of the equilibrium points by noting that $f'(y) = 2y$. Then, $f'(\pm\sqrt{\mu}) = \pm 2\sqrt{\mu}$ for $\mu \geq 0$. Therefore, the equilibria $y = +\sqrt{\mu}$ are unstable equilibria for $\mu > 0$. Similarly, the equilibria $y = -\sqrt{\mu}$ are stable equilibria for $\mu > 0$.



(a) (b)

Figure 3.6: (a) The typical phase lines for $y' = y^2 - \mu$. (b) Bifurcation diagram for $y' = y^2 - \mu$. This is an example of a saddle-node bifurcation.

We can combine these results for the phase lines into one diagram known as a bifurcation diagram. We will plot the equilibrium solutions and their phase lines $y = \pm\sqrt{\mu}$ in the μy-plane. We begin by lining up the phase lines for various μ's. These are shown on the left side of Figure 3.6. Note the pattern of equilibrium points lies on the parabolic curve $y^2 = \mu$. The upper branch of this curve is a collection of unstable equilibria and the bottom is a stable branch. So, we can dispose of the phase lines and just keep the equilibria. However, we will draw the unstable branch as a dashed line and the stable branch as a solid line.

The bifurcation diagram is displayed on the right side of Figure 3.6. This type of bifurcation is called a *saddle-node bifurcation*. The point $\mu = 0$ at which the behavior changes is the *bifurcation point. As μ changes from negative to positive values, the system goes from having no equilibria to having one stable and one unstable equilibrium point.*

Example 3.8. $y' = y^2 - \mu y$.

Writing this equation in factored form, $y' = y(y - \mu)$, we see that there are two equilibrium points, $y = 0$ and $y = \mu$. The behavior of the solutions depends upon the sign of $y' = y(y - \mu)$. This leads to four cases with the indicated signs of the derivative. The regions indicating the signs of y' are shown in Figure 3.7.

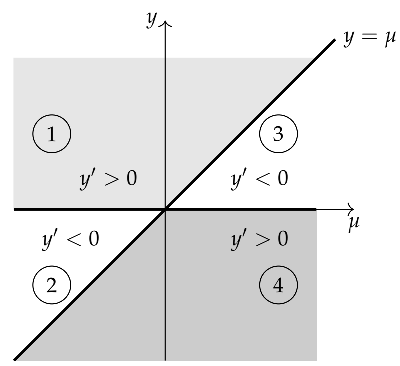


Figure 3.7: The regions indicating the different signs of the derivative for $y' = y^2 - \mu y$.

1. $y > 0, y - \mu > 0 \Rightarrow y' > 0$.
2. $y < 0, y - \mu > 0 \Rightarrow y' < 0$.
3. $y > 0, y - \mu < 0 \Rightarrow y' < 0$.
4. $y < 0, y - \mu < 0 \Rightarrow y' > 0$.

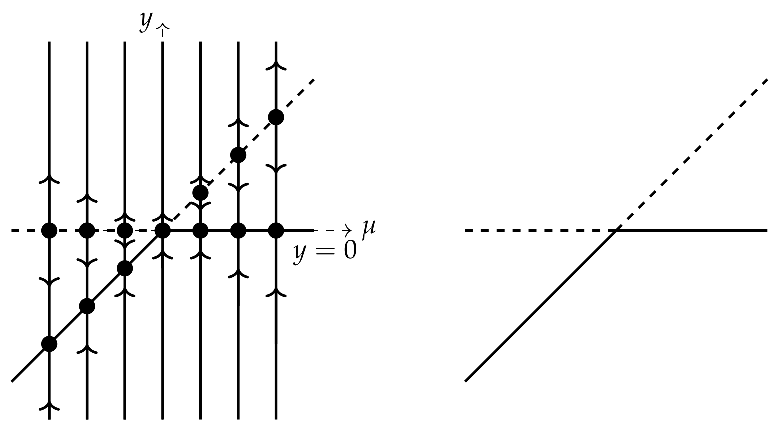
The corresponding phase lines and superimposed bifurcation diagram are shown in figure 3.8. The bifurcation diagram is on the right side of Figure 3.8 and this type of bifurcation is called a *transcritical bifurcation*.

Again, the stability can be determined from the derivative $f'(y) = 2y - \mu$ evaluated at $y = 0, \mu$. From $f'(0) = -\mu$, we see that $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. Similarly, $f'(\mu) = \mu$ implies that $y = \mu$ is unstable for $\mu > 0$ and stable for $\mu < 0$. These results are consistent with the phase line plots.

Example 3.9. $y' = y^3 - \mu y$.

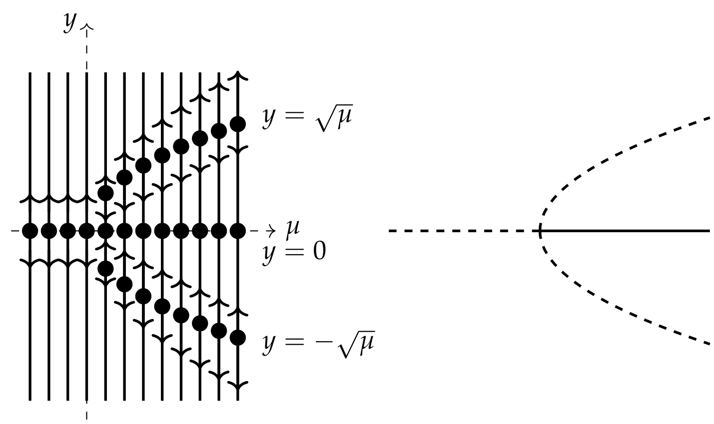
For this last example, we find from $y^3 - \mu y = y(y^2 - \mu) = 0$ that there are two cases.

1. $\mu < 0$. In this case there is only one equilibrium point at $y = 0$. For positive values of y we have that $y' > 0$ and for negative values of y we have that $y' < 0$. Therefore, this is an unstable equilibrium point.
2. $\mu > 0$. Here we have three equilibria, $y = 0, \pm\sqrt{\mu}$. A careful investigation shows that $y = 0$ is a stable equilibrium point and that the other two equilibria are unstable.



(a) (b)

Figure 3.8: (a) Collection of phase lines for $y' = y^2 - \mu y$. (b) Bifurcation diagram for $y' = y^2 - \mu y$. This is an example of a transcritical bifurcation.



(a) (b)

Figure 3.9: (a) The phase lines for $y' = y^3 - \mu y$. The left one corresponds to $\mu < 0$ and the right phase line is for $\mu > 0$. (b)Bifurcation diagram for $y' = y^3 - \mu y$. This is an example of a pitchfork bifurcation.

In Figure 3.9 we show the phase lines for these two cases. The corresponding bifurcation diagram is then sketched on the right side of Figure 3.9. For obvious reasons this has been labeled a *pitchfork bifurcation*.

When two of the prongs of the pitchfork are unstable branches, the bifurcation is called a subcritical pitchfork bifurcation. When two prongs are stable branches, the bifurcation is a supercritical pitchfork bifurcation.

Since $f'(y) = 3y^2 - \mu$, the stability analysis gives that $f'(0) = -\mu$. So, $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. For $\mu > 0$, we have that $f'(\pm\sqrt{\mu}) = 2\mu$. Therefore, $y = \pm\sqrt{\mu}$, $\mu > 0$, is unstable. Thus, we have a subcritical pitchfork bifurcation.

3.5 *The Stability of Fixed Points in Nonlinear Systems*

We next investigate the stability of the equilibrium solutions of the nonlinear pendulum which we first encountered in Section 1.4.2. Along the way we will develop some basic methods for studying the stability of equilibria in nonlinear systems in general.

Recall that the derivation of the pendulum equation was based upon a simple point mass m hanging on a string of length L from some support as shown in Figure 3.10. One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time, $\theta(t)$.

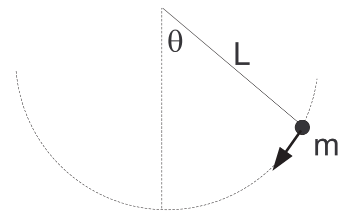


Figure 3.10: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

In Chapter 2 we derived the nonlinear pendulum equation,

$$L\ddot{\theta} + g\sin\theta = 0. \tag{3.11}$$

There are several variations of Equation (3.11) which we have used in this text. The first one is the linear pendulum, which was obtained using a small angle approximation,

$$L\ddot{\theta} + g\theta = 0. \tag{3.12}$$

We also made the system more realistic by adding damping and forcing. A variety of these oscillation problems are summarized in the table below.

Equations for Pendulum Motion

1. Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0.$
2. Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0.$
3. Linear Pendulum: $L\ddot{\theta} + g\theta = 0.$
4. Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0.$
5. Forced Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t.$
6. Forced Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t.$

There are two simple systems that we will consider, the damped linear pendulum, in the form

$$x'' + bx' + \omega^2 x = 0$$

and the the damped nonlinear pendulum, in the form

$$x'' + bx' + \omega^2 \sin x = 0.$$

These are second order differential equations and can be cast as a system of two first order differential equations using the methods of Chapter 6.

The linear equation can be written as

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 x. \end{aligned} \tag{3.13}$$

This system has only one equilibrium solution, $x = 0, y = 0$.

The damped nonlinear pendulum takes the form

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.14}$$

This system also has the equilibrium solution $x = 0$, $y = 0$. However, there are actually an infinite number of solutions. The equilibria are determined from

$$y = 0 \text{ and } -by - \omega^2 \sin x = 0. \tag{3.15}$$

These equations imply that $y = 0$ and $\sin x = 0$. There are an infinite number of solutions to the latter equation: $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So, this system has an infinite number of equilibria, $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

The next step is to determine the stability of the equilibrium solutions these systems. This can be accomplished just as we had done for first order equations. To do this we need a more general theory for nonlinear systems. So, we will develop the needed machinery.

We begin with the $n-$dimensional system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n. \tag{3.16}$$

Linear stability analysis of systems.

Here $\mathbf{f}: \mathrm{R}^n \to \mathrm{R}^n$ is a mapping from R^n to R^n. We define the equilibrium solutions, or fixed points, of this system as the points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

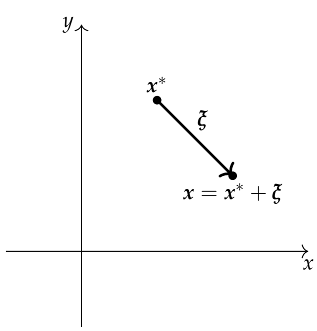


Figure 3.11: A general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$,

The stability in the neighborhood of equilibria will now be determined. We are interested in what happens to solutions of the system with initial conditions starting near a fixed point. We will represent a general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$. We note that the length of $\boldsymbol{\xi}$ gives an indication of how close we are to the fixed point. So, we consider that initially, $|\boldsymbol{\xi}| \ll 1$.

As the system evolves, $\boldsymbol{\xi}$ will change. The change of $\boldsymbol{\xi}$ in time is in turn governed by a system of equations. We can approximate this evolution as follows. First, we note that

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

We can expand the right side about the fixed point using a multidimensional version of Taylor's Theorem. Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

The Jacobian matrix.

Here $D\mathbf{f}(\mathbf{x})$ is the *Jacobian matrix*, defined as

$$D\mathbf{f}(\mathbf{x}^*) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Noting that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we then have that system (3.16) becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi}. \tag{3.17}$$

It is this equation which describes the behavior of the system near the fixed point. As with first order equations, we say that system (3.16) has been linearized or that Equation (3.17) is the linearization of system (3.16).

Linearization of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

The stability of the equilibrium point of the nonlinear system is now reduced to analyzing the behavior of the linearized system given by Equation (3.17). We can use the methods from the last two chapters to investigate the eigenvalues of the Jacobian matrix evaluated at each equilibrium point. We will demonstrate this procedure with several examples.

Example 3.10. Determine the equilibrium points and their stability for the system

$$\begin{aligned} x' &= -2x - 3xy, \\ y' &= 3y - y^2. \end{aligned} \tag{3.18}$$

We first determine the fixed points. Setting the right-hand side equal to zero and factoring, we have

$$\begin{aligned} -x(2+3y) &= 0, \\ y(3-y) &= 0. \end{aligned} \tag{3.19}$$

From the second equation, we see that either $y = 0$ or $y = 3$. The first equation then gives $x = 0$ in either case. So, there are two fixed points: $(0,0)$ and $(0,3)$.

Next, we linearize the system of differential equations about each fixed point. First, we note that the Jacobian matrix is given by

$$D\mathbf{f}(x,y) = \begin{pmatrix} -2-3y & -3x \\ 0 & 3-2y \end{pmatrix}. \tag{3.20}$$

1. Case I Equilibrium point $(0,0)$.

   In this case we find that

   $$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3.21}$$

   Therefore, the linearized equation becomes

   $$\xi' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{\xi}. \tag{3.22}$$

   This is equivalently written out as the system

   $$\begin{aligned} \xi_1' &= -2\xi_1, \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{3.23}$$

   This is the linearized system about the origin. Note the similarity with the original system.

   We should emphasize that the linearized equations are constant coefficient equations and we can use matrix methods to determine the nature of the equilibrium point. The eigenvalues of this system are obviously $\lambda = -2, 3$. Therefore, we have that the origin is a saddle point.

2. Case II Equilibrium point $(0, 3)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}(0,3) = \begin{pmatrix} -11 & 0 \\ 0 & -3 \end{pmatrix}. \tag{3.24}$$

   The eigenvalues are $\lambda = -11, -3$. So, this fixed point is a stable node.

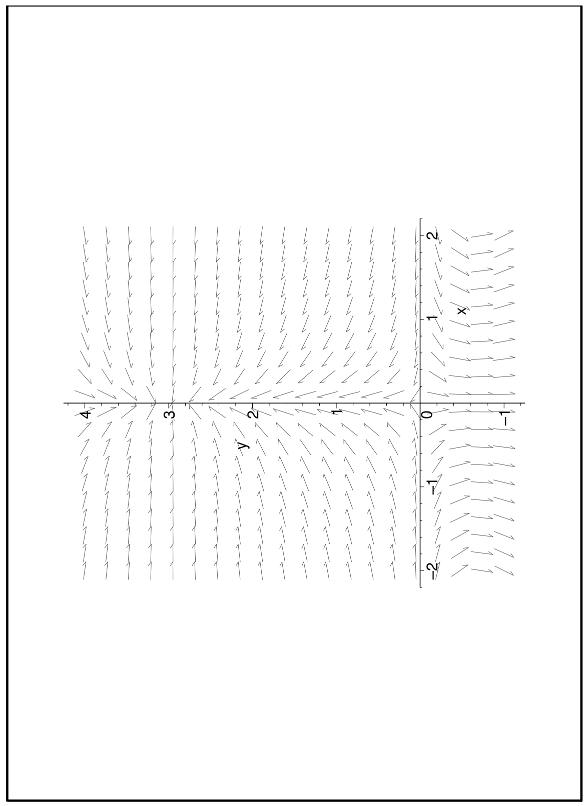


Figure 3.12: Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

This analysis has given us a saddle and a stable node. We know what the behavior is like near each fixed point, but we have to resort to other means to say anything about the behavior far from these points. The phase portrait for this system is given in Figure 3.12. You should be able to locate the saddle point and the node in the figure. Notice how solutions behave in regions far from these points.

We can expect to be able to perform a linearization under general conditions. These are given in the *Hartman-Großman Theorem*:

Theorem 3.1. *A continuous map exists between the linear and nonlinear systems when $D\mathbf{f}(\mathbf{x}^*)$ does not have any eigenvalues with zero real part.*

Generally, there are several types of behavior that one can see in nonlinear systems. One can see sinks or sources, hyperbolic (saddle) points, elliptic points (centers) or foci. We have defined some of these for planar systems. In general, if at least two eigenvalues have real parts with opposite signs, then the fixed point is a *hyperbolic point*. If the real part of a nonzero eigenvalue is zero, then we have a center, or *elliptic point.*

For linear systems in the plane, this classification was done in Chapter 6. The Jacobian matrix evaluated at the equilibrium points is simply the 2×2 coefficient matrix we had called A.

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.25}$$

Here we are using $J = D\mathbf{f}(\mathbf{x}^*)$.

The eigenvalue equation is given by

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

However, $a + d$ is the trace, $\operatorname{tr}(J)$ and $\det(J) = ad - bc$. Therefore, we can write the eigenvalue equation as

$$\lambda^2 - \operatorname{tr}(J)\lambda + \det(J) = 0.$$

The solution of this equation is found using the quadratic formula,

$$\lambda = \frac{1}{2}\left[-\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}^2(J) - 4\det(J)}\right].$$

We had seen in previous chapter that equilibrium points in planar systems can be classified as nodes, saddles, centers, or spirals (foci). The type of behavior can be determined from solutions of the eigenvalue equation. Since the nature of the eigenvalues depends on the trace and determinant of the Jacobian matrix at the equilibrium point, we can relate the types of equilibria to points in the det-tr plane. This is shown in Figure 3.13, which is similar to Figure 2.25.

In Figure 3.13 the parabola $\operatorname{tr}^2(J) = 4\det(J)$ divides the det-tr plane. Points on this curve give a vanishing discriminant in the computation of the eigenvalues. In these cases one finds repeated roots, or eigenvalues. Along this curve one can find stable and unstable degenerate nodes. Also along this line are stable and unstable proper nodes, called star nodes. These arise from systems of the form $x' = ax$, $y' = ay$.

In the case that $\det(J) < 0$, we have that the discriminant

$$\Delta \equiv \operatorname{tr}^2(J) - 4\det(J)$$

is positive. Not only that, $\Delta > \operatorname{tr}^2(J)$. Thus, we obtain two real and distinct eigenvalues with opposite signs. These lead to saddle points.

In the case that $\det(J) > 0$, we can have either $\Delta > 0$ or $\Delta < 0$. The discriminant is negative for points inside the parabolic curve. It is in this region that one finds centers and spirals, corresponding to complex eigenvalues. When $\operatorname{tr}(J) > 0$, there are unstable spirals. There are stable spirals when $\operatorname{tr}(J) < 0$. For the case that $\operatorname{tr}(J) = 0$, the eigenvalues are pure imaginary, giving centers.

There are several other types of behavior depicted in the figure, but we will now turn to studying a few of examples.

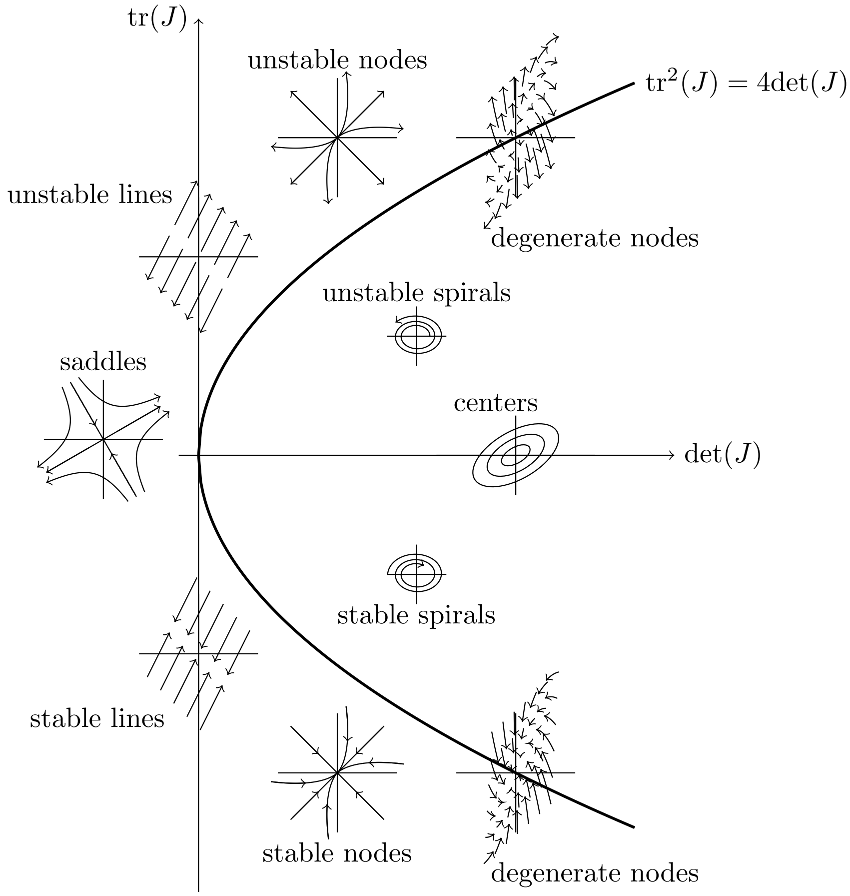


Figure 3.13: Diagram indicating the behavior of equilibrium points in the det − tr plane. The parabolic curve

$$\text{tr}^2(J) = 4\det(J)$$

indicates where the discriminant vanishes.

Example 3.11. Find and classify all of the equilibrium solutions of the nonlinear system

$$\begin{aligned} x' &= 2x - y + 2xy + 3(x^2 - y^2), \\ y' &= x - 3y + xy - 3(x^2 - y^2). \end{aligned} \tag{3.26}$$

In Figure 3.14 we show the direction field for this system. Try to locate and classify the equilibrium points visually. After the stability analysis, you should return to this figure and determine if you identified the equilibrium points correctly.

We will first determine the equilibrium points. Setting the right-hand side of each differential equation to zero, we have

$$\begin{aligned} 2x - y + 2xy + 3(x^2 - y^2) &= 0, \\ x - 3y + xy - 3(x^2 - y^2) &= 0. \end{aligned} \tag{3.27}$$

This system of algebraic equations can be solved exactly. Adding the equations, we have

$$3x - 4y + 3xy = 0.$$

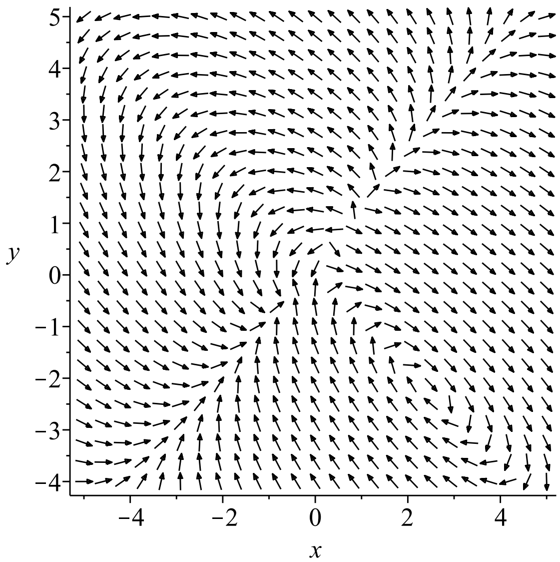


Figure 3.14: Phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$
$$y' = x - 3y + xy - 3(x^2 - y^2).$$

Solving for x,

$$x = \frac{4y}{3(1+y)},$$

and substituting the result for x into the first algebraic equation, we find an equation for y :

$$\frac{y(1-y)(9y^2+22y+5)}{3(1+y)^2} = 0.$$

The solutions to this equation are

$$y = 0, 1, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}.$$

The corresponding values for x are

$$x = 0, \frac{2}{3}, 1 \mp \frac{\sqrt{19}}{3}.$$

Now that we have located the equilibria, we can classify them. The Jacobian matrix is given by

$$D\mathbf{f}(x,y) = \begin{pmatrix} 6x+2y+2 & 2x-6y-1 \\ -6x+y+1 & x+6y-3 \end{pmatrix}. \tag{3.28}$$

Now, we evaluate the Jacobian at each equilibrium point and find the eigenvalues.

1. Case I. Equilibrium point $(0,0)$.
   In this case we find that

$$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix}. \tag{3.29}$$

   The eigenvalues of this matrix are $\lambda = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$. Therefore, the origin is a saddle point.

2. Case II. Equilibrium point $(\frac{2}{3}, 1)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(\frac{2}{3}, 1\right) = \begin{pmatrix} 8 & -\frac{17}{3} \\ -2 & \frac{11}{3} \end{pmatrix}. \tag{3.30}$$

   The eigenvalues are $\lambda = \frac{35}{6} \pm \frac{\sqrt{577}}{6} \approx 9.84, 1.83$. This fixed point is an unstable node.

3. Case III. Equilibrium point $(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19})$.

   The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}\right) = \begin{pmatrix} \frac{50}{9} \mp \frac{14}{9}\sqrt{19} & \frac{25}{3} \mp 2\sqrt{19} \\ -\frac{56}{9} \pm \frac{20}{9}\sqrt{19} & -\frac{28}{3} \pm \sqrt{19} \end{pmatrix}. \tag{3.31}$$

   There are two equilibrium points under this case. The first is given by

$$(1 - \frac{\sqrt{19}}{3}, -\frac{11}{9} + \frac{2}{9}\sqrt{19}) \approx (0.453, -0.254).$$

   The eigenvalues for this point are

$$\lambda = -\frac{17}{9} - \frac{5}{18}\sqrt{19} \pm \frac{1}{18}\sqrt{3868\sqrt{19} - 16153}.$$

   These are approximately -4.58 and -1.62 So, this equilibrium point is a stable node.

   The other equilibrium is $(1 + \frac{\sqrt{19}}{3}, -\frac{11}{9} - \frac{2}{9}\sqrt{19}) \approx (2.45, -2.19)$. The corresponding eigenvalues are complex with negative real parts,

$$\lambda = -\frac{17}{9} + \frac{5}{18}\sqrt{19} \pm \frac{i}{18}\sqrt{16153 + 3868\sqrt{19}},$$

   or $\lambda \approx -0.678 \pm 10.1i$. This point is a stable spiral.

Plots of the phase plane are given in Figures 3.12 and 3.14. The reader can look at the direction field and verify these results for the behavior of equilibrium solutions. A zoomed in view is shown in Figure 3.15 with several orbits indicated.

Example 3.12. Damped Nonlinear Pendulum Equilibria

We are now ready to establish the behavior of the fixed points of the damped nonlinear pendulum system in Equation (3.14). Recall that the system for the damped nonlinear pendulum was given by

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.32}$$

For a damped system, we will need $b > 0$. We had found that there are an infinite number of equilibrium points at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

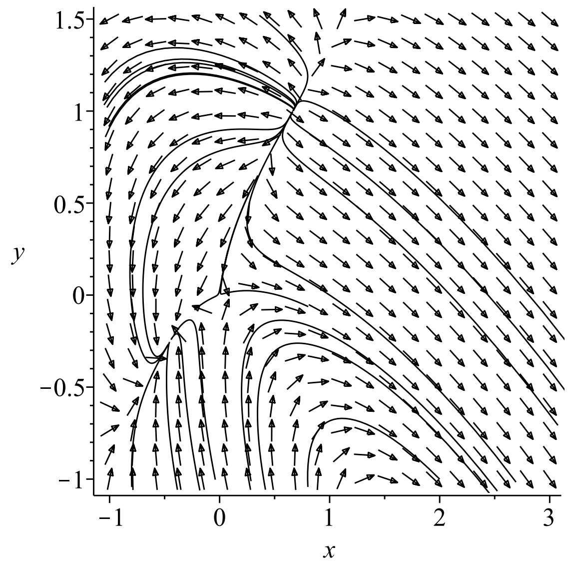


Figure 3.15: A closer look at the phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$
$$y' = x - 3y + xy - 3(x^2 - y^2)$$

with a few trajectories shown.

The Jacobian matrix for this systems is

$$Df(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}. \tag{3.33}$$

Evaluating this matrix at the fixed points, we find that

$$Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & -b \end{pmatrix}. \tag{3.34}$$

The eigenvalue equation is given by

$$\lambda^2 + b\lambda + (-1)^n \omega^2 = 0.$$

There are two cases to consider: n even and n odd. For the first case, we find the eigenvalues

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even n values. If there is no damping, then we obtain centers ($\lambda = \pm i\omega$).

In the second case, n odd, we find

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have hyperbolic points, or saddles. If there is no damping, the eigenvalues reduce to $\lambda = \pm\omega$.

In Figure (3.16) we show the phase plane for the undamped nonlinear pendulum with $\omega = 1.25$. We see that we have a mixture of centers

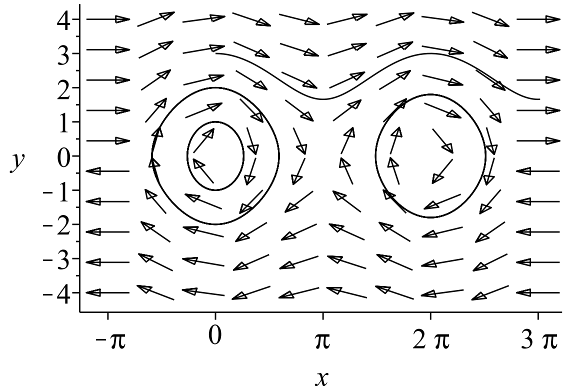


Figure 3.16: Phase plane for the undamped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

and saddles. There are orbits for which there is periodic motion. In the case that $\theta = \pi$ we have an inverted pendulum. This is an unstable position and this is reflected in the presence of saddle points, especially if the pendulum is constructed using a massless rod.

There are also unbounded orbits, going through all possible angles. These correspond to the mass spinning around the pivot in one direction forever due to initially having large enough energies.

We have indicated in the figure solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$. These show the various types of motions that we have described.

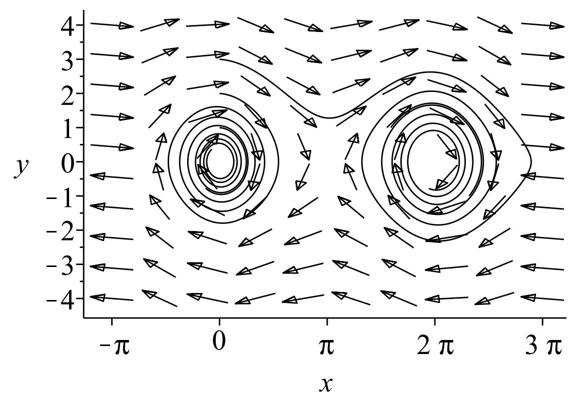


Figure 3.17: Phase plane for the damped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

When there is damping, we see that we can have a variety of other behaviors as seen in Figure (3.17). In this example we have set $b = 0.08$ and $\omega = 1.25$. We see that energy loss results in the mass settling around one of the stable fixed points. This leads to an understanding as to why there are an infinite number of equilibria, even though physically the mass traces out a bound set of Cartesian points. We have indicated in the Figure (3.17) solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

In Figure 3.18 we show a region of the phase plane which corresponds to oscillations about $x = 0$. For small angles the pendulum oscillates following somewhat elliptical orbits. As the angles get larger, due to greater initial

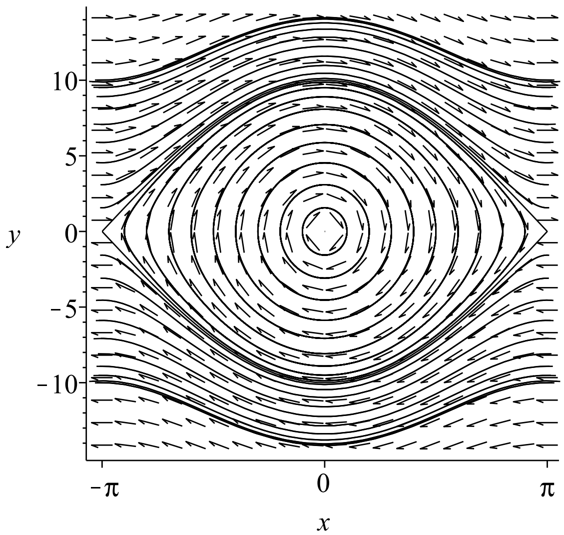


Figure 3.18: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$. The orbits surround a center at $(0,0)$. At the edges there are saddle points, $(\pm\pi, 0)$.

energies, these orbits begin to change from ellipses to other periodic orbits. There is a limiting orbit, beyond which one has unbounded motion. The limiting orbit connects the saddle points on either side of the center. The curve is called a separatrix and being that these trajectories connect two saddles, they are often referred to as heteroclinic orbits.

Heteroclinc orbits and separatrices.

In Figures 3.19-3.19 we show more orbits, including both bound and unbound motion beyond the interval $x \in [-\pi, \pi]$. For both plots we have chosen $\omega = 5$ and the same set of initial conditions, $x(0) = \pi k/10$, $k = -20, \ldots, 20$. for $y(0) = 0, \pm 10$. The time interval is taken for $t \in [-3, 3]$. The only difference is that in the damped case we have $b = 0.5$. In these plots one can see what happens to the heteroclinic orbits and nearby unbounded orbits under damping.

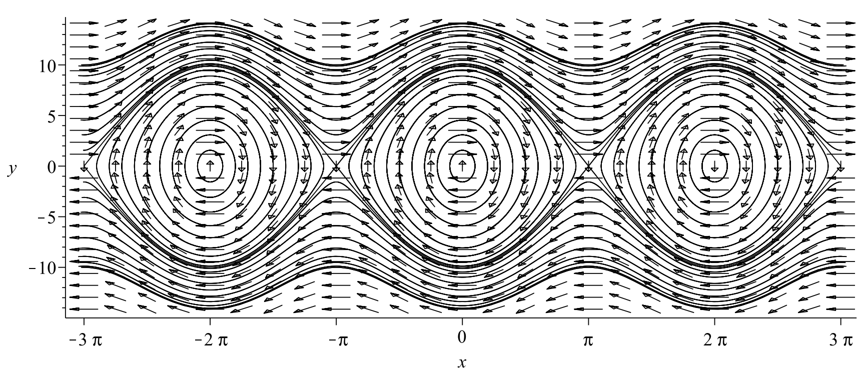


Figure 3.19: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$.

Before leaving this problem, we should note that the orbits in the phase plane for the undamped nonlinear pendulum can be obtained graphically. Recall from Equation (3.70), the total mechanical energy for the nonlinear

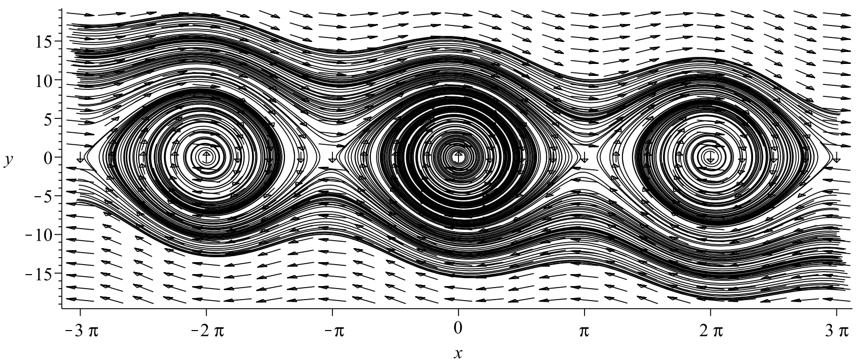


Figure 3.20: Several orbits in the phase plane for the damped nonlinear pendulum with $\omega = 5.0$ and $b = 0.5$.

pendulum is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta).$$

From this equation we obtained Equation (3.71),

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

Letting $y = \dot{\theta}$, $x = \theta$, and defining $z = -\omega^2\cos\theta_0$, this equation can be written as

$$\frac{1}{2}y^2 - \omega^2\cos x = z. \tag{3.35}$$

For each energy (z), this gives a constant energy curve. Plotting the family of energy curves we obtain the phase portrait shown in Figure 3.21.

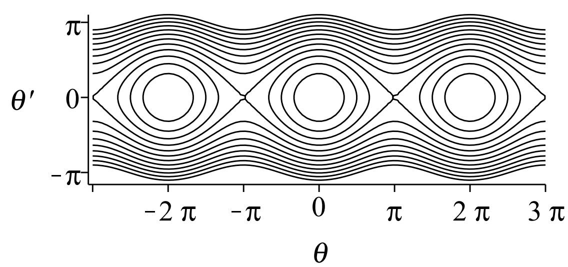


Figure 3.21: A family of energy curves in the phase plane for $\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = z$. Here we took $\omega = 1.0$ and $z \in [-5, 15]$.

3.6 *Nonlinear Population Models*

We have already encountered several models of population dynamics in this and previous chapters. Of course, one could dream up several other examples. While such models might seem far from applications in physics, it turns out that these models lead to systems od differential equations which also appear in physical systems such as the coupling of waves in lasers, in plasma physics, and in chemical reactions.

Two well-known nonlinear population models are the predator-prey and competing species models. In the predator-prey model, one typically has one species, the predator, feeding on the other, the prey. We will look at the standard Lotka-Volterra model in this section. The competing species model looks similar, except there are a few sign changes, since one species is not feeding on the other. Also, we can build in logistic terms into our model. We will save this latter type of model for the homework.

The Lotka-Volterra model is named after Alfred James Lotka (1880-1949) and Vito Volterra (1860-1940).

The Lotka-Volterra model takes the form

The Lotka-Volterra model of population dynamics.

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= -dy + cxy, \end{aligned} \tag{3.36}$$

where a, b, c, and d are positive constants. In this model, we can think of x as the population of rabbits (prey) and y is the population of foxes (predators). Choosing all constants to be positive, we can describe the terms.

- ax: When left alone, the rabbit population will grow. Thus a is the natural growth rate without predators.
- $-dy$: When there are no rabbits, the fox population should decay. Thus, the coefficient needs to be negative.
- $-bxy$: We add a nonlinear term corresponding to the depletion of the rabbits when the foxes are around.
- cxy: The more rabbits there are, the more food for the foxes. So, we add a nonlinear term giving rise to an increase in fox population.

Example 3.13. Determine the equilibrium points and their stability for the Lotka-Volterra system.

The analysis of the Lotka-Volterra model begins with determining the fixed points. So, we have from Equation (3.36)

$$\begin{aligned} x(a - by) &= 0, \\ y(-d + cx) &= 0. \end{aligned} \tag{3.37}$$

Therefore, the origin, $(0,0)$, and $(\frac{d}{c}, \frac{a}{b})$ are the fixed points.

Next, we determine their stability, by linearization about the fixed points. We can use the Jacobian matrix, or we could just expand the right-hand side of each equation in (3.36) about the equilibrium points as shown in he next example. The Jacobian matrix for this system is

$$Df(x,y) = \begin{pmatrix} a - by & -bx \\ cy & -d + cx \end{pmatrix}.$$

Evaluating at each fixed point, we have

$$Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.38}$$

$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}. \tag{3.39}$$

The eigenvalues of (3.38) are $\lambda = a, -d$. So, the origin is a saddle point.

The eigenvalues of (3.39) satisfy $\lambda^2 + ad = 0$. So, the other point is a center. In Figure 3.22 we show a sample direction field for the Lotka-Volterra system.

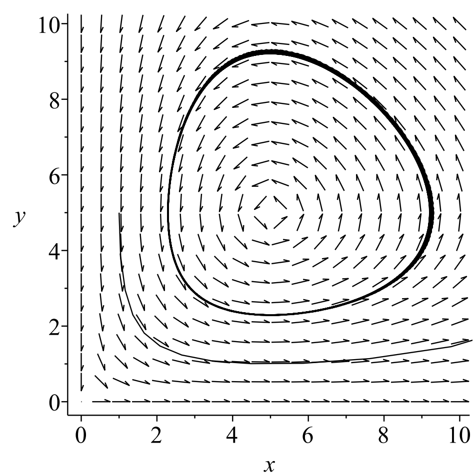


Figure 3.22: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$. Solution curves are shown for initial conditions $(x_0, y_0) = (8,3), (1,5)$.

Another way to carry out the linearization of the system of differential equations is to expand the equations about the fixed points. For fixed points (x^*, y^*), we let

$$(x, y) = (x^* + u, y^* + v).$$

Inserting this translation of the origin into the equations of the system, and dropping nonlinear terms in u and v, results in the linearized system. This method is equivalent to analyzing the Jacobian matrix for each fixed point.

Direct linearization of a system is carried out by introducing $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$, or $(x, y) = (x^* + u, y^* + v)$ into the system and dropping nonlinear terms in u and v.

Example 3.14. Expand the Lotka-Volterra system about the equilibrium points.

For the origin $(0,0)$ the linearization about the origin amounts to simply dropping the nonlinear terms. In this case we have

$$\begin{aligned} \dot{u} &= au, \\ \dot{v} &= -dv. \end{aligned} \tag{3.40}$$

The coefficient matrix for this system is the same as $Df(0,0)$.

For the second fixed point, we let

$$(x, y) = \left(\frac{d}{c} + u, \frac{a}{b} + v\right).$$

Inserting this transformation into the system gives

$$\begin{aligned} \dot{u} &= a\left(\frac{d}{c} + u\right) - b\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right), \\ \dot{v} &= -d\left(\frac{a}{b} + v\right) + c\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right). \end{aligned} \tag{3.41}$$

Expanding, we obtain

$$\begin{aligned} \dot{u} &= \frac{ad}{c} + au - b\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right), \\ \dot{v} &= -\frac{ad}{b} - dv + c\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right). \end{aligned} \tag{3.42}$$

In both equations the constant terms cancel and linearization is simply getting rid of the uv terms. This leaves the linearized system

$$\begin{aligned} \dot{u} &= au - b\left(\frac{d}{c}v + \frac{a}{b}u\right), \\ \dot{v} &= -dv + c\left(+\frac{d}{c}v + \frac{a}{b}u\right), \end{aligned} \tag{3.43}$$

or

$$\begin{aligned} \dot{u} &= -\frac{bd}{c}v, \\ \dot{v} &= \frac{ac}{b}u. \end{aligned} \tag{3.44}$$

The coefficient matrix for this linearized system is the same as $Df\left(\frac{d}{c}, \frac{a}{b}\right)$. In fact, for nearby orbits, they are almost circular orbits. From this linearized system, we have $\ddot{u} + adu = 0$.

We can take $u = A\cos(\sqrt{ad}t + \phi)$, where A and ϕ can be determined from the initial conditions. Then,

$$\begin{aligned} v &= -\frac{c}{bd}\dot{u} \\ &= \frac{c}{bd}A\sqrt{ad}\sin(\sqrt{ad}t + \phi) \\ &= \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi). \end{aligned} \tag{3.45}$$

Therefore, the solutions near the center are given by

$$(x, y) = \left(\frac{d}{c} + A\cos(\sqrt{ad}t + \phi), \frac{a}{b} + \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi)\right).$$

For $a = d = 1$, $b = c = 0.2$, and initial values of $(x_0, y_0) = (5.5, 5)$, these solutions become

$$x(t) = 5.0 + 0.5\cos t, \quad y(t) = 5.0 + 0.5\sin t.$$

Plots of these solutions are shown in Figure (3.23).

It is also possible to find a first integral of the Lotka-Volterra system whose level curves give the phase portrait of the system. As we had done in Chapter 2, we can write

$$\begin{aligned} \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} \\ &= \frac{-dy + cxy}{ax - bxy} \\ &= \frac{y(-d + cx)}{x(a - by)}. \end{aligned} \tag{3.46}$$

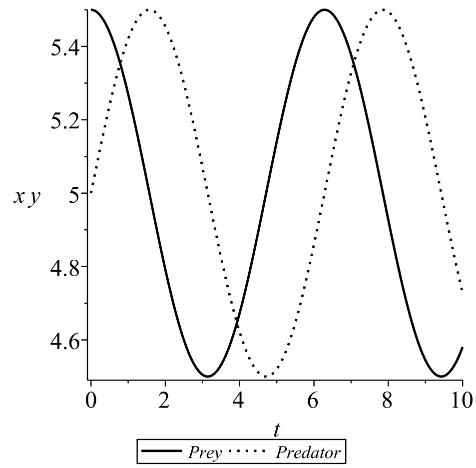


Figure 3.23: The linearized solutions of Lotka-Volterra system $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ for the initial conditions $(x_0, y_0) = (5.5, 5)$.

This is an equation of the form seen in Problem 2.**??**. This equation is now a separable differential equation. The solution this differential equation is given in implicit form as

$$a \ln y + d \ln x - cx - by = C,$$

where C is an arbitrary constant. This expression is known as the first integral of the Lotka-Volterra system. These level curves are shown in Figure 3.24.

The first integral of the Lotka-Volterra system.

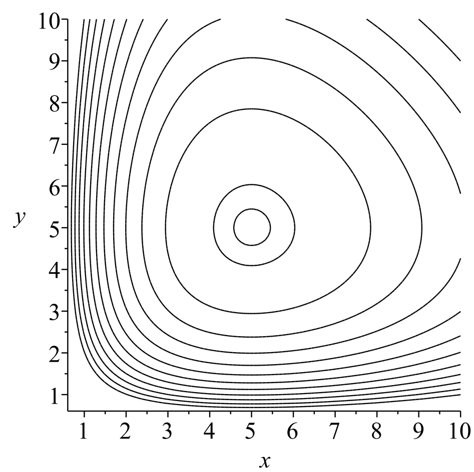


Figure 3.24: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ based upon the first integral of the system.

3.7 *Limit Cycles*

So far we have just been concerned with equilibrium solutions and their behavior. However, asymptotically stable fixed points are not the only attractors. There are other types of solutions, known as limit cycles, towards which a solution may tend. In this section we will look at some examples of these periodic solutions.

Such solutions are common in nature. Rayleigh investigated the problem

$$x'' + c\left(\frac{1}{3}(x')^2 - 1\right)x' + x = 0 \tag{3.47}$$

in the study of the vibrations of a violin string. Balthasar van der Pol (1889-1959) studied an electrical circuit, modeling this behavior. Others have looked into biological systems, such as neural systems, chemical reactions, such as Michaelis-Menten kinetics, and other chemical systems leading to chemical oscillations. One of the most important models in the historical study of dynamical systems is that of planetary motion and investigating the stability of planetary orbits. As is well known, these orbits are periodic.

Limit cycles are isolated periodic solutions towards which neighboring states might tend when stable. A key example exhibiting a limit cycle is given in the next example.

Example 3.15. Find the limit cycle in the system

$$\begin{aligned} x' &= \mu x - y - x(x^2+y^2) \\ y' &= x + \mu y - y(x^2+y^2). \end{aligned} \tag{3.48}$$

It is clear that the origin is a fixed point. The Jacobian matrix is given as

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}. \tag{3.49}$$

The eigenvalues are found to be $\lambda = \mu \pm i$. For $\mu = 0$ we have a center. For $\mu < 0$ we have a stable spiral and for $\mu > 0$ we have an unstable spiral. However, this spiral does not wander off to infinity. We see in Figure 3.25 that the equilibrium point is a spiral. However, in Figure 3.26 it is clear that the solution does not spiral out to infinity. It is bounded by a circle.

One can actually find the radius of this circle. This requires rewriting the system in polar form. Recall from Chapter 2 that we can change derivatives of Cartesian coordinates to derivatives of polar coordinates by using the relations

$$rr' = xx' + yy', \tag{3.50}$$

$$r^2\theta' = xy' - yx'. \tag{3.51}$$

Inserting the system (3.48) into these expressions, we have

$$rr' = \mu r^2 - r^4, \qquad r^2\theta' = r^2.$$

This leads to the system

$$\begin{aligned} r' &= \mu r - r^3, \\ \theta' &= 1. \end{aligned} \tag{3.52}$$

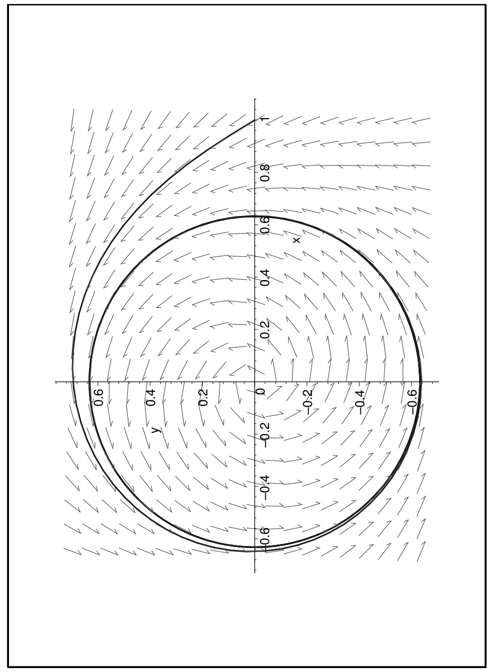


Figure 3.25: Phase plane for system (3.48) with $\mu = 0.4$.

Of course, for a circle the radius is constant, $r = \text{const}$. Therefore, in order to find the limit cycle, we need to look at the equilibrium solutions of Equation (3.52). This amounts to finding the constant solutions of $\mu r - r^3 = 0$. The equilibrium solutions are $r = 0, \pm\sqrt{\mu}$. The limit cycle corresponds to the positive radius solution, $r = \sqrt{\mu}$.

In Figures 3.25-3.26 we take $\mu = 0.4$. In this case we expect a circle with $r = \sqrt{0.4} \approx 0.63$. From the θ equation, we have that $\theta' > 0$. This means that we follow the limit cycle in a counterclockwise direction as time increases.

Limit cycles are not always circles. In Figures 3.27-3.28 we show the behavior of the Rayleigh system (3.47) for $c = 0.4$ and $c = 2.0$. In this case we see that solutions tend towards a noncircular limit cycle in a clockwise direction.

A slight change of the Rayleigh system leads to the van der Pol equation:

$$x'' + c(x^2 - 1)x' + x = 0 \tag{3.53}$$

The van der Pol system.

The limit cycle for $c = 2.0$ is shown in Figure 3.29.

Can one determine ahead of time if a given nonlinear system will have a limit cycle? In order to answer this question, we will introduce some definitions.

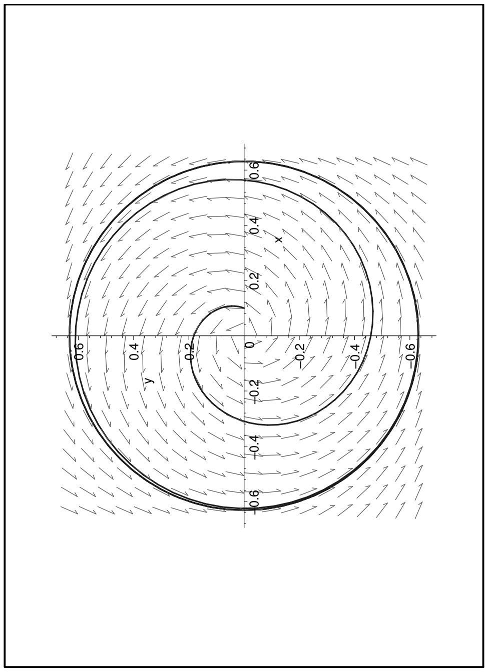


Figure 3.26: Phase plane for system (3.48) with $\mu = 0.4$ showing that the inner spiral is bounded by a limit cycle.

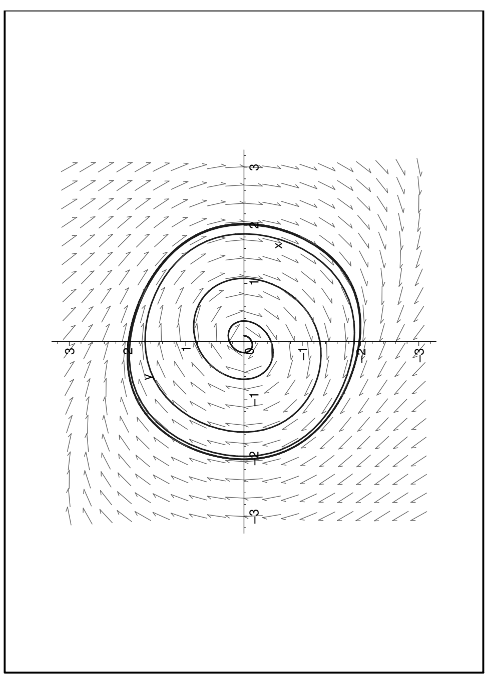


Figure 3.27: Phase plane for the Rayleigh system (3.47) with $c = 0.4$.

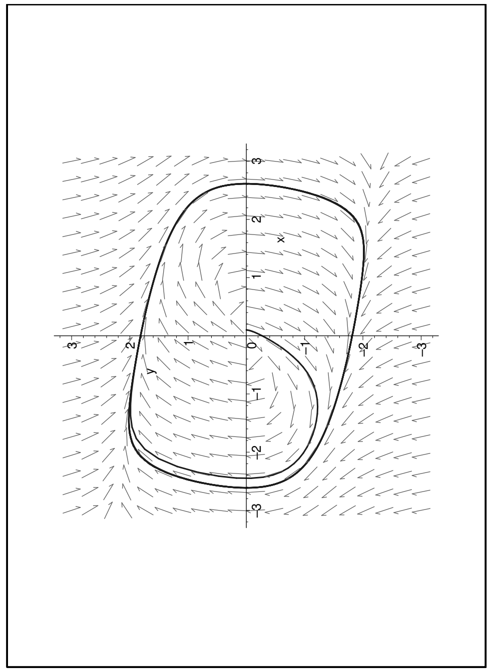


Figure 3.28: Phase plane for the van der Pol system (3.53) with $c = 2.0$.

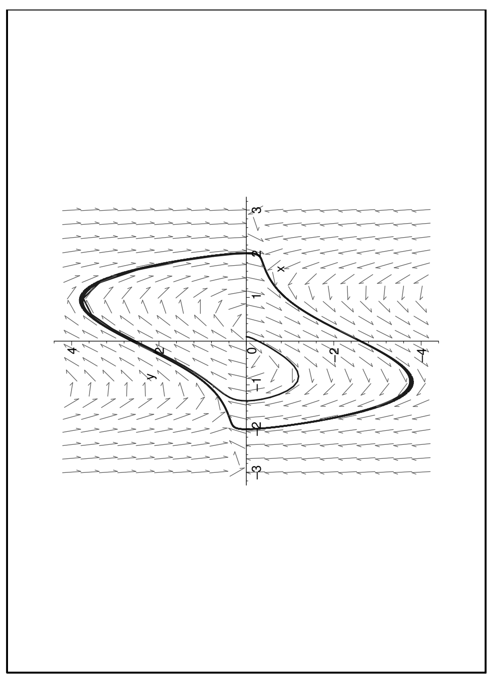


Figure 3.29: Phase plane for the van der Pol system (3.53) with $c = 0.4$.

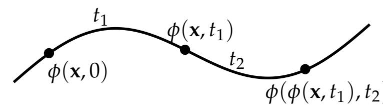


Figure 3.30: A sketch depicting the idea of trajectory, or orbit, passing through $\mathbf{x}$.

Flows.

We first describe different trajectories and families of trajectories. A *flow* on R^2 is a function ϕ that satisfies the following

1. $\phi(\mathbf{x}, t)$ is continuous in both arguments.

2. $\phi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in R^2$.

3. $\phi(\phi(\mathbf{x}, t_1), t_2) = \phi(\mathbf{x}, t_1 + t_2)$.

Orbits and trajectories.

The *orbit*, or *trajectory*, through $\mathbf{x}$ is defined as $\gamma = \{\phi(\mathbf{x}, t) | t \in I\}$. In Figure 3.30 we demonstrate these properties. For $t = 0$, $\phi(\mathbf{x}, 0) = \mathbf{x}$. Increasing t, one follows the trajectory until one reaches the point $\phi(\mathbf{x}, t_1)$. Continuing t_2 further, one is then at $\phi(\phi(\mathbf{x}, t_1), t_2)$. By the third property, this is the same as going from $\mathbf{x}$ to $\phi(\mathbf{x}, t_1 + t_2)$ for $t = t_1 + t_2$.

Having defined the orbits, we need to define the asymptotic behavior of the orbit for both positive and negative large times. We define the *positive semiorbit* through $\mathbf{x}$ as $\gamma^+ = \{\phi(\mathbf{x}, t) | t > 0\}$. The *negative semiorbit* through $\mathbf{x}$ is defined as $\gamma^- = \{\phi(\mathbf{x}, t) | t < 0\}$. Thus, we have $\gamma = \gamma^+ \cup \gamma^-$.

Limit sets and limit points.

The *positive limit set*, or *ω-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^+ = \{\mathbf{y}|\text{ there exists a sequence of } t_n \to \infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}.$$

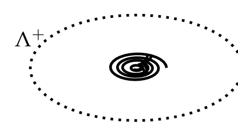


Figure 3.31: A sketch depicting an ω-limit set. Note that the orbits tend towards the set as t increases.

The $\mathbf{y}$'s are referred to as *ω-limit points*. This is shown in Figure 3.31.

Similarly, we define the *negative limit set*, or the *alpha-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^- = \{\mathbf{y}|\text{ there exists a sequences of } t_n \to -\infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}$$

and the corresponding $\mathbf{y}$'s are *α-limit points*. This is shown in Figure 3.32.

Cycles and periodic orbits.

There are several types of orbits that a system might possess. A *cycle* or *periodic orbit* is any closed orbit which is not an equilibrium point. A periodic orbit is stable if for every neighborhood of the orbit such that all nearby orbits stay inside the neighborhood. Otherwise, it is unstable. The orbit is asymptotically stable if all nearby orbits converge to the periodic orbit.

A limit cycle is a cycle which is the α or ω-limit set of some trajectory other than the limit cycle. A limit cycle Γ is stable if $\Lambda^+ = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. A limit cycle Γ is unstable if $\Lambda^- = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. Finally, a limit cycle is semistable if it is attracting on one side and repelling on the other side. In the previous examples, we saw limit cycles that were stable. Figures 3.31 and 3.32 depict stable and unstable limit cycles, respectively.

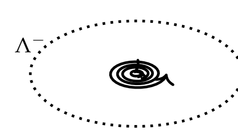


Figure 3.32: A sketch depicting an α-limit set. Note that the orbits tend away from the set as t increases.

We now state a theorem which describes the type of orbits we might find in our system.

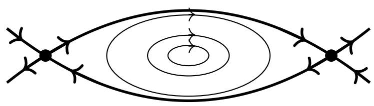


Figure 3.33: A heteroclinic orbit connecting two critical points.

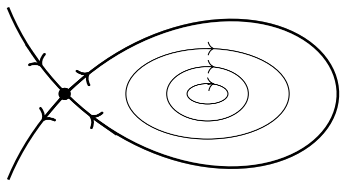


Figure 3.34: A homoclinic orbit returning to the point it left.

Theorem 3.2. Poincaré-Bendixon Theorem *Let γ^+ be contained in a bounded region in which there are finitely many critical points. Then Λ^+ is either*

1. *a single critical point;*
2. *a single closed orbit;*
3. *a set of critical points joined by heteroclinic orbits. [Compare Figures 3.33 and 3.34.]*

We are interested in determining when limit cycles may, or may not, exist. A consequence of the Poincaré-Bendixon Theorem is given by the following corollary.

Corollary Let D be a bounded closed set containing no critical points and suppose that $\gamma^+ \subset D$. Then there exists a limit cycle contained in D.

More specific criteria allow us to determine if there is a limit cycle in a given region. These are given by Dulac's Criteria and Bendixon's Criteria.

Dulac's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

and a continuously differentiable function ψ defined on an annular region D contained in some open set. If

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g)$$

does not change sign in D, then there is at most one limit cycle contained entirely in D.

Bendixon's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

defined on a simply connected domain D such that

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g) \neq 0$$

in D. Then, there are no limit cycles entirely in D.

Proof. These are easily proved using Green's Theorem in the Plane. (See your calculus text.) We prove Bendixon's Criteria. Let $\mathbf{f} = (f,g)$. Assume that Γ is a closed orbit lying in D. Let S be the interior of Γ. Then

$$\begin{aligned} \int_S \nabla \cdot \mathbf{f}\, dxdy &= \oint_\Gamma (f\,dy - g\,dx) \\ &= \int_0^T (f\dot{y} - g\dot{x})dt \\ &= \int_0^T (fg - gf)dt = 0. \end{aligned} \tag{3.54}$$

So, if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in S, then from the continuity of $\nabla \cdot \mathbf{f}$ in S we have that the right side above is either positive or negative. Thus, we have a contradiction and there is no closed orbit lying in D □

Example 3.16. Consider the earlier example in (3.48) with $\mu = 1$.

$$\begin{aligned} x' &= x - y - x(x^2 + y^2) \\ y' &= x + y - y(x^2 + y^2). \end{aligned} \tag{3.55}$$

We already know that a limit cycle exists at $x^2 + y^2 = 1$. A simple computation gives that

$$\nabla \cdot \mathbf{f} = 2 - 4x^2 - 4y^2.$$

For an arbitrary annulus $a < x^2 + y^2 < b$, we have

$$2 - 4b < \nabla \cdot \mathbf{f} < 2 - 4a.$$

For $a = 3/4$ and $b = 5/4$, $-3 < \nabla \cdot \mathbf{f} < -1$. Thus, $\nabla \cdot \mathbf{f} < 0$ in the annulus $3/4 < x^2 + y^2 < 5/4$. Therefore, by Dulac's Criteria there is at most one limit cycle in this annulus.

Example 3.17. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -ax - by + cx^2 + dy^2. \end{aligned} \tag{3.56}$$

Let $\psi(x, y) = e^{-2dx}$. Then,

$$\frac{\partial}{\partial x}(\psi y) + \frac{\partial}{\partial y}(\psi(-ax - by + cx^2 + dy^2)) = -be^{-2dx} \neq 0.$$

We conclude by Bendixon's Criteria that there are no limit cycles for this system.

3.8 Nonautonomous Nonlinear Systems

IN THIS SECTION WE DISCUSS NONAUTONOMOUS SYSTEMS. Recall that an autonomous system is one in which there is no explicit time dependence. A simple example is the forced nonlinear pendulum given by the nonhomogeneous equation

$$\ddot{x} + \omega^2 \sin x = f(t). \tag{3.57}$$

We can set this up as a system of two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(t). \end{aligned} \tag{3.58}$$

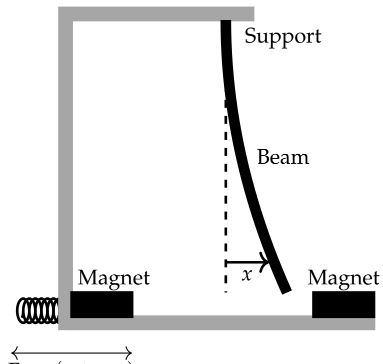


Figure 3.35: One model of the Duffing equation describes a periodically forced beam which interacts with two magnets.

This system is not in a form for which we could use the earlier methods. Namely, it is a nonautonomous system. However, we introduce a new variable $z(t) = t$ and turn it into an autonomous system in one more dimension. The new system takes the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(z). \\ \dot{z} &= 1. \end{aligned} \tag{3.59}$$

The system is now a three dimensional autonomous, possibly nonlinear, system and can be explored using methods from Chapters 2 and 3.

A more interesting model is provided by the Duffing Equation. This equation, named after Georg Wilhelm Christian Caspar Duffing (1861-1944), models hard spring and soft spring oscillations. It also models a periodically forced beam as shown in Figure 3.35. It is of interest because it is a simple system which exhibits chaotic dynamics and will motivate us towards using new visualization methods for nonautonomous systems.

The most general form of Duffing's equation is given by the damped, forced system

$$\ddot{x} + k\dot{x} + (\beta x^3 \pm \omega_0^2 x) = \Gamma \cos(\omega t + \phi). \tag{3.60}$$

This equation models hard spring, $(\beta > 0)$, and soft spring, $(\beta < 0)$, oscillations. However, we will use the simpler version of the Duffing equation:

$$\ddot{x} + k\dot{x} + x^3 - x = \Gamma \cos \omega t. \tag{3.61}$$

An equation of this form can be obtained by setting $\phi = 0$ and rescaling x and t in the original equation. We will explore the behavior of the system as we vary the remaining parameters. In Figures 3.36-3.39 we show some typical solution plots superimposed on the direction field.

The undamped, unforced Duffing equation.

We start with the undamped ($k = 0$) and unforced ($\Gamma = 0$) Duffing equation,

$$\ddot{x} + x^3 - x = 0.$$

We can write this second order equation as the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2). \end{aligned} \tag{3.62}$$

We see that there are three equilibrium points at $(0, 0)$ and $(\pm 1, 0)$. In Figure 3.36 we plot several orbits for $k = 0$, and $\Gamma = 0$. We see that the three equilibrium points consist of two centers and a saddle.

The unforced Duffing equation.

We now turn on the damping. The system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky + x(1 - x^2). \end{aligned} \tag{3.63}$$

In Figures 3.37 and 3.38 we show what happens when $k = 0.1$. These plots are reminiscent of the plots for the nonlinear pendulum; however, there are fewer equilibria. Note that the centers become stable spirals for $k > 0$.

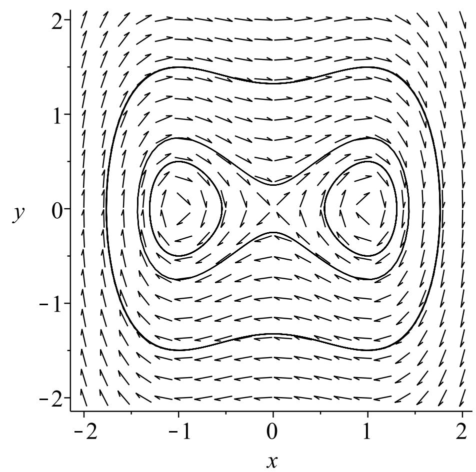


Figure 3.36: Phase plane for the undamped, unforced Duffing equation ($k = 0, \Gamma = 0$).

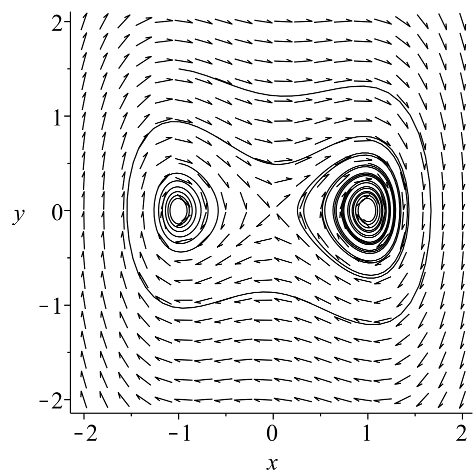


Figure 3.37: Phase plane for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

Next we turn on the forcing to obtain a damped, forced Duffing equation. The system is now nonautonomous.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-x^2) + \Gamma \cos \omega t. \end{aligned} \tag{3.64}$$

The damped, forced Duffing equation.

In Figure 3.39 we only show one orbit with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. The solution intersects itself and look a bit messy. We can imagine what we would get if we added any more orbits. For completeness, we show in Figure 3.40 an example with four different orbits.

In cases for which one has periodic orbits such as the Duffing equation, Poincaré introduced the notion of *surfaces of section*. One embeds the orbit in a higher dimensional space so that there are no self intersections, like we saw in Figures 3.39 and 3.40. In Figure 3.42 we show an example where a simple orbit is shown as it periodically pierces a given surface.

In order to simplify the resulting pictures, one only plots the points at which the orbit pierces the surface as sketched in Figure 3.41. In practice, there is a natural frequency, such as ω in the forced Duffing equation. Then,

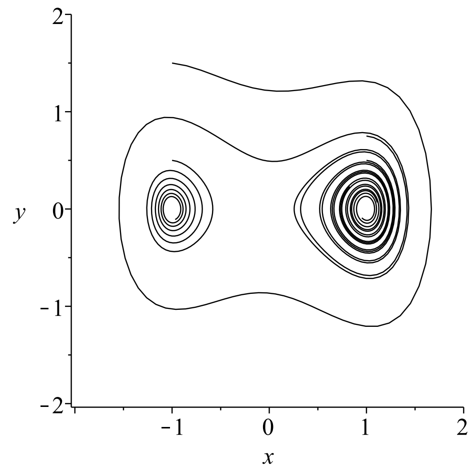


Figure 3.38: Display of two orbits for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

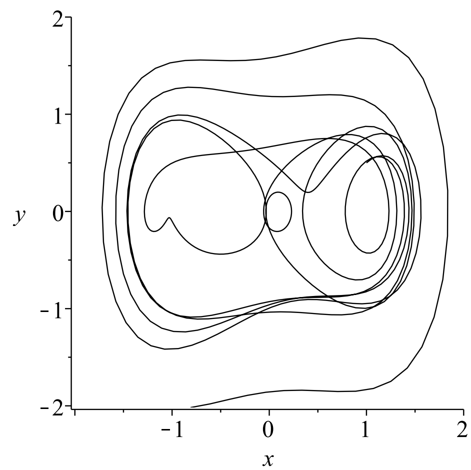


Figure 3.39: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case we show only one orbit which was generated from the initial condition $(x_0 = 1.0, \quad y_0 = 0.5)$.

one plots points at times that are multiples of the period, $T = \frac{2\pi}{\omega}$. In Figure 3.43 we show what the plot for one orbit would look like for the damped, unforced Duffing equation.

The more interesting case, is when there is forcing and damping. In this case the surface of section plot is given in Figure 3.44. While this is not as busy as the solution plot in Figure 3.39, it still provides some interesting behavior. What one finds is what is called a strange attractor. Plotting many orbits, we find that after a long time, all of the orbits are attracted to a small region in the plane, much like a stable node attracts nearby orbits. However, this set consists of more than one point. Also, the flow on the attractor is chaotic in nature. Thus, points wander in an irregular way throughout the attractor. This is one of the interesting topics in chaos theory and this whole theory of dynamical systems has only been touched in this text leaving the reader to wander of into further depth into this fascinating field.

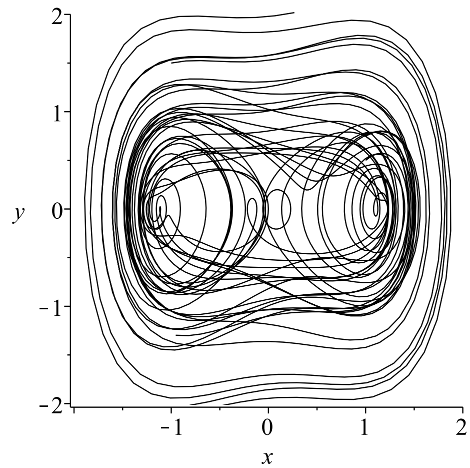


Figure 3.40: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case four initial conditions were used to generate four orbits.

The surface of section plots at the end of the last section were obtained using code from S. Lynch's book *Dynamical Systems with Applications Using Maple*. For reference, the plots in Figures 3.36 and 3.37 were generated in Maple using the following commands:

```
> with(DEtools):
> Gamma:=0.5:omega:=1.25:k:=0.1:
> DEplot([diff(x(t),t)=y(t), diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t)], [x(t),y(t)],t=0..500,[[x(0)=1,y(0)=0.5],
  [x(0)=-1,y(0)=0.5], [x(0)=1,y(0)=0.75], [x(0)=-1,y(0)=1.5]],
  x=-2..2,y=-2..2, stepsize=0.1, linecolor=blue, thickness=1,
  color=black);
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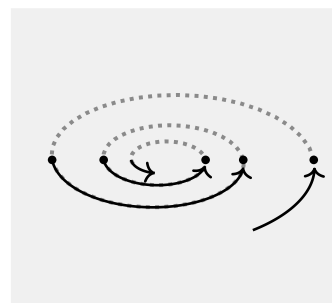


Figure 3.41: As an orbit pierces the surface of section, one plots the point of intersection in that plane to produce the surface of section plot.

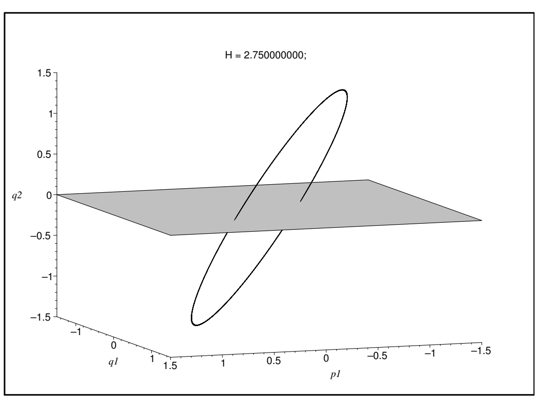


Figure 3.42: Poincaré's surface of section. One notes each time the orbit pierces the surface.

Figure 3.43: Poincaré's surface of section plot for the damped, unforced Duffing equation.

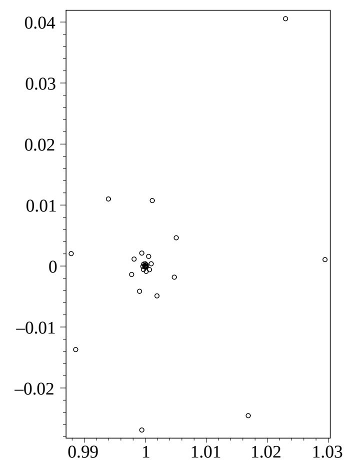
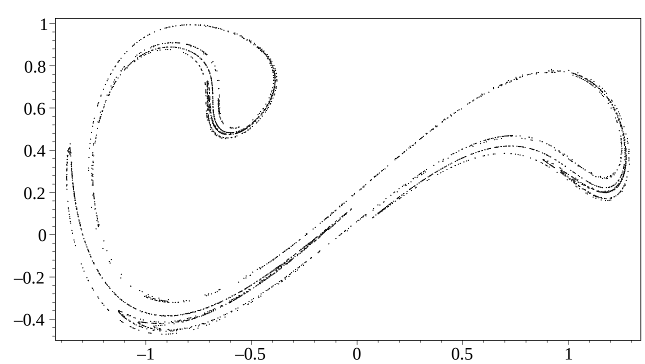


Figure 3.44: Poincaré's surface of section plot for the damped, forced Duffing equation. This leads to what is known as a strange attractor.



3.9 *The Period of the Nonlinear Pendulum**

Recall that the period of the simple pendulum is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.65}$$

for

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{3.66}$$

This was based upon the solving the linear pendulum equation (3.12). This equation was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle?

We recall that the Taylor series approximation of $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{3.67}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \left|\frac{\sin\theta - \theta}{\sin\theta}\right|.$$

A plot of the relative error is given in Figure 3.45. Thus for $\theta \approx 0.4$ radians (or, 23^o) we have that the relative error is about 2.6%.

Relative error in $\sin\theta$ approximation.

We would like to do better than this. So, we now turn to the nonlinear pendulum equation (3.11) in the simpler form

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{3.68}$$

Solution of nonlinear pendulum equation.

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\,d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For the nonlinear pendulum problem, we multiply Equation (3.68) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{3.69}$$

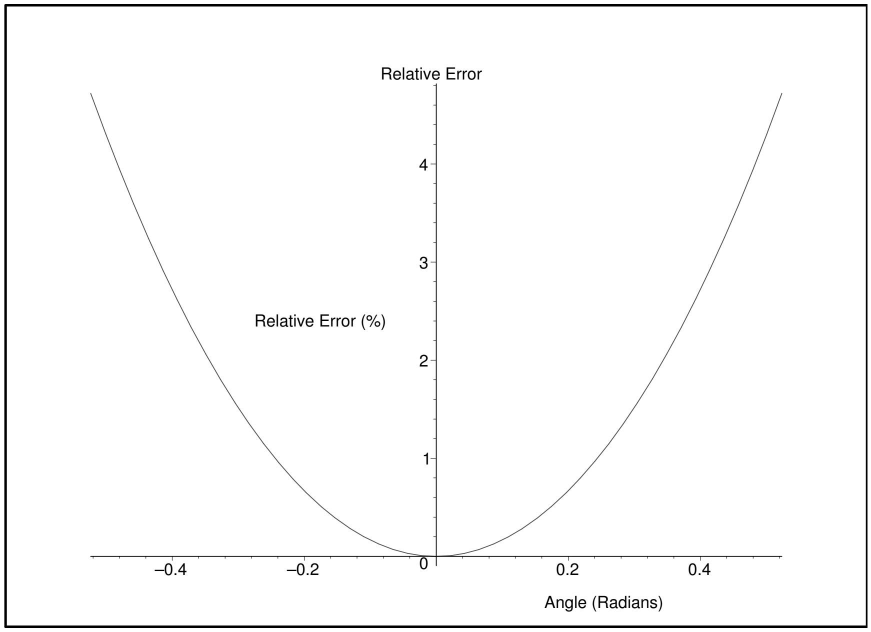


Figure 3.45: The relative error in percent when approximating $\sin\theta$ by θ.

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2 \cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2 \cos\theta)}}.$$

Of course, we need to be able to do the integral. When one finds a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

In fact, the above integral can be transformed into what is known as an elliptic integral of the first kind. We will rewrite this result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (3.69). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the mass on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1-\cos\theta)$. So,

$$U = mgL(1-\cos\theta).$$

So, the total mechanical energy is

Total mechanical energy for the nonlinear pendulum.

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta). \tag{3.70}$$

We note that a little rearranging shows that we can relate this result to Equation (3.69). Dividing by m and L^2 and using the definition of $\omega^2 = g/L$, we have

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2.$$

Therefore, we have determined the integration constant in terms of the total mechanical energy,

$$c = \frac{1}{mL^2}E - \omega^2.$$

We can use Equation (3.70) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total mechanical energy is pure potential energy. Letting θ_0 denote the angle at the highest angular position, we have that

$$E = mgL(1-\cos\theta_0) = mL^2\omega^2(1-\cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0. \tag{3.71}$$

We can solve for $\dot{\theta}$ and integrate the differential equation to obtain

$$t = \int dt = \int \frac{d\theta}{\omega\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta),$$

we can rewrite the argument in the radical as

$$\cos\theta - \cos\theta_0 = 2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right].$$

Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{3.72}$$

This result can now be transformed into an elliptic integral.[2] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

[2] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate,

$$(x^2+y^2)^2 = x^2 - y^2.$$

and

$$k = \sin \frac{\theta_0}{2}.$$

Then, Equation (3.72) becomes

$$T = \frac{4}{\omega} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.73}$$

This is done by noting that $dz = \frac{1}{2k} \cos \frac{\theta}{2}\, d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\, d\theta$ and that $\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is called the complete elliptic integral of the first kind.

We note that the incomplete elliptic integral of the first kind is defined as

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The complete and incomplete elliptic integrals of the first kind.

Then, the complete elliptic integral of the first kind is given by $K(k) = F(\frac{\pi}{2}, k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Therefore, the period of the nonlinear pendulum is given by

$$T = \frac{4}{\omega} K\left(\sin\frac{\theta_0}{2}\right). \tag{3.74}$$

There are table of values for elliptic integrals. However, one can use a computer algebra system to compute values of such integrals. We will look for small angle approximations.

For small angles ($\theta_0 \ll \frac{\pi}{2}$), we have that k is small. So, we can develop a series expansion for the period, T, for small k. This is simply done by using the binomial expansion,

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

Inserting this expansion into the integrand for the complete elliptic integral and integrating term by term, we find that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right]. \tag{3.75}$$

The first term of the expansion gives the well known period of the simple pendulum for small angles. The next terms in the expression give further corrections to the linear result which are useful for larger amplitudes of oscillation. In Figure 3.46 we show the relative errors incurred when keeping the k^2 (quadratic) and k^4 (quartic) terms as compared to the exact value of the period.

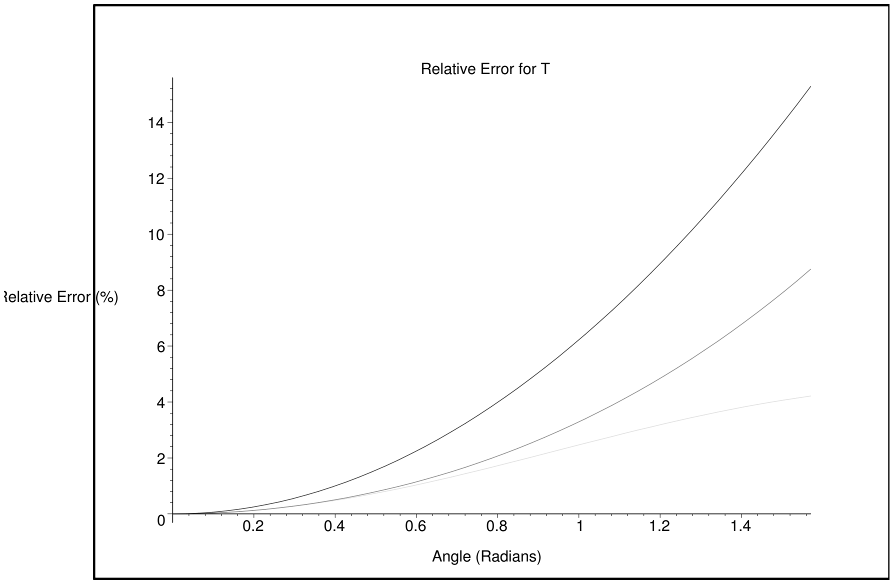


Figure 3.46: The relative error in percent when approximating the exact period of a nonlinear pendulum with one (solid), two (dashed), or three (dotted) terms in Equation (3.75).

3.10 *Exact Solutions Using Elliptic Functions**

THE SOLUTION IN EQUATION (3.73) OF THE NONLINEAR PENDULUM EQUATION led to the introduction of elliptic integrals. The incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.76}$$

The complete integral of the first kind is given by $K(k) = F(\frac{\pi}{2}, k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Elliptic integrals of the second kind are defined as

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{\sin\phi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{3.77}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{1} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{3.78}$$

Recall, a first integration of the nonlinear pendulum equation from Equation (3.70),

$$\left(\frac{d\theta}{dt}\right)^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega^2\left[\sin^2\frac{\theta}{2} - \sin^2\frac{\theta_0}{2}\right].$$

Letting

$$kz = \sin\frac{\theta}{2} \text{ and } k = \sin\frac{\theta_0}{2},$$

the differential equation becomes

$$\frac{dz}{d\tau} = \pm\omega\sqrt{1-z^2}\sqrt{1-k^2z^2}.$$

Applying separation of variables, we find

$$\begin{aligned} \pm\omega(t-t_0) &= \frac{1}{\omega}\int_1^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (3.79)\\ &= \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} - \int_0^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (3.80)\\ &= K(k) - F(\sin^{-1}(k^{-1}\sin\theta), k). && (3.81) \end{aligned}$$

The solution, $\theta(t)$, is then found by solving for z and using $kz = \sin\frac{\theta}{2}$ to solve for θ. This requires that we know how to invert the elliptic integral, $F(z,k)$.

Elliptic functions result from the inversion of elliptic integrals. Consider

$$\begin{aligned} u(\sin\phi, k) = F(\phi, k) &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. && (3.82)\\ &= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. && (3.83) \end{aligned}$$

Note:$F(\phi,0) = \phi$ and $F(\phi,1) = \ln(\sec\phi + \tan\phi)$. In these cases F is obviously monotone increasing and thus there must be an inverse.

The inverse of Equation (3.76) is defined as $\phi = F^{-1}(u,k) = \mathrm{am}(u,k)$, where $u = \sin\phi$. The function $\mathrm{am}(u,k)$ is called the Jacobi amplitude function and k is the elliptic modulus. [In some references and software like MATLAB packages, $m = k^2$ is used as the parameter.] Three of the Jacobi elliptic functions, shown in Figure 3.47, can be defined in terms of the amplitude function by

$$\mathrm{sn}(u,k) = \sin\mathrm{am}(u,k) = \sin\phi,$$

$$\mathrm{cn}(u,k) = \cos\mathrm{am}(u,k) = \cos\phi,$$

Jacobi elliptic functions.

and the delta amplitude

$$\mathrm{dn}(u,k) = \sqrt{1-k^2\sin^2\phi}.$$

They are related through the identities

The elliptic functions can be extended to the complex plane. In this case the functions are doubly periodic. However, we will not need to consider this in the current text.

$$\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1, \quad (3.84)$$

$$\mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1. \quad (3.85)$$

Also, we see that these functions are periodic. The period is given in terms of the complete elliptic integral of the first kind, $K(k)$. Consider

$$\begin{aligned} F(\phi+2\pi, k) &= \int_0^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.\\ &= \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_\phi^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \end{aligned}$$

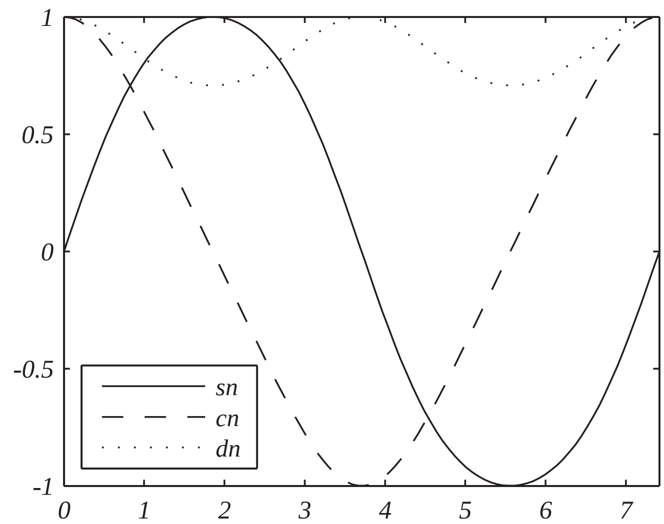


Figure 3.47: Plots of the Jacobi elliptic functions $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, and $\operatorname{dn}(u,k)$ for $m = k^2 = 0.5$. Here $K(k) = 1.8541$.

$$\begin{aligned} &= F(\phi,k) + \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= F(\phi,k) + 4K(k). \end{aligned} \tag{3.86}$$

Since $F(\phi + 2\pi, k) = u + 4K$, we have

$$\operatorname{sn}(u+4K) = \sin(\operatorname{am}(u+4K)) = \sin(\operatorname{am}(u) + 2\pi) = \sin\operatorname{am}(u) = \operatorname{sn} u.$$

In general, we have

$$\operatorname{sn}(u+2K,k) = -\operatorname{sn}(u,k) \tag{3.87}$$

$$\operatorname{cn}(u+2K,k) = -\operatorname{cn}(u,k) \tag{3.88}$$

$$\operatorname{dn}(u+2K,k) = \operatorname{dn}(u,k). \tag{3.89}$$

The plots of $\operatorname{sn}(u)$, $\operatorname{cn}(u)$, and $dn(u)$, are shown in Figures 3.48-3.50.

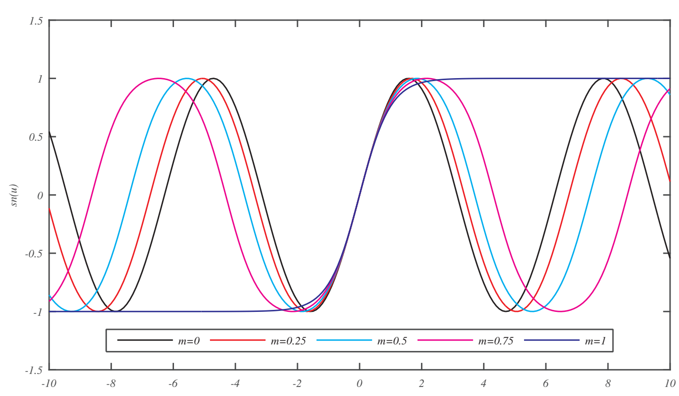


Figure 3.48: Plots of $\operatorname{sn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

Namely,

$$\operatorname{sn}(u+K,k) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \operatorname{sn}(u+2K,k) = -\operatorname{sn} u,$$

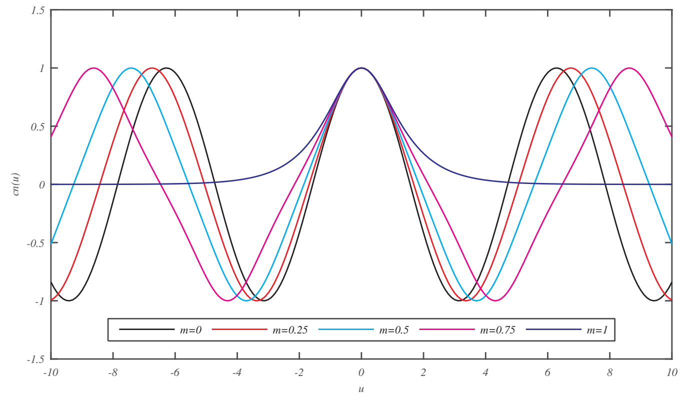


Figure 3.49: Plots of $\operatorname{cn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

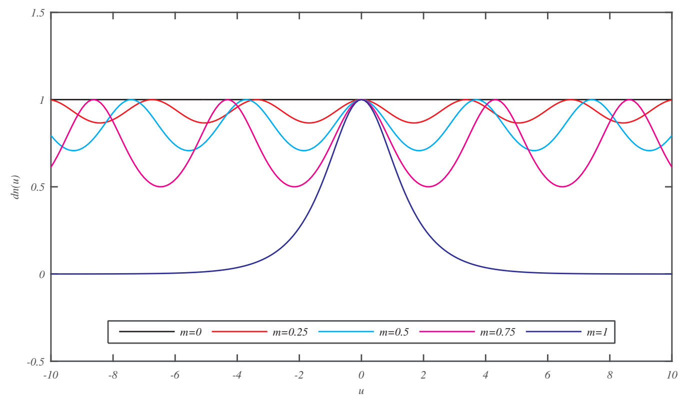


Figure 3.50: Plots of $\operatorname{dn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

$$\operatorname{cn}(u+K,k) = -\sqrt{1-k^2}\frac{\operatorname{sn} u}{\operatorname{dn} u}, \quad \operatorname{dn}(u+2K,k) = -\operatorname{cn} u,$$

$$\operatorname{dn}(u+K,k) = \frac{\sqrt{1-k^2}}{\operatorname{dn} u}, \quad \operatorname{dn}(u+2K,k) = \operatorname{dn} u.$$

Therefore, dn and cn have a period of $4K$ and dn has a period of $2K$.

Special values found in Figure 3.47 are seen as

$$\operatorname{sn}(K,k) = 1,$$

$$\operatorname{cn}(K,k) = 0,$$

$$\operatorname{dn}(K,k) = \sqrt{1-k^2} = k',$$

where k' is called the complementary modulus.

Important to this section are the derivatives of these elliptic functions,

$$\frac{\partial}{\partial u}\operatorname{sn}(u,k) = \operatorname{cn}(u,k)\operatorname{dn}(u,k),$$

$$\frac{\partial}{\partial u}\operatorname{cn}(u,k) = -\operatorname{sn}(u,k)\operatorname{dn}(u,k),$$

$$\frac{\partial}{\partial u}\operatorname{dn}(u,k) = -k^2\operatorname{sn}(u,k)\operatorname{cn}(u,k),$$

and the amplitude function

$$\frac{\partial}{\partial u}\operatorname{am}(u,k) = \operatorname{dn}(u,k).$$

Sometimes the Jacobi elliptic functions are displayed without reference to the elliptic modulus, such as $\operatorname{sn}(u) = \operatorname{sn}(u, k)$. When k is understood, we can do the same.

Example 3.18. Show that $\operatorname{sn}(u)$ satisfies the differential equation

$$y'' + (1 + k^2)y = 2k^2y^3.$$

From the above derivatives, we have that

$$\begin{aligned} \frac{d^2}{du^2}\operatorname{sn}(u) &= \frac{d}{du}(\operatorname{cn}(u)\operatorname{dn}(u)) \\ &= -\operatorname{sn}(u)\operatorname{dn}^2(u) - k^2\operatorname{sn}(u)\operatorname{cn}^2(u). \end{aligned} \tag{3.90}$$

Letting $y(u) = \operatorname{sn}(u)$ and using the identities (3.84)-(3.85), we have that

$$y'' = -y(1 - k^2y^2) - k^2y(1 - y^2) = -(1 + k^2)y + 2k^2y^3.$$

This is the desired result.

Example 3.19. Show that $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$ is a solution of the equation $\ddot{\theta} + \sin\theta = 0$.

Differentiating $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$, we have

$$\begin{aligned} \frac{d^2}{dt^2}\left(2\sin^{-1}(k\operatorname{sn} t)\right) &= \frac{d}{dt}\left(2\frac{k\operatorname{cn} t\operatorname{dn} t}{\sqrt{1 - k^2\operatorname{sn}^2 t}}\right) \\ &= \frac{d}{dt}(2k\operatorname{cn} t) \\ &= -2k\operatorname{sn} t\operatorname{dn} t. \end{aligned} \tag{3.91}$$

However, we can evaluate $\sin\theta$ for a range of θ. Thus, we have

$$\begin{aligned} \sin\theta &= \sin(2\sin^{-1}(k\operatorname{sn} t)) \\ &= 2\sin(\sin^{-1}(k\operatorname{sn} t))\cos(\sin^{-1}(k\operatorname{sn} t)) \\ &= 2k\operatorname{sn} t\sqrt{1 - k^2\operatorname{sn}^2 t} \\ &= 2k\operatorname{sn} t\operatorname{dn} t. \end{aligned} \tag{3.92}$$

Comparing these results, we have shown that $\ddot{\theta} + \sin\theta = 0$.

The solution to the last example can be used to obtain the exact solution to the nonlinear pendulum problem, $\ddot{\theta} + \omega^2\sin\theta = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$. The general solution is given by $\theta(t) = 2\sin^{-1}(k\operatorname{sn}(\omega t + \phi))$ where ϕ has to be determined from the initial conditions. We note that

$$\begin{aligned} \frac{d\operatorname{sn}(u + K)}{du} &= \operatorname{cn}(u + K)\operatorname{dn}(u + K) \\ &= \left(-\sqrt{1 - k^2}\frac{\operatorname{sn} u}{\operatorname{dn} u}\right)\left(\frac{\sqrt{1 - k^2}}{\operatorname{dn} u}\right) \\ &= -(1 - k^2)\frac{\operatorname{sn} u}{\operatorname{dn}^2 u}. \end{aligned} \tag{3.93}$$

Evaluating at $u = 0$, we have $\operatorname{sn}'(K) = 0$.

Therefore, if we pick $\phi = K$, then $\dot{\theta}(0) = 0$ and the solution is

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\omega t + K)).$$

Furthermore, the other initial value is found to be

$$\theta(0) = 2\sin^{-1}(k\,\mathrm{sn}\,K) = 2\sin^{-1}k.$$

Thus, $k = \sin\frac{\theta_0}{2}$, as we had seen in the earlier derivation of the elliptic integral solution. The solution is given as

$$\theta(t) = 2\sin^{-1}(\sin\frac{\theta_0}{2}\,\mathrm{sn}(\omega t + K)).$$

In Figures 3.51-3.52 we show comparisons of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and initial angles $\theta_0 = 10^o$ and $\theta_0 = 50^o$.

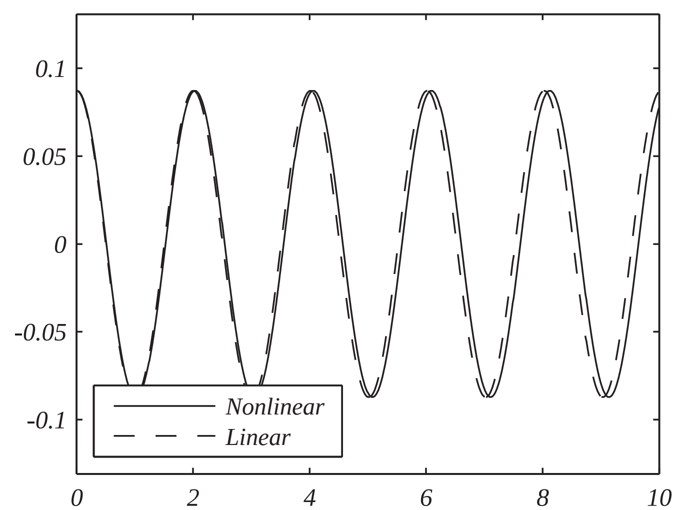


Figure 3.51: Comparison of exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 10^o$.

Problems

1. Solve the general logistic problem,

$$\frac{dy}{dt} = ky - cy^2, \quad y(0) = y_0 \tag{3.94}$$

using separation of variables.

2. Find the equilibrium solutions and determine their stability for the following systems. For each case draw representative solutions and phase lines.

a. $y' = y^2 - 6y - 16$.

b. $y' = \cos y$.

c. $y' = y(y-2)(y+3)$.

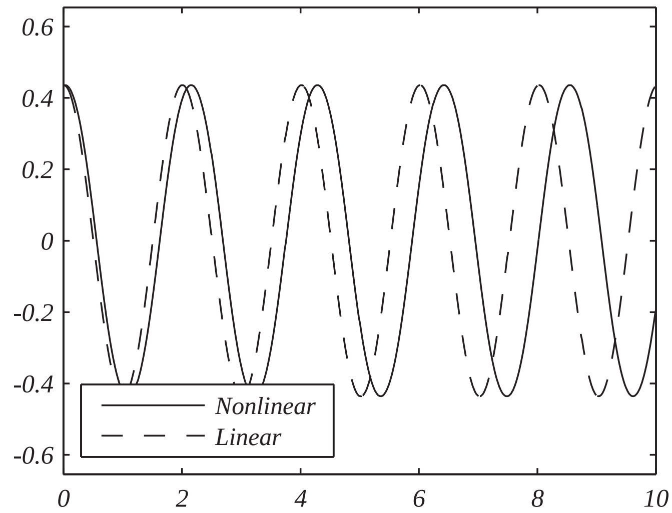


Figure 3.52: Comparison of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 50^o$.

d. $y' = y^2(y+1)(y-4)$.

3. For $y' = y - y^2$, find the general solution corresponding to $y(0) = y_0$. Provide specific solutions for the following initial conditions and sketch them: a. $y(0) = 0.25$, b. $y(0) = 1.5$, and c. $y(0) = -0.5$.

4. For each problem determine equilibrium points, bifurcation points and construct a bifurcation diagram. Discuss the different behaviors in each system.

a. $y' = y - \mu y^2$

b. $y' = y(\mu - y)(\mu - 2y)$

c. $x' = \mu - x^3$

d. $x' = x - \frac{\mu x}{1+x^2}$

5. Consider the family of differential equations $x' = x^3 + \delta x^2 - \mu x$.

a. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta = 0$.

b. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta > 0$.

Hint: Pick a few values of δ and μ in order to get a feel for how this system behaves.

6. System 3.52 can be solved exactly. Integrate the r-equation using separation of variables. For initial conditions a) $r(0) = 0.25$, $\theta(0) = 0$, and b) $r(0) = 1.5$, $\theta(0) = 0$, and $\mu = 1.0$, find and plot the solutions in the xy-plane showing the approach to a limit cycle.

7. Consider the system

$$\begin{aligned} x' &= -y + x\left[\mu - x^2 - y^2\right], \\ y' &= x + y\left[\mu - x^2 - y^2\right]. \end{aligned}$$

Rewrite this system in polar form. Look at the behavior of the r equation and construct a bifurcation diagram in μr space. What might this diagram look like in the three dimensional μxy space? (Think about the symmetry in this problem.) This leads to what is called a *Hopf bifurcation.*

8. Find the fixed points of the following systems. Linearize the system about each fixed point and determine the nature and stability in the neighborhood of each fixed point, when possible. Verify your findings by plotting phase portraits using a computer.

a.

$$\begin{aligned} x' &= x(100 - x - 2y), \\ y' &= y(150 - x - 6y). \end{aligned}$$

b.

$$\begin{aligned} x' &= x + x^3, \\ y' &= y + y^3. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - x^2 + xy, \\ y' &= 2y - xy - 6y^2. \end{aligned}$$

d.

$$\begin{aligned} x' &= -2xy, \\ y' &= -x + y + xy - y^3. \end{aligned}$$

9. Plot phase portraits for the Lienard system

$$\begin{aligned} x' &= y - \mu(x^3 - x) \\ y' &= -x. \end{aligned}$$

for a small and a not so small value of μ. Describe what happens as one varies μ.

10. Consider the period of a nonlinear pendulum. Let the length be $L = 1.0$ m and $g = 9.8$ m/s^2. Sketch T vs the initial angle θ_0 and compare the linear and nonlinear values for the period. For what angles can you use the linear approximation confidently?

11. Another population model is one in which species compete for resources, such as a limited food supply. Such a model is given by

$$\begin{aligned} x' &= ax - bx^2 - cxy, \\ y' &= dy - ey^2 - fxy. \end{aligned}$$

In this case, assume that all constants are positive.

a Describe the effects/purpose of each terms.

b Find the fixed points of the model.

c Linearize the system about each fixed point and determine the stability.

d From the above, describe the types of solution behavior you might expect, in terms of the model.

12. Consider a model of a food chain of three species. Assume that each population on its own can be modeled by logistic growth. Let the species be labeled by $x(t)$, $y(t)$, and $z(t)$. Assume that population x is at the bottom of the chain. That population will be depleted by population y. Population y is sustained by x's, but eaten by z's. A simple, but scaled, model for this system can be given by the system

$$\begin{aligned} x' &= x(1-x) - xy \\ y' &= y(1-y) + xy - yz \\ z' &= z(1-z) + yz. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Find the eigenvalues and eigenvectors.

d. Describe the solution behavior near each equilibrium point.

e. Which of these equilibria are important in the study of the population model and describe the interactions of the species in the neighborhood of these point(s).

13. Derive the first integral of the Lotka-Volterra system, $a \ln y + d \ln x - cx - by = C$.

14. Show that the system $x' = x - y - x^3$, $y' = x + y - y^3$, has a unique limit cycle by picking an appropriate $\psi(x, y)$ in Dulac's Criteria.

15. The Lorenz model is a simple model for atmospheric convection developed by Edward Lorenz in 1963. The system is given by the three equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Determine any bifurcation points and describe what happens near the bifurcation point(s). Consider $\sigma = 10$, $\beta = 8/3$, and vary ρ.

d. This system is know to exhibit chaotic behavior. Lorenz found a so-called strange attractor for parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. Using a computer, locate this strange attractor.

16. The Michaelis-Menten kinetics reaction is given by

$$\boxed{E + S \underset{k_1}{\overset{k_3}{\rightleftarrows}} ES \underset{k_2}{\longrightarrow} E + P.}$$

The resulting system of equations for the chemical concentrations is

$$\begin{aligned} \frac{d[S]}{dt} &= -k_1[E][S] + k_3[ES], \\ \frac{d[E]}{dt} &= -k_1[E][S] + (k_2 + k_3)[ES], \\ \frac{d[ES]}{dt} &= k_1[E][S] - (k_2 + k_3)[ES], \\ \frac{d[P]}{dt} &= k_2[ES]. \end{aligned} \tag{3.95}$$

In chemical kinetics one seeks to determine the rate of product formation ($v = d[P]/dt = k_3[ES]$). Assuming that $[ES]$ is a constant, find v as a function of $[S]$ and the total enzyme concentration $[E_T] = [E] + [ES]$. As a nonlinear dynamical system, what are the equilibrium points?

17. In Equation (2.58) we saw a linear version of an epidemic model. The commonly used nonlinear SIR model is given by

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{3.96}$$

where S is the number of susceptible individuals, I is the number of infected individuals, and R are the number who have been removed from the the other groups, either by recovering or dying.

a. Let $N = S + I + R$ be the total population. Prove that $N =$ constant. Thus, one need only solve the first two equations and find $R = N - S - I$ afterwards.

b. Find and classify the equilibria. Describe the equilibria in terms of the population behavior.

c. Let $\beta = 0.05$ and $\gamma = 0.2$. Assume that in a population of 100 there is one infected person. Numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

d. The equations can be modified by adding constant birth and death rates. Assuming these are te same, one would have a new system.

$$\frac{dS}{dt} = -\beta SI + \mu(N - S)$$

$$\begin{aligned} \frac{dI}{dt} &= \beta SI - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{3.97}$$

How does this affect any equilibrium solutions?

e. Again, let $\beta = 0.05$ and $\gamma = 0.2$. Let $\mu = 0.1$ For a population of 100 with one infected person numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

18. An undamped, unforced Duffing equation, $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$, can be solved exactly in terms of elliptic functions. Using the results of Exercise 3.18, determine the solution of this equation and determine if there are any restrictions on the parameters.

19. Determine the circumference of an ellipse in terms of an elliptic integral.

20. Evaluate the following in terms of elliptic integrals and compute the values to four decimal places.

a. $\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$.

b. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}$.

c. $\int_0^{2} \frac{dx}{\sqrt{(9-x^2)(4-x^2)}}$.

d. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$.

e. $\int_1^{\infty} \frac{dx}{\sqrt{x^4-1}}$.