MAT 463/563 Final (Boundary Value Problems) Review

I. Preliminary Material
   a. Constant Coefficient Equations
   b. Cauchy-Euler Equations
   c. Integration Methods

II. Boundary Value Problems
   a. Direct Solution
   b. Separation of Variables – Heat Equation
   c. Solution of Eigenvalue Equations with Homogeneous BCs

III. Vector Spaces and Function Spaces
   a. Bases/Eigenfunctions
   b. Scalar Product $\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx$
   c. Orthogonal, Orthonormal, Normalization
   d. Determination of Expansion Coefficients

IV. Fourier Series
   a. Trigonometric Fourier Series Expansion on $[-L, L]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

   b. Fourier Coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

   c. Orthogonality of $\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots\}$ on $[-L, L]$, etc.
   d. Half Range Expansions on $[0, L]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx$$

   e. Fourier Sine and Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

   f. Even Periodic Extension, Odd Periodic Extension, Periodic Extension

V. Sturm-Liouville Eigenvalue Problems
   a. Sturm-Liouville Operator: $L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x).$
   b. Conver SOLDEs to Self-Adjoint Form, $p(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)}dx\right)$

c. Types of Boundary Conditions:
   i. Regular BCs $\alpha u(a)+\beta u'(a)=0, \delta u(b)+\gamma u'(b)=0$
   ii. Dirichlet: $u(a)=u(b)=0.$
   iii. Neumann: $u'(a)=u'(b)=0.$
   iv. Periodic: $u(a)=u(b), u'(a)=u'(b).$

d. Sturm-Liouville Eigenvalue Problems:
$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right]+q(x)u(x)=-\lambda\sigma(x)u(x)$ plus BCs

e. Lagrange's Identity: $uLv-vLu=\frac{d}{dx}\left[p(uv'-vu')\right]$

f. Green's Identity: $\int_a^b (uLv-vLu)\,dx==\left[p(uv'-vu')\right]_a^b.$

g. Adjoint Operators $\langle v,Lu\rangle=\left\langle L^\dagger v,u\right\rangle$

h. Proof that eigenvalues are real and eigenfunctions are orthogonal for self-adjoint problems. – Using Green's Identity

i. Eigenfunction Expansion Method:
Assume $u(x)=\sum c_n\phi_n(x)$ and find coefficients.

VI. Special Functions

a. Classical Orthogonal Polynomials

b. Gram-Schmidt Orthogonalization Process

c. Legendre Polynomials
   i. Rodrigues Formula $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
   ii. Three Term Recursion Formula:
   $nP_n(x)=(2n+1)xP_{n-1}(x)+(n-1)P_{n-2}(x).$
   iii. Generating Function:
   $$g(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n,\quad |t|\le 1, |x|\le 1.$$
   iv. Binomial expansion: $(1+x)^p=\sum_{n=0}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{n!}x^n$

d. Gamma Function: $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt,\quad x>0.$

e. Factorials, double factorials, Stirling Approximation $n!\sim\sqrt{2\pi n}n^n e^{-n}.$

f. Iterate recursion (finite difference) formulae, like $I_{n+1}=I_n$, given I_0.

g. Bessel Functions – General Properties, types

h. Series solutions of differential equations, $y(x)=\sum_{n=0}^{\infty}a_n x^{n+s}.$

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VII. Green's Functions

a. Variation of Parameters $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

$$c_1' = \frac{1}{W}\begin{vmatrix} 0 & y_2 \\ f/p & y_2' \end{vmatrix}, \quad c_2' = \frac{1}{W}\begin{vmatrix} y_1 & 0 \\ y_1' & f/p \end{vmatrix}$$

b. $Lu = f \Rightarrow u = \int G(x,\xi) f(\xi)\, d\xi.$

c. Initial Value Green's Function

$$u(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau, \qquad G(t,\tau) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{pW}$$

d. Boundary Value Green's Function

$$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi \qquad G(x,\xi) = \begin{cases} Cy_1(\xi)y_2(x), & \xi \le x \\ Cy_1(x)y_2(\xi), & x \le \xi \end{cases}$$

e. Properties of BVP Green's Functions

i. $LG(x,\xi) = \delta(x-\xi)$

ii. Satisfies Homogeneous BCs

iii. Symmetry: $G(x,\xi) = G(\xi,x)$

iv. Continuity: $G(x^+,\xi) = G(x^-,\xi)$

v. Jump Discontinuity of Derivative: $\dfrac{dG(x^+,\xi)}{dx} - \dfrac{dG(x^-,\xi)}{dx} = \dfrac{1}{p(\xi)}$

f. Dirac Delta Function

i. $\delta(x) = 0,\ x \ne 0$ and $\displaystyle\int_{-\infty}^{\infty} \delta(x)\, dx = 0$

ii. $\displaystyle\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$

iii. For the case that a function has multiple simple roots, $f(x_i) = 0,\ f'(x_i) \ne 0,$ for $i = 1, \ldots, n,$ one has that that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|f'(x_i)|}.$$

g. Green's Functions from Eigenfunction Expansions

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n \|\phi_n\|^2}$$