Green's Function Example

We consider solving a problem of the form

$$y'' + 2y' + 2y = f(x), \quad y(0) = 0, y\left(\frac{\pi}{2}\right) = 0. \tag{1}$$

This problem takes the form $Ly = f$, where $L = D^2 + 2D + 2$ is not a self-adjoint operator. The adjoint operator is given by $L^\dagger = D^2 - 2D + 2$. We can also put this problem in Sturm-Liouville form,

$$\mathcal{L}y(x) = \frac{d}{dx}\left(e^{2x}\frac{dy(x)}{dx}\right) + 2e^{2x}y(x) = e^{2x}f(x). \tag{2}$$

We seek the Green's functions associated with each operator satisfying homogeneous boundary conditions and the equations

$$\begin{aligned} LG(x,\xi) &= \delta(x-\xi) \\ L^\dagger G^A(x,\xi) &= \delta(x-\xi) \\ \mathcal{L}\mathcal{G}(x,\xi) &= \delta(x-\xi). \end{aligned} \tag{3}$$

We then use $G(x,\xi)$ and $\mathcal{G}(x,\xi)$ to construct solutions to the boundary value problem.

We first seek the Green's function, $G(x,\xi)$, satisfying

$$\frac{\partial^2 G(x,\xi)}{\partial x^2} + 2\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi) = \delta(x-\xi), \tag{4}$$

and the boundary conditions $G(0,\xi) = 0$, $G\left(\frac{\pi}{2},\xi\right) = 0$. We will see that this Green's function is not symmetric.

We will then find the adjoint Green's function, $G^A(x,\xi)$, satisfying

$$\frac{\partial^2 G^A(x,\xi)}{\partial x^2} - 2\frac{\partial G^A(x,\xi)}{\partial x} + 2G^A(x,\xi) = \delta(x-\xi), \tag{5}$$

and the boundary conditions $G^A(0,\xi) = 0, G^A\left(\frac{\pi}{2},\xi\right) = 0$. We will show that $G(\xi,x) = G^A(x,\xi)$ and use both functions to find the solution to the boundary value problem. We will then show that this solution is the same as using the Sturm-Liouville operator.

Example 1. Find the Green's function satisfying Equation (4).

Defining $g(x) = G(x,\xi)$, then for $x \neq \xi$,

$$g'' + 2g' + 2g = 0, \quad g(0) = 0, g\left(\frac{\pi}{2}\right) = 0.$$

The characteristic equation is $r^2 + 2r + 2 = 0$. So, $r = -1 \pm i$. This gives the general solution as

$$g(x) = e^{-x}(a\cos x + b\sin x).$$

For $0 \leq x \leq \xi$, we find the solution $g_1(x)$ satisfying the boundary condition $g_1(0) = 0$.

$$g_1(0) = e^0(a\cos 0 + b\sin 0) = a = 0.$$

So, $g_1(x) = be^{-x}\sin x$.

Similarly, we find the solution $g_2(x)$, $\xi \leq x \leq \frac{\pi}{2}$, satisfying the boundary condition $g_2\left(\frac{\pi}{2}\right) = 0$. In this case we find $g_2(x) = ae^{-x}\cos x$.

Now we construct the Green's function. So far, we have the piecewise defined function

$$G(x,\xi) = \begin{cases} be^{-x}\sin x, & 0 \leq x \leq \xi, \\ ae^{-x}\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{6}$$

The first condition is that $G(x,\xi)$ be continuous at $x = \xi$. This gives

$$be^{-\xi}\sin\xi = ae^{-\xi}\cos\xi.$$

This can be satisfied by defining

$$a = c\sin\xi, \quad b = c\cos\xi.$$

So, we have

$$G(x,\xi) = \begin{cases} ce^{-x}\sin x\cos\xi, & 0 \leq x \leq \xi, \\ ce^{-x}\sin\xi\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{7}$$

The next condition is that $\frac{\partial G(x,\xi)}{\partial x}$ is discontinuous at $x = \xi$. We show this by integrating Equation (4) over the interval $x \in [\xi - \epsilon, \xi + \epsilon]$. Using the definition of the Dirac delta function and continuity of $G(x,\xi)$, we let ϵ approach zero to obtain

$$\begin{aligned} \int_{\xi-\epsilon}^{\xi+\epsilon} \left[\frac{\partial^2 G(x,\xi)}{\partial x^2} + 2\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi)\right] dx &= \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx \\ \lim_{\epsilon\to 0} \left[\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi)\right]_{\xi-\epsilon}^{\xi+\epsilon} &= \int_{-\infty}^{\infty} \delta(x-\xi)\, dx \\ \left[\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} &= 1 \end{aligned} \tag{8}$$

This gives a jump condition for the discontinuity of the derivative at $x = \xi$ where ξ^+ is the value above $x = \xi$ and ξ^- is the value below $x = \xi$.

We can apply this to $G(x,\xi)$. Namely, we have

$$\begin{aligned} 1 &= \left[\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} = ce^{-\xi}\left[-\sin\xi\cos\xi - \sin^2\xi - (-\sin\xi\cos\xi + \cos^2\xi)\right] \\ &= -ce^{-\xi}. \end{aligned}$$

So, $c = -e^{\xi}$ and the Green's function is

$$G(x,\xi) = \begin{cases} -e^{\xi-x}\sin x\cos\xi, & 0 \leq x \leq \xi, \\ -e^{\xi-x}\sin\xi\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{9}$$

We see that $G(x,\xi)$ is not symmetric, $G(x,\xi) \neq G(\xi,x)$. Thus, it seems that $G(x,\xi)$ does not satisfy a reciprocity condition.

Example 2. Find the adjoint Green's function satisfying Equation (5).

The derivation parallels that for $G(x,\xi)$. However, in this case we start with the general solution $g^A(x) = G^A(x,\xi)$, for

$$g^{A''} - 2g^{A'} + 2g^A = 0, \quad x \neq \xi, \quad g^A(0) = 0, g^A\left(\frac{\pi}{2}\right) = 0$$

as

$$g^A(x) = e^x(a\cos x + b\sin x).$$

Following the same steps as before, we find the adjoint Green's function,

$$G^A(x,\xi) = \begin{cases} -e^{x-\xi}\sin x\cos\xi, & 0 \leq x \leq \xi, \\ -e^{x-\xi}\sin\xi\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{10}$$

While $G^A(x,\xi) \neq G^A(\xi,x)$, we do have that $G(x,\xi) = G^A(\xi,x)$ gives the form of a reciprocity condition.

Example 3. Derive a general relation between $G(x,\xi)$ and $G^A(x,\xi)$.

From the last example we have found $G(x,\xi) = G^A(\xi,x)$ for a specific problem. In general these Green's functions satisfy the equations for $x \in [a,b]$:

$$\begin{aligned} LG(x,\xi) &= \delta(x-\xi) \\ L^A G^A(x,\xi') &= \delta(x-\xi'). \end{aligned} \tag{11}$$

Multiply the first equation by $G^A(x,\xi')$ and the second equation by $G(x,\xi)$. Subtract and integrate

$$\int_a^b \left[G^A(x,\xi')LG(x,\xi) - G(x,\xi)L^A G^A(x,\xi')\right]\,dx = \int_a^b \left[G^A(x,\xi')\delta(x-\xi) - G(x,\xi)\delta(x-\xi')\right]\,dx.$$

Assuming appropriate boundary conditions, we have

$$\int_a^b G^A(x,\xi')LG(x,\xi)\,dx = \int_a^b G(x,\xi)L^A G^A(x,\xi')\,dx.$$

So, after applying the Dirac delta function integrations, we have

$$G^A(\xi,\xi') = G(\xi',\xi).$$

Now we can return to the original problem but adding nonhomogeneous boundary conditions.

Example 4. Use the adjoint Green's function to solve

$$y'' + 2y' + 2y = f(x), \quad y(0) = A, y\left(\frac{\pi}{2}\right) = B. \tag{12}$$

Defining $L = D^2 + 2D + 2$, we have

$$\begin{aligned} Ly(x) &= f(x) \\ L^A G^A(x,\xi) &= \delta(x-\xi). \end{aligned} \tag{13}$$

As with the previous example, we multiply the first equation by $G^A(x,\xi)$ and the second equation by $y(x)$. Subtracting and integrating we have

$$\begin{aligned} &\int_0^{\pi/2} \left[G^A(x,\xi)\left(y''(x) + 2y'(x) + 2y(x)\right) - y\left(\frac{\partial^2 G^A(x,\xi)}{\partial x^2} - 2\frac{\partial G^A(x,\xi)}{\partial x} + 2G^A(x,\xi)\right)\right]\,dx \\ = &\int_0^{\pi/2} \left[G^A(x,\xi)f(x) - y(x)\delta(x-\xi)\right]\,dx \end{aligned} \tag{14}$$

Cleaning this up, we find that

$$\int_0^{\pi/2} \frac{\partial}{\partial x}\left[G^A(x,\xi)y'(x) - \frac{\partial G^A(x,\xi)}{\partial x}y(x) + 2y(x)G^A(x,\xi)\right]\,dx = \int_0^{\pi/2} G^A(x,\xi)f(x)\,dx - y(\xi),$$

or, after applying boundary conditions,

$$\begin{aligned} y(\xi) &= \int_0^{\pi/2} G^A(x,\xi)f(x)\,dx - \left[G^A(x,\xi)y'(x) - \frac{\partial G^A(x,\xi)}{\partial x}y(x) + 2y(x)G^A(x,\xi)\right]_0^{\pi/2} \\ &= \int_0^{\pi/2} G^A(x,\xi)f(x)\,dx + \frac{\partial G^A(\frac{\pi}{2},\xi)}{\partial x}B - \frac{\partial G^A(0,\xi)}{\partial x}A. \end{aligned} \tag{15}$$

Since $G^A(x,\xi) = G(\xi,x)$, we can exchange variables to obtain the solution in terms of the Green's function, $G(x,\xi)$:

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G^A(\xi,x)f(\xi)\,d\xi + \frac{\partial G^A(\frac{\pi}{2},x)}{\partial \xi}B - \frac{\partial G^A(0,x)}{\partial \xi}A \\ &= \int_0^{\pi/2} G(x,\xi)f(\xi)\,d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi}B - \frac{\partial G(x,0)}{\partial \xi}A. \end{aligned} \tag{16}$$

We now apply this general solution to a specific problem.

Example 5. Use Equation (16) with the Green's function in Equation (9) to solve

$$y'' + 2y' + 2y = 5\sin x, \quad y(0) = 2e, y\left(\frac{\pi}{2}\right) = 0. \tag{17}$$

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G(x,\xi)f(\xi)\,d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi}B - \frac{\partial G(x,0)}{\partial \xi}A \\ &= \int_0^{x} G(x,\xi)f(\xi)\,d\xi + \int_x^{\pi/2} G(x,\xi)f(\xi)\,d\xi - 2e\frac{\partial G(x,0)}{\partial \xi} \\ &= \int_0^{x} \left[-e^{\xi-x}\sin\xi\cos x\right]5\sin(\xi)\,d\xi + \int_x^{\pi/2} \left[-e^{\xi-x}\sin x\cos\xi\right]5\sin(\xi)\,d\xi \\ &\quad -2e\frac{\partial}{\partial \xi}\left[-e^{\xi-x}\sin\xi\cos x\right]_{\xi=0} \\ &= -5e^{-x}\cos x\int_0^{x} e^{\xi}\sin^2\xi\,d\xi - 5e^{-x}\sin x\int_x^{\pi/2} e^{\xi}\sin\xi\cos\xi\,d\xi + 2e^{1-x}\cos x \\ &= -5e^{-x}\left[\left(-\frac{2}{5} + \frac{1}{5}e^x\sin^2 x - \frac{2}{5}e^x\sin x\cos x + \frac{2}{5}e^x\right)\cos x\right. \\ &\quad \left. + \left(-\frac{1}{10}e^x\sin 2x + (\frac{1}{5}e^x\cos 2x + \frac{1}{5}e^{\frac{\pi}{2}}\right)\sin x\right] + 2e^{1-x}\cos x \\ &= 2e^{-x}\cos x - 2\cos x + \sin x - e^{\frac{\pi}{2}-x}\sin x + 2e^{1-x}\cos x \\ &= 2\left[e^{-x}(1+e) - 1\right]\cos x + \left(1 - e^{\frac{\pi}{2}-x}\right)\sin x. \end{aligned} \tag{18}$$

We can check this solution by using the Method of Undetermined Coefficients to obtain the solution. We know the solution to the homogeneous equation is

$$y_h(x) = e^{-x}(c_1 \sin x + c_2 \cos x).$$

We seek a particular solution,

$$y_p(x) = c_3 \sin x + c_4 \cos x.$$

Inserting into the differential equation, we have

$$[c_3 - 2c_4] \sin x + [c_4 + 2c_3] \cos x = 5 \sin x.$$

This is true when $c_3 = 1$ and $c_4 = -2$. So, the general solution to the nonhomogeneous equation is

$$y(x) = e^{-x}(c_1 \sin x + c_2 \cos x) + \sin x - 2 \cos x.$$

For the solution of the boundary value problem, we need to satisfy the boundary conditions.

$$\begin{aligned} y(0) &= c_2 - 2 = 2e, \\ y\left(\frac{\pi}{2}\right) &= e^{-\frac{\pi}{2}} c_1 + 1 = 0. \end{aligned} \tag{19}$$

So, $c_2 = 2(1 + e)$ and $c_1 = -e^{\frac{\pi}{2}}$ and the solution is

$$\begin{aligned} y(x) &= e^{-x}(-e^{\frac{\pi}{2}} \sin x + 2(1 + e) \cos x) + \sin x - 2 \cos x \\ &= 2\left[e^{-x}(1 + e) - 1\right] \cos x + \left(1 - e^{\frac{\pi}{2} - x}\right) \sin x. \end{aligned} \tag{20}$$

So, the solutions agree.

We have seen how we can solve for and use the Green's function and adjoint Green's function in an example inolving a non-Hermitian operator. However, we also know that we can cast the problem in Sturm-Liouville form. So, how do these methods differ if we used the Sturm-Liouville operator and its Green's function?

Example 6. Consider the boundary value problem

$$y'' + 2y' + 2y = f(x), \quad y(0) = A, y\left(\frac{\pi}{2}\right) = B. \tag{21}$$

Put this in Sturm-Liouville form, find its Green's function, and write the solution in terms of the Green's function.

The Sturm-Liouville form of the differential equation is

$$\frac{d}{dx}\left(e^{2x} \frac{dy(x)}{dx}\right) + 2e^{2x} y(x) = e^{2x} f(x). \tag{22}$$

The associated Green's function would then satisfy

$$\frac{\partial}{\partial x}\left(e^{2x} \frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right) + 2e^{2x} \mathcal{G}(x,\xi) = \delta(x - \xi). \tag{23}$$

Following Example 1, we find the Green's function satisfying the homogeneous boundary conditions, $G(0,\xi)=0$, $G\left(\frac{\pi}{2},\xi\right)=0$, takes the form

$$\mathcal{G}(x,\xi)=\begin{cases} ce^{-x}\sin x\cos\xi, & 0\le x\le\xi,\\ ce^{-x}\sin\xi\cos x, & \xi\le x\le\frac{\pi}{2}.\end{cases} \tag{24}$$

So far, this function is continuous at $x=\xi$.

We need to derive the jump condition for the discontinuity of the derivative of the Green's function. As before, we integrate the Green's function equation over the interval $x\in[\xi-\epsilon,\xi+\epsilon]$ to obtain

$$\begin{aligned}\int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right)+2e^{2x}\mathcal{G}(x,\xi)\right]dx &= \int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\,dx\\ \lim_{\epsilon\to 0}\left[e^{2x}\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right]_{\xi-\epsilon}^{\xi+\epsilon} &= \int_{-\infty}^{\infty}\delta(x-\xi)\,dx\\ \left[e^{2x}\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} &= 1 \end{aligned}\tag{25}$$

We can apply this to $\mathcal{G}(x,\xi)$. Namely, we have

$$\begin{aligned}1 &= \left[e^{2x}\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+}=ce^{2\xi}e^{-\xi}\left[-\sin\xi\cos\xi-\sin^2\xi-(-\sin\xi\cos\xi+\cos^2\xi)\right]\\ &= -ce^{\xi}.\end{aligned}$$

So, $c=-e^{-\xi}$ and the Green's function is

$$\mathcal{G}(x,\xi)=\begin{cases} -e^{-x-\xi}\sin x\cos\xi, & 0\le x\le\xi,\\ -e^{-x-\xi}\sin\xi\cos x, & \xi\le x\le\frac{\pi}{2}.\end{cases} \tag{26}$$

We see that $\mathcal{G}(x,\xi)$ is symmetric, $\mathcal{G}(x,\xi)=\mathcal{G}(\xi,x)$.

We can now find the solution to Equation (21) using this Green's function. Defining

$$\mathcal{L}=\frac{d}{dx}\left(e^{2x}\frac{d}{dx}\right)+2e^{2x},$$

$y(x)$ and $\mathcal{G}$ satisfy the differential equations

$$\begin{aligned}\mathcal{L}y(x) &= e^{2x}f(x)\\ \mathcal{L}\mathcal{G}(x,\xi) &= \delta(x-\xi).\end{aligned}\tag{27}$$

As with the previous example, we multiply the first equation by $\mathcal{G}(x,\xi)$ and the second equation by $y(x)$. Subtracting and integrating we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\left[\mathcal{G}(x,\xi)\mathcal{L}y(x)-y(x)\mathcal{L}\mathcal{G}(x,\xi)\right]dx &= \int_0^{\frac{\pi}{2}}\left[e^{2x}f(x)\mathcal{G}(x,\xi)-y(x)\delta(x-\xi)\right]dx\\ \left[e^{2x}\left(\mathcal{G}(x,\xi)\frac{dy(x)}{dx}-y(x)\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right)\right]_0^{\frac{\pi}{2}} &= \int_0^{\frac{\pi}{2}}e^{2x}f(x)\mathcal{G}(x,\xi)\,dx-y(\xi)\\ y(\xi) &= \int_0^{\frac{\pi}{2}}e^{2x}f(x)\mathcal{G}(x,\xi)\,dx+e^{\pi}B\frac{\partial\mathcal{G}(\frac{\pi}{2},\xi)}{\partial x}-A\frac{\partial\mathcal{G}(0,\xi)}{\partial x}\end{aligned}\tag{28}$$

Computing the derivative of the Green's function,

$$\frac{\partial \mathcal{G}(x,\xi)}{\partial x} = \begin{cases} e^{-x-\xi}(\sin x\cos\xi - \cos x\cos\xi), & 0 \le x \le \xi, \\ e^{-x-\xi}(\sin\xi\cos x + \sin\xi\sin x), & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{29}$$

and evaluating it at the boundary points, we have

$$\frac{\partial \mathcal{G}(0,\xi)}{\partial x} = -e^{-\xi}\cos\xi, \quad \frac{\partial \mathcal{G}(\frac{\pi}{2},\xi)}{\partial x} = e^{-\frac{\pi}{2}-\xi}\sin\xi.$$

The solution can be written as

$$y(\xi) = \int_0^{\frac{\pi}{2}} e^{2x} f(x)\mathcal{G}(x,\xi)\,dx + Be^{\frac{\pi}{2}-\xi}\sin\xi + Ae^{-\xi}\cos\xi$$

or

$$y(x) = \int_0^{\frac{\pi}{2}} e^{2\xi} f(\xi)\mathcal{G}(x,\xi)\,dx + Be^{\frac{\pi}{2}-x}\sin x + Ae^{-x}\cos x.$$

Let's compare this with the solution in Equation (16). We need

$$\frac{\partial G(x,\xi)}{\partial \xi} = \begin{cases} -e^{\xi-x}(\sin x\cos\xi - \sin x\sin\xi), & 0 \le x \le \xi, \\ -e^{\xi-x}(\sin\xi\cos x + \cos\xi\cos x), & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{30}$$

Then,

$$\frac{\partial G(x,0)}{\partial \xi} = -e^{-x}\cos x, \quad \frac{\partial G(x,\frac{\pi}{2})}{\partial x} = e^{\frac{\pi}{2}-x}\sin x.$$

Inserting these values and noting that $G(x,\xi) = e^{2\xi}\mathcal{G}(x,\xi)$

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G(x,\xi)f(\xi)\,d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi}B - \frac{\partial G(x,0)}{\partial \xi}A \\ &= \int_0^{\pi/2} e^{2\xi}\mathcal{G}(x,\xi)f(\xi)\,d\xi + Be^{\frac{\pi}{2}-x}\sin x + Ae^{-x}\cos x. \end{aligned} \tag{31}$$

Thus, the solutions using the Sturm-Liouville form of the problem and the original form are the same.