Ordinary Differential Equations — Spring 2025
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Fourier-Legendre Series Example

Example: Find the Fourier-Legendre series expansion of $f(x) = 1 - |x|$.

We seek the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Since $f(x)$ is an even function, $c_n = 0$ for n odd. Computing c_0, we have

$$c_0 = \frac{1}{2} \int_{-1}^{1} (1 - |x|)\, dx = \int_0^1 (1-x)\, dx = \frac{1}{2}.$$

In order to compute the nonzero coefficients, we need the identities

$$\begin{aligned} (2n+1) x P_n(x) &= (n+1) P_{n+1}(x) + n P_{n-1}(x), && (1)\\ (2n+1) P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x). && (2) \end{aligned}$$

We seek to compute for n even

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} (1 - |x|) P_n(x)\, dx = (2n+1) \int_0^1 (1-x) P_n(x)\, dx. \tag{3}$$

We have already seen that identity (2) gives

$$(2n+1) \int_0^1 P_n(x)\, dx = \left[P_{n+1}(x) - P_{n-1}(x).\right]_0^1 = P_{n-1}(0) - P_{n+1}(0).$$

Since n is even, $P_{n-1}(0) = P_{n+1}(0) = 0$. So, this integral vanishes.

The second integral in Equation (3) makes use of both identities (1)-(2). First, we note that

$$(2n+1) \int_0^1 x P_n(x)\, dx = \int_0^1 \left[(n+1) P_{n+1}(x) + n P_{n-1}(x)\right] dx.$$

Making use of identity (2), we have the separate integrals

$$\begin{aligned} \int_0^1 P_{n+1}(x)\, dx &= \frac{1}{2n+3} \left[P_{n+2}(x) - P_n(x)\right]_0^1 \\ \int_0^1 P_{n+1}(x)\, dx &= \frac{1}{2n-1} \left[P_n(x) - P_{n-2}(x)\right]_0^1. && (4) \end{aligned}$$

Therefore,

$$\begin{aligned}(2n+1)\int_0^1 xP_n(x)\,dx &= \frac{n+1}{2n+1}\left[P_n(0)-P_{n+2}(0)\right] \\ &\quad +\frac{n}{2n-1}\left[P_{n-2}(0)-P_n(0)\right].\end{aligned}$$

Letting $n=2\ell$, $\ell=1,2,3,\ldots$, we then have

$$\begin{aligned}c_{2\ell} &= -(4\ell+1)\int_0^1 xP_{2\ell}(x)\,dx \\ &= \frac{2\ell+1}{4\ell+1}\left[P_{2\ell+2}(0)-P_{2\ell}(0)\right] \\ &\quad +\frac{2\ell}{4\ell-1}\left[P_{2\ell}(0)-P_{2\ell-2}(0)\right]. \\ &= \frac{2\ell+1}{4\ell+1}P_{2\ell+2}(0)+\frac{4\ell+1}{(4\ell+3)(4\ell+1)}P_{2\ell}(0)-\frac{2\ell}{4\ell-1}P_{2\ell-2}(0).\end{aligned}\tag{5}$$

We can attempt to simplify the Fourier-Legendre coefficients using the identities

$$\begin{aligned}(2n)!! &= 2^n n! \\ (2n-1)!! &= \frac{(2n)!}{2^n n!}.\end{aligned}\tag{6}$$

Then, we have

$$\begin{aligned}P_{2\ell}(0) &= (-1)^\ell\frac{(2\ell-1)!!}{(2\ell)!!} \\ &= (-1)^\ell\frac{(2\ell)!}{2^{2\ell}(\ell!)^2} \\ P_{2\ell+2}(0) &= (-1)^{\ell+1}\frac{(2\ell+2)!}{2^{2\ell+2}((\ell+1)!)^2} \\ P_{2\ell-2}(0) &= (-1)^{\ell-1}\frac{(2\ell-2)!}{2^{2\ell-2}((\ell-1)!)^2}.\end{aligned}\tag{7}$$

This gives

$$c_{2\ell}=\frac{4^{-\ell}(4\ell+1)(-1)^\ell\Gamma(2\ell-1)}{\ell(\ell+1)\Gamma(\ell)^2},$$

Therefore,

$$f(x)=\frac{1}{2}+\sum_{\ell=1}^\infty\frac{4^{-\ell}(4\ell+1)(-1)^\ell\Gamma(2\ell-1)}{\ell(\ell+1)\Gamma(\ell)^2}P_{2\ell}(x).\tag{8}$$

The plot of this series with that of $f(x)$ is shown in Figure 1.

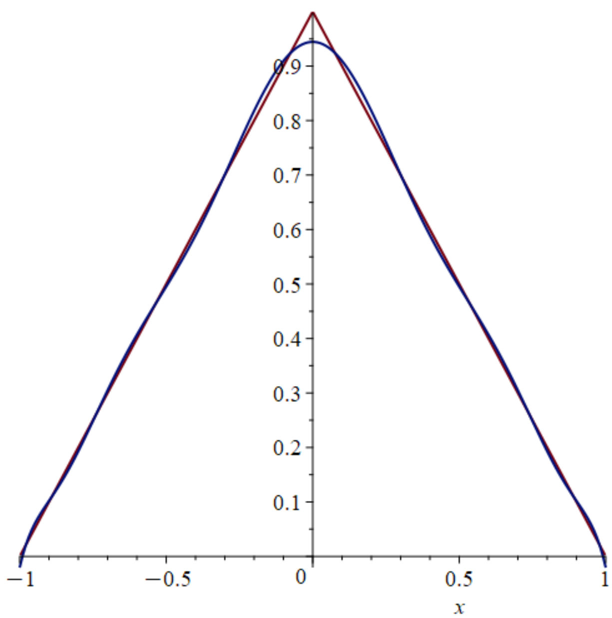

Figure 1: A plot of the Fourier-Legendre expansion of $f(x) = 1 - |x|$ with the plot of $f(x)$ using five terms from the sum in Equation (8).