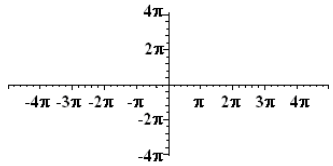
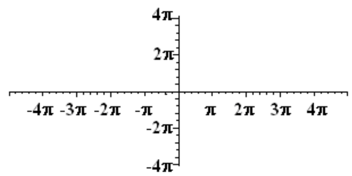
MAT 463/563 Practice Problems for Final Exam

1. Consider the ODE $xy''+(1-x)y'+\lambda y=0,\ 0\le x<\infty$, when answering a-c.
   a. Put this equation in self-adjoint form.
   b. Write out the Lagrange identity for this equation.
   c. Prove that solutions corresponding to different eigenvalues are orthogonal under appropriate conditions.
2. Determine the Fourier coefficients for $f(x)=3\sin 2x-\cos^2 x,\ x\in[-\pi,\pi]$ without doing any integration.
3. Let $f(x)=2\pi-x, 0<x<2\pi$.
   a. Determine the Fourier series representation of $f(x)$.
   b. Find the Fourier cosine series of $f(x)$ on $[0,2\pi$.
   c. Sketch several periods showing what you would expect a graph of this Fourier series.



   d. Sketch the even periodic extension of $f(x)$.



4. Use the generating function $g(x,t)=\dfrac{1}{\sqrt{1-2xt+t^2}}$ to find $P_n(-1)$.
5. Recall the Green's function $G(x,\xi)=\begin{cases} C\xi(1-x), & 0\le\xi\le x \\ Cx(1-\xi), & x\le\xi\le 1\end{cases}$ is used to solve the problem $y''=f(x)$ for some specific boundary conditions.
   a. What are the boundary conditions?
   b. Prove that the Green's function is symmetric.
   c. Determine C.
6. Consider the boundary value problem $\dfrac{d^2y}{dx^2}=6\sin\dfrac{3x}{2},\ y(0)=0,\ y'(\pi)=0.$
   a. Solve this problem by direct integration.
   b. Determine the Green's function for this problem, $\dfrac{\partial^2 G(x,\xi)}{\partial x^2}=\delta(x-\xi),\ \ G(0,\xi)=0,$ $\dfrac{\partial}{\partial x}G(\pi,\xi)=0.$
   c. Use the Green's function to solve the nonhomogeneous problem.
   d. Find the eigenfunctions of the corresponding eignenvalue problem, $\phi''(x)+\lambda\phi(x)=0,$ $\phi(0)=0,\ \phi'(\pi)=0.$
   e. Use these eigenfunctions to solve the nonhomogeneous problem; i.e., assume that $y(x)=\sum c_n\phi_n(x)$ and find the expansion coefficients.
7. Consider the boundary value problem $x^2y''-2xy+2y=0,\ \ y(1)=0,\ y(2)=1.$
   a. Solve the given problem.
   b. Using what you have learned in this course, describe how you would solve the nonhomogeneous problem: $x^2y''-2xy'+2y=f(x),\ \ y(1)=0,\ y(2)=1.$
8. Use the Method of Variation of Parameters to solve: $y''+4y=\sec 2x$

MAT 463/563 Practice Problems for Final Exam

9. Show that the set $\{\cos 3nx\}$, $n = 0, 1, 2, \ldots$ is orthogonal on $|x| \le \pi$. Determine the corresponding orthonormal set.
10. Write $n!!$ in terms of Gamma functions.
11. Find the Fourier-Legendre series expansion of $f(x) = x$ on [-1,1]. Some possible relations that you might need are: $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \;\; n = 1, 2, \ldots$;
$(2n+1)P_n(x) = P_{n+1}{}'(x) - P_{n-1}{}'(x), \;\; n = 1, 2, \ldots$; $\int_{-1}^{1} P_n^2(x)\,dx = \dfrac{2}{2n+1}$, $n = 0, 1, 2, \ldots$
12. Construct an orthonormal basis for R^3 using the vectors $<1,1,0>, <2,0,1>, <2,2,1>$.
13. Find and expression for $\Gamma\left(\frac{1}{2} - n\right)$ in terms of $\sqrt{\pi}$.
14. Use a specific Fourier series expansion to show that $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$.
15. Let $g(x,t) = \dfrac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$ be the generating function for Laguerre polynomials. Show that $L_n{}'(0) = -n$ and $L_n{}''(0) = \frac{1}{2} n(n-1)$.
16. Show that the sequence $\left\{\dfrac{\sin nx}{\pi x}\right\}$ approaches $\delta(x)$.