Fourier-Bessel Series Examples

We consider examples of the Fourier-Bessel function series on $[0, b]]$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x), \tag{1}$$

where

$$c_i = \frac{\int_0^b x J_n(\alpha_i x) f(x)\, dx}{\|J_n(\alpha_i x)\|^2}, \quad i = 1, 2, \ldots. \tag{2}$$

Here, $J_n(\alpha x)$ is a solution of Bessel's equation in the form

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0. \tag{3}$$

We assume that one boundary condition is that the solution is finite at the origin. The other homogeneous boundary conditions at $x = b$ take the form

$$Ay(b) + By'(b) = 0.$$

Noting that from the Chain Rule we have

$$\frac{dJ_n(\alpha x)}{dx} = \alpha J_n'(\alpha x),$$

then this boundary condition can be written as

$$AJ_n(\alpha b) + \alpha B J_n'(\alpha b) = 0.$$

Solutions of this equation yield the values α_i, for $i = 1, 2, \ldots$. This also impacts the value of $\|J_n(\alpha_i x)\|^2$ as well as providing the orthogonality condition

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x)\, dx = \|J_n(\alpha_i x)\|^2 \delta_{ij}.$$

One can derive the expansion coefficients for different types of boundary conditions. The main affect is in the value of $\|J_n(\alpha_i x)\|^2$.

- For $J_n(\alpha b) = 0$, we have that
$$\|J_n(\alpha_i x)\|^2 = \frac{b^2}{2} \left[J_{n+1}(\alpha_i b)\right]^2.$$
- For $J_0'(\alpha b) = 0$, we have that
$$\|J_0(\alpha_i x)\|^2 = \frac{b^2}{2} \left[J_0(\alpha_i b)\right]^2,$$
for $i > 1$. For $i = 1$, we have
$$\|J_0(\alpha_i x)\|^2 = \frac{b^2}{2}.$$

- For $AJ_n(\alpha b) + \alpha BJ_n'(\alpha b) = 0$, we have that

$$\|J_n(\alpha_i x)\|^2 = \frac{\alpha_i^2 b^2 - n^2 + A^2}{2\alpha_i^2} \left[J_n(\alpha_i b)\right]^2.$$

We now look at a couple of examples of Fourier-Bessel series. We note that several identities may be useful.

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \tag{4}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n} J_{n+1}(x). \tag{5}$$

Example 1. Consider the function $f(x) = x^2$, $x \in [0,1]$. Find the Fourier-Bessel series expansion in $J_2(\alpha_i x)$ for $J_2(\alpha_i) = 0$.

Here we have that

$$c_i = \frac{2}{\left[J_3(\alpha_i)\right]^2} \int_0^1 x^3 J_2(\alpha_i x)\, dx.$$

We can compute the integral using a substitution of $t = \alpha_i x$ and the identity in Equation (4).

$$\begin{aligned} \int_0^1 x^3 J_2(\alpha_i x)\, dx &= \frac{1}{\alpha_i^4} \int_0^{\alpha_i} t^3 J_2(t)\, dt \\ &= \frac{1}{\alpha_i^4} \int_0^{\alpha_i} \frac{d}{dt}\left[t^3 J_3(x)\right]\, dt \\ &= \frac{1}{\alpha_i^4} \left[t^3 J_3(x)\right]_0^{\alpha_i} \\ &= \frac{1}{\alpha_i} J_3(\alpha_i). \end{aligned} \tag{6}$$

This gives

$$c_i = \frac{2}{J_3(\alpha_i)}$$

and

$$x^2 = \sum_{i=1}^{\infty} \frac{2}{J_3(\alpha_i)} J_2(\alpha_i x), \tag{7}$$

where $J_2(\alpha_i) = 0$. In Figure 1 is a plot of this function with 100 terms.

Example 2. Consider the function $f(x) = 1$, $x \in [0,2]$. Find the Fourier-Bessel series expansion in $J_0(\alpha_i x)$ for $J_0'(2\alpha_i) = 0$.

Here we have that for $i > 1$,

$$c_i = \frac{1}{2\left[J_0(2\alpha_i)\right]^2} \int_0^2 x J_0(\alpha_i x)\, dx.$$

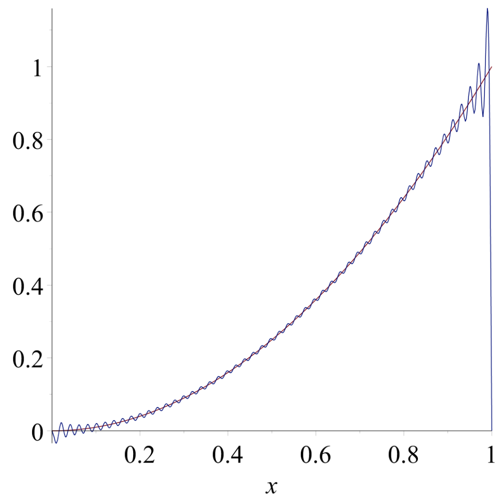

Figure 1: Plot of Fourier-Bessel series expansion for $f(x) = x^2$ using 100 terms.

We can compute the integral using a substitution of $t = \alpha_i x$ and the identity in Equation (**??**).

$$\begin{aligned} \int_0^2 x J_0(\alpha_i x)\, dx &= \frac{1}{\alpha_i^2} \int_0^{2\alpha_i} t J_0(t)\, dt \\ &= \frac{1}{\alpha_i^2} \int_0^{2\alpha_i} \frac{d}{dt} \left[t J_1(x) \right] dt \\ &= \frac{1}{\alpha_i^2} \left[t J_1(x) \right]_0^{2\alpha_i} \\ &= \frac{2}{\alpha_i} J_1(2\alpha_i). \end{aligned} \tag{8}$$

This gives

$$c_i = \frac{1}{2\left[J_0(2\alpha_i)\right]^2} \frac{2}{\alpha_i} J_1(2\alpha_i) = \frac{J_1(2\alpha_i)}{\alpha_i \left[J_0(2\alpha_i)\right]^2}.$$

We note that for $i = 1$, we also have

$$c_1 = \frac{2}{b^2} \int_0^b x f(x)\, dx = \frac{1}{2} \int_0^2 x\, dx = 1.$$

Then, we find that

$$1 = 1 + \sum_{i=1}^{\infty} \frac{J_1(2\alpha_i)}{\alpha_i \left[J_0(2\alpha_i)\right]^2} J_0(\alpha_i x), \tag{9}$$

In Figure 2 is a plot of this function with 10 terms. where $J_0'(2\alpha_i) = 0$.

The theory and history of series involving Bessel function is given in Chapter XVIII of Watson's *A Treatise on the Theory of Bessel Functions.* Series of the form

$$f(x) = \sum_{m=1}^{\infty} a_m J_0(j_m x), \quad 0 < x < 1,$$

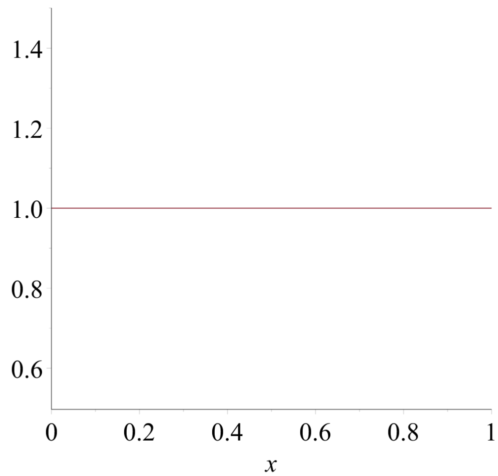

Figure 2: Plot of Fourier-Bessel series expansion for $f(x) = 1$ using 10 terms.

where the j_m are positive zeros of $J_0(x)$ were treated in studies by Fourier of the conduction of heat, Daniel Bernoulli's vibrating chain, and Euler's vibrating circular membrane.

This can be extended to the Fourier-Bessel series

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(j_m x), \quad 0 < x < 1,$$

where the j_m are positive zeros of $J_n(x)$. Multiplying the series by $xJ_n(j_k x)$ and integrating, one has

$$\begin{aligned}
\int_0^1 xf(x)J_n(j_k x)\,dx &= \sum_{m=1}^{\infty} a_m \int_0^1 xJ_n(j_m x)J_n(j_k x)\,dx \\
&= a_k \|J_n(j_k x)\|^2 \delta_{k,m}. \qquad (10)
\end{aligned}$$

According to Watson, the expansion coefficients can be determined from

$$2\lambda_k^2 \int_0^1 xf(x)J_n(\lambda_k x)\,dx = [(\lambda_k^2 - n^2)J_n^2(\lambda_k) + \lambda_k^2 {J_n'}^2(\lambda_k)]a_k$$

So, if $\lambda_k = j_k$ are the zeros of $J_n(x)$, then

$$\|J_n(j_k x)\|^2 = \frac{1}{2}{J_n'}^2(\lambda_k)$$

and

$$a_k = \frac{2}{{J_n'}^2(\lambda_k)} \int_0^1 xf(x)J_n(j_k x)\,dx.$$

In the more general case, one can have mixed boundary conditions, $f'(1) + Af(1) = 0$. Then, the λ_n's are roots of $z^{-n}[zJ_n(z) + AJ_n(z)]$. The resulting series is called a Dini series. In the next example we derive the formula giving for $\|J_n(\alpha x)\|^2$.

Example 3. Show that

$$\int_0^b x[J_n(\alpha x)]^2\,dx = \frac{1}{2\alpha^2}\left[\left(\alpha^2 b^2 - n^2\right) J_n^2(\alpha b) + \alpha^2 b^2 J_n'^2(\alpha b)\right].$$

We begin with the Bessel equation (3) for different values of α. Consider the equations

$$\frac{d}{dx}\left(x\frac{dJ_n(\alpha x)}{dx}\right) + \left(\alpha^2 x - \frac{n^2}{x}\right) J_n(\alpha x) = 0. \tag{11}$$

$$\frac{d}{dx}\left(x\frac{dJ_n(\beta x)}{dx}\right) + \left(\beta^2 x - \frac{n^2}{x}\right) J_n(\beta x) = 0. \tag{12}$$

Multiply Equation (11) by $J_n(\beta x)$, Equation (12) by $J_n(\alpha x)$, subtract and rearrange to find

$$\begin{aligned} J_n(\beta x)\left[\frac{d}{dx}\left(x\frac{dJ_n(\alpha x)}{dx}\right)\right] - J_n(\alpha x)\left[\frac{d}{dx}\left(x\frac{dJ_n(\beta x)}{dx}\right)\right] &= (\beta^2-\alpha^2)xJ_n(\alpha x)J_n(\beta x) \\ \frac{d}{dx}\left[xJ_n(\beta x)\frac{dJ_n(\alpha x)}{dx} - xJ_n(\alpha x)\frac{dJ_n(\beta x)}{dx}\right] &= (\beta^2-\alpha^2)xJ_n(\alpha x)J_n(\beta x). \qquad (13) \end{aligned}$$

Integrate from $x = 0$ to $x = b$, to obtain

$$\begin{aligned} (\beta^2-\alpha^2)\int_0^b xJ_n(\alpha x)J_n(\beta x)\,dx &= \left[xJ_n(\beta x)\frac{dJ_n(\alpha x)}{dx} - xJ_n(\alpha x)\frac{dJ_n(\beta x)}{dx}\right]_0^b \\ &= bJ_n(\beta b)\frac{dJ_n(\alpha x)}{dx} - bJ_n(\alpha b)\frac{dJ_n(\beta x)}{dx}. \qquad (14) \end{aligned}$$

So, if $\alpha \neq \beta$ and $AJ_n(\alpha b) + \alpha BJ_n'(\alpha b) = 0$, then

$$\int_0^b xJ_n(\alpha x)J_n(\beta x)\,dx = 0.$$

However, what if $\alpha = \beta$? Then, we need to compute

$$L = \lim_{\alpha\to\beta}\frac{1}{\beta^2-\alpha^2}\left[xJ_n(\beta x)\frac{dJ_n(\alpha x)}{dx} - xJ_n(\alpha x)\frac{dJ_n(\beta x)}{dx}\right]_0^b$$

using l'Hopital's Rule. First, we simplify the numerator using the identity

$$\frac{dJ_n(z)}{dz} = \frac{n}{z}J_n(z) - J_{n+1}(z).$$

Let $z = \alpha x$. Then,

$$\begin{aligned} \frac{dJ_n(z)}{dx} &= \frac{dJ_n(z)}{dz}\frac{dz}{dx} \\ &= \alpha\left[\frac{n}{z}J_n(z) - J_{n+1}(z)\right] \\ \frac{dJ_n(\alpha x)}{dx} &= \frac{n}{x}J_n(\alpha x) - \alpha J_{n+1}(\alpha x). \qquad (15) \end{aligned}$$

So, we have

$$\begin{aligned}\left[xJ_n(\beta x)\frac{dJ_n(\alpha x)}{dx} - xJ_n(\alpha x)\frac{dJ_n(\beta x)}{dx}\right]_0^b &= bJ_n(\beta b)\left[\frac{n}{b}J_n(\alpha b) - \alpha J_{n+1}(\alpha b)\right] \\ &\quad -bJ_n(\alpha b)\left[\frac{n}{b}J_n(\beta b) - \beta J_{n+1}(\beta b)\right] \\ &= \beta b J_n(\alpha b)J_{n+1}(\beta b) - \alpha b J_{n+1}(\alpha b)J_n(\beta b) \qquad (16)\end{aligned}$$

We also have the identity

$$\frac{dJ_n(z)}{dz} = J_{n-1}(z) - \frac{n}{z}J_n(z).$$

Therefore, we can use the result

$$\frac{dJ_{n+1}(\alpha x)}{d\alpha} = J_n(\alpha x) - \frac{n+1}{\alpha}J_{n+1}(\alpha x)$$

to simplify the numerator in the limit:

$$\begin{aligned}\frac{d}{d\alpha}\left[\beta b J_n(\alpha b)J_{n+1}(\beta b) - \alpha b J_{n+1}(\alpha b)J_n(\beta b)\right] &= \beta b J_{n+1}(\beta b)\left(\frac{n}{\alpha}J_n(\alpha b) - xJ_{n+1}(\alpha b)\right) \\ &\quad -bJ_{n+1}(\alpha b)J_n(\beta b) \\ &\quad -\alpha b J_n(\beta b)\left(bJ_n(\alpha b) - \frac{n+1}{\alpha}J_{n+1}(\alpha b)\right), \qquad (17)\end{aligned}$$

Then the limit simplifies as

$$\begin{aligned}L &= \lim_{\alpha\to\beta}\frac{\frac{d}{d\alpha}\left[\beta b J_n(\alpha b)J_{n+1}(\beta b) - \alpha b J_{n+1}(\alpha b)J_n(\beta b)\right]}{-2\alpha} \\ &= b\frac{J_{n+1}(\beta b)\left[nJ_n(\beta b) - \beta b J_{n+1}(\beta b)\right] - J_{n+1}(\beta b)J_n(\beta b) - J_n(\beta b)\left[\beta b J_n(\beta b) - (n+1)J_{n+1}(\beta b)\right]}{-2\beta} \\ &= \frac{b}{2\beta}\left[\beta b J_n^2(\beta b) - 2nJ_n(\beta b)J_{n+1}(\beta b) + \beta b J_{n+1}^2(\beta b)\right]. \qquad (18)\end{aligned}$$

Since

$$J_{n+1}(z) = \frac{n}{z}J_n(z) - J_n'(z),$$

we can write the result in terms of $J_n(\beta b)$ and $J_n'(\beta b)$ as

$$\begin{aligned}L &= \frac{b}{2\beta}\left[\beta b J_n^2(\beta b) - 2nJ_n(\beta b)\left(\frac{n}{\beta b}J_n(\beta b) - J_n'(\beta b)\right) + \beta b\left(\frac{n}{\beta b}J_n(\beta b) - J_n'(\beta b)\right)^2\right] \\ &= \frac{b}{2\beta}\left[\left(\beta b - \frac{2n^2}{\beta b} + \frac{n^2}{\beta b}\right)J_n^2(\beta b) + (2n-2n)J_n(\beta b)J_n'(\beta b) + \beta b J_n'^2(\beta b)\right] \\ &= \frac{1}{2\beta^2}\left[\left(\beta^2 b^2 - n^2\right)J_n^2(\beta b) + \beta^2 b^2 J_n'^2(\beta b)\right]. \qquad (19)\end{aligned}$$

We have shown that

$$\int_0^b x[J_n(\alpha x)]^2\,dx = \frac{1}{2\alpha^2}\left[\left(\alpha^2 b^2 - n^2\right)J_n^2(\alpha b) + \alpha^2 b^2 J_n'^2(\alpha b)\right]. \qquad (20)$$