The Duffing Equation

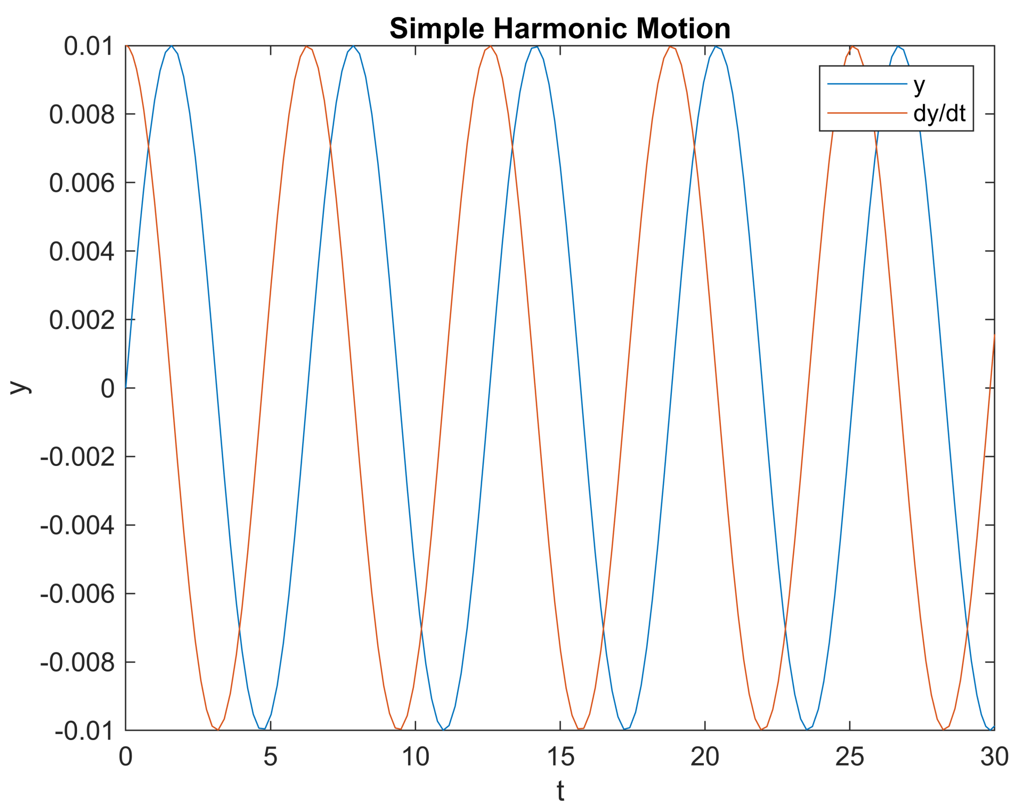
The Duffing equation is an externally forced and damped oscillator equation. It is given by

$\ddot{x} + \delta\,\dot{x} + \alpha\,x + \beta\,x^3 = \gamma\cos\omega\,.\,.$

We will investigate the equation by turning on a few constants at a time. We begin with simple harmonic motion:

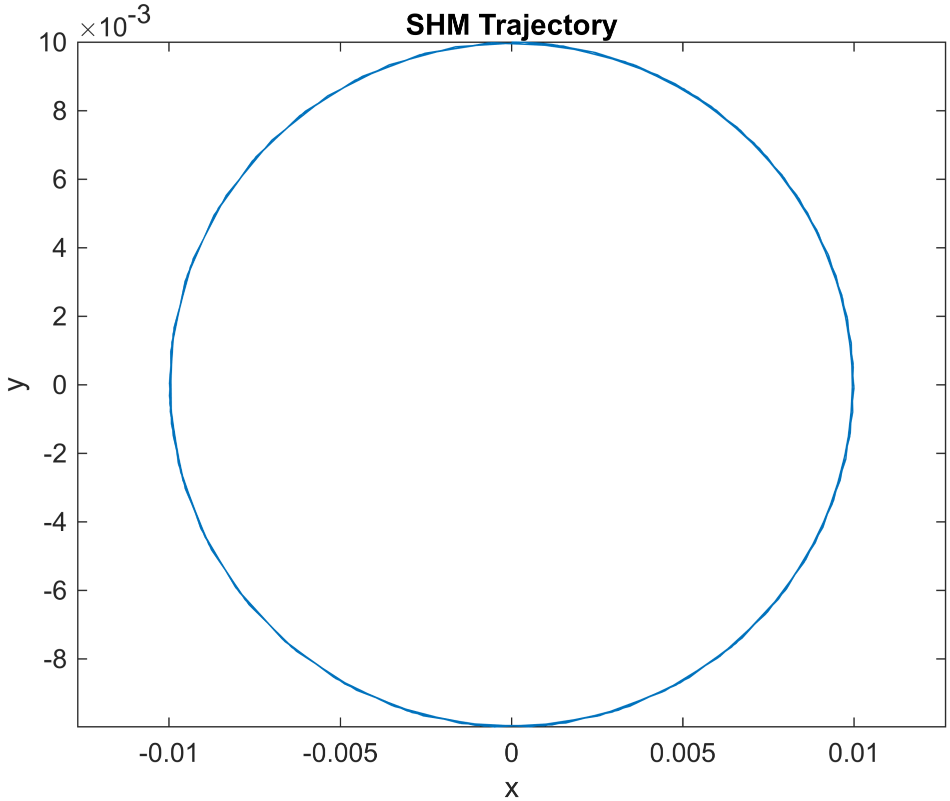
$\ddot{x} + \alpha\,x = 0.$

```
clear
alpha = 1; delta = 0; beta = 0; % SHM
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3];
tspan = [0 30];
y0 = [0 0.01]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Simple Harmonic Motion')
```



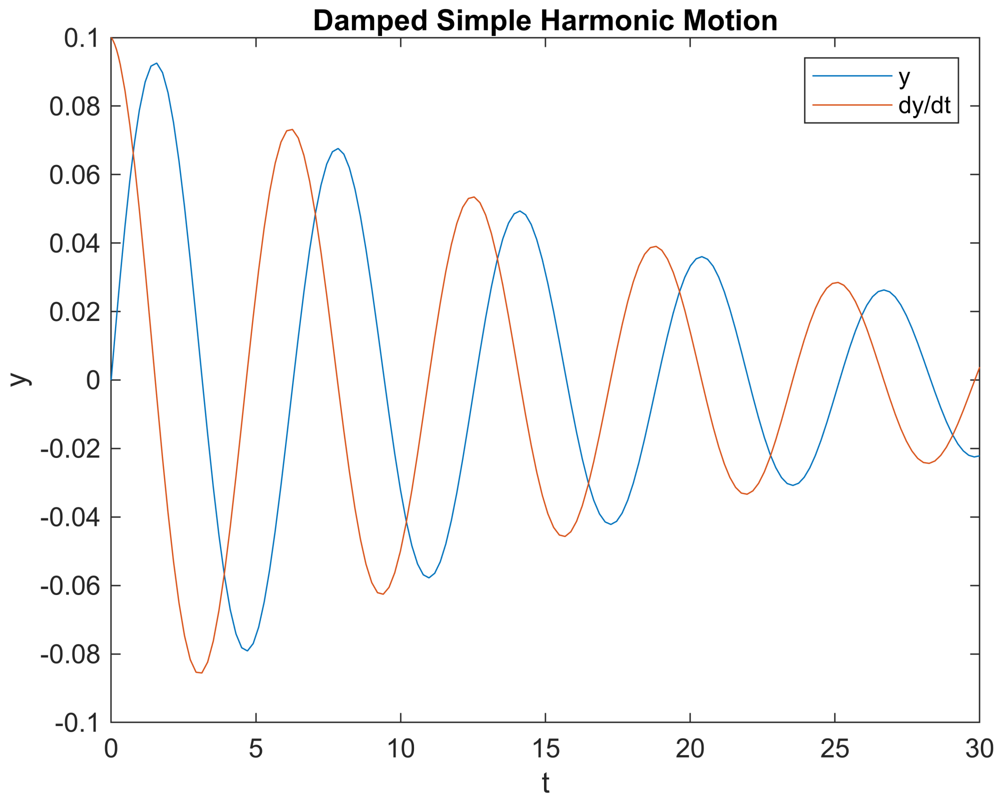

```
figure
plot(y(:,1),y(:,2))
xlabel('x')
```

```
ylabel('y')
title('SHM Trajectory')
axis equal
```

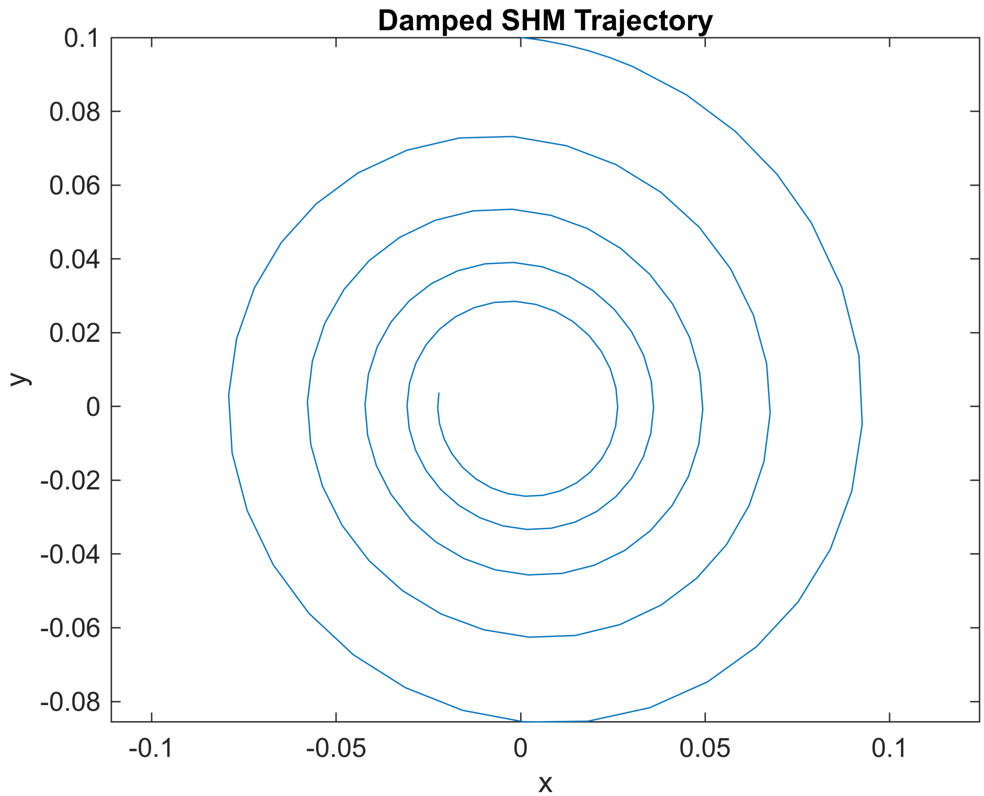


Now add damping, $\delta > 0$.

```
clear
alpha = 1; delta = 0.1; beta = 0;  % damped SHM
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3];
tspan = [0 30];  % tspan = 0:.1:30;  % Makes solution smoother
y0 = [0 0.1]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Damped Simple Harmonic Motion')
```




```
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
title('Damped SHM Trajectory')
axis equal
```

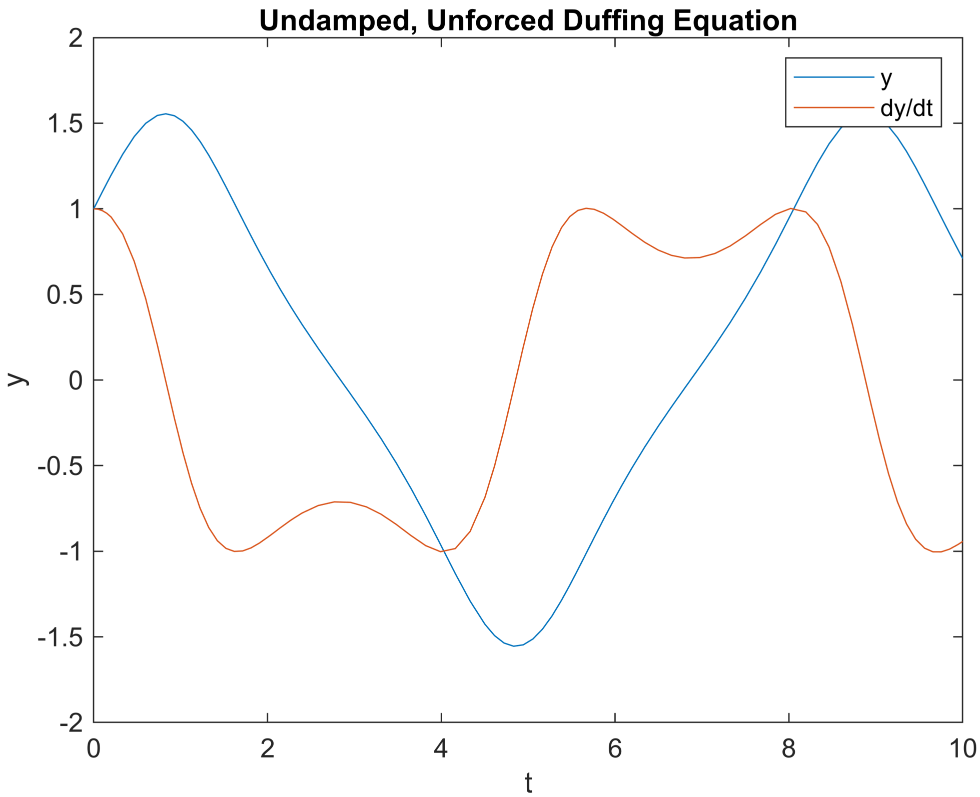


Now, consider the undamped Duffing equation $\ddot{x} + \alpha x + \beta x^3 = 0$. As a first order system, we have

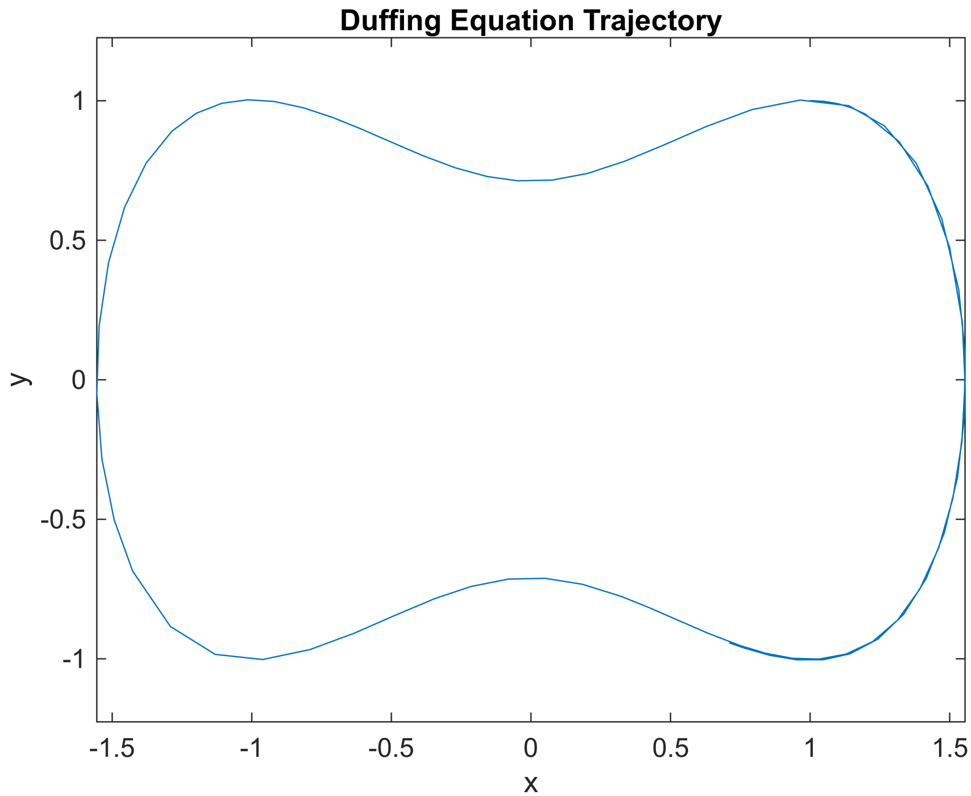
$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\alpha x - \beta x^3\end{aligned}$$

Note the equilibrium points are $(0,0)$, $\left(\pm\sqrt{\frac{-\alpha}{\beta}}, 0\right)$.

```
clear
alpha = -1; delta = 0; beta = 1; % Undamped Duffing Equation
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3];
tspan = [0 10];
y0 = [1 1]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Undamped, Unforced Duffing Equation')
```

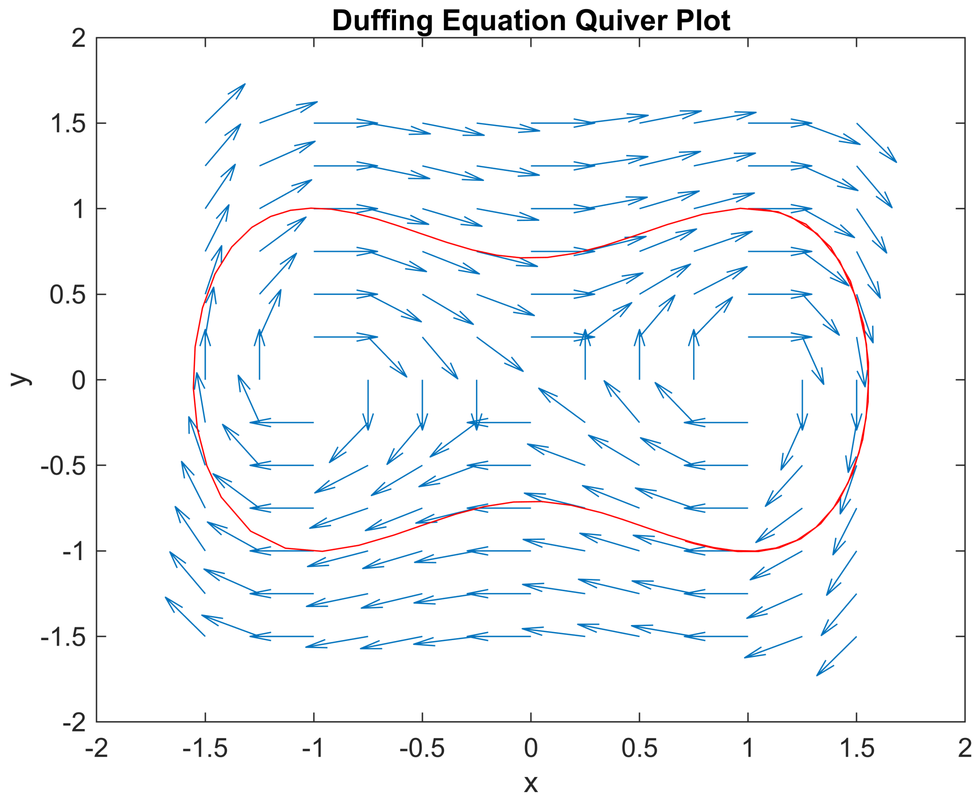



```
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
title('Duffing Equation Trajectory')
axis equal
```



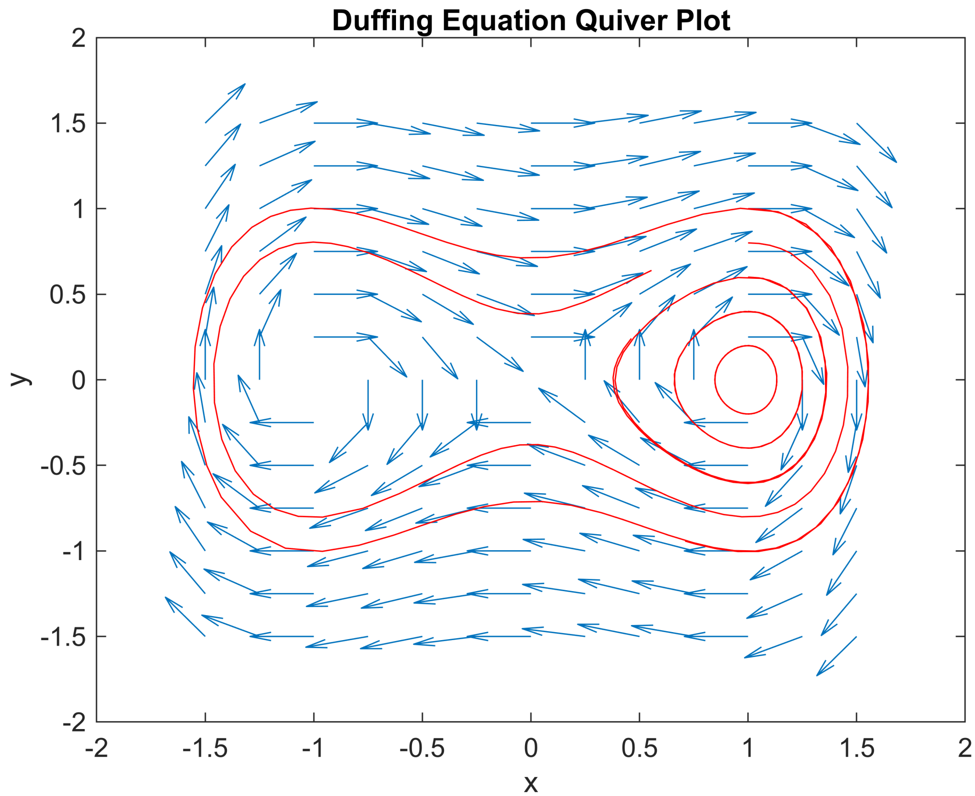
Its useful to look at the phase plot using the quiver function. N is used to normalize the arrows in the direction field. The first plot shows one trajectory.

```
[X, Y] = meshgrid(-1.5:0.25:1.5, -1.5:0.25:1.5);
U = Y; V=-delta*Y-alpha*X-beta*X.^3;
N=sqrt(U.^2+V.^2);
quiver(X, Y, U./N, V./N)

hold on
plot(y(:,1),y(:,2),'r')
hold off
xlabel('x')
ylabel('y')
title('Duffing Equation Quiver Plot')
```



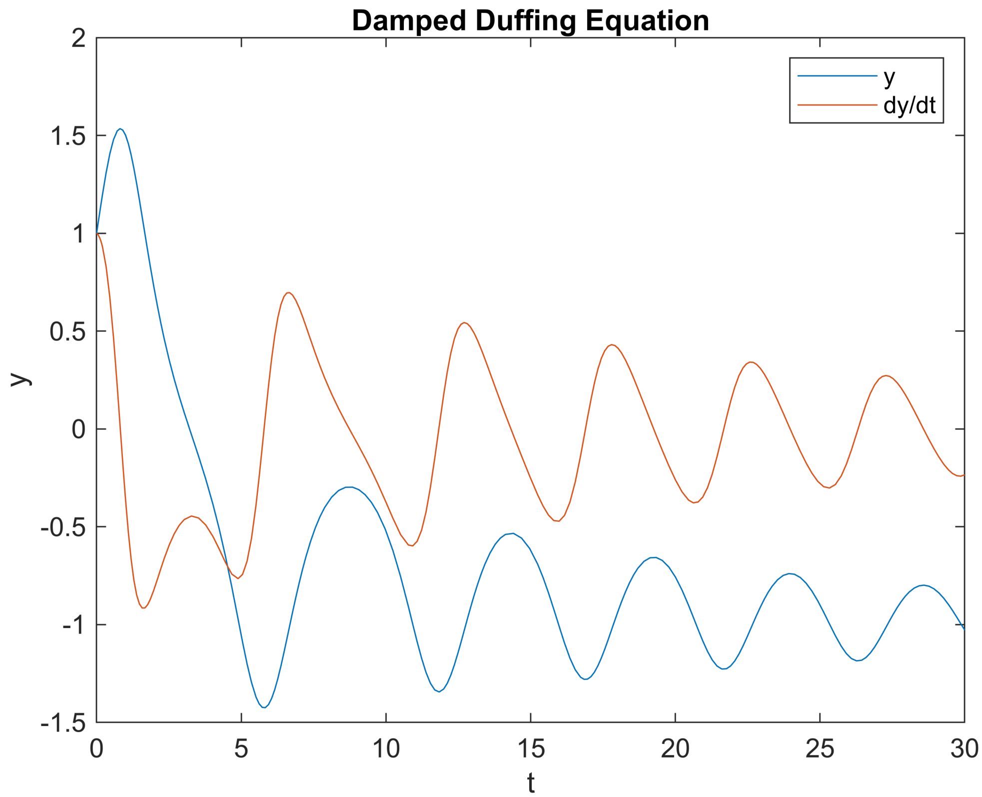
The following shows how to add several trajectories.

```
hold on
for j=1:5
    y0 = [1 j/5]';
    [t,y] = ode45(duffing,tspan,y0);
    plot(y(:,1),y(:,2),'r')
end
hold off
```

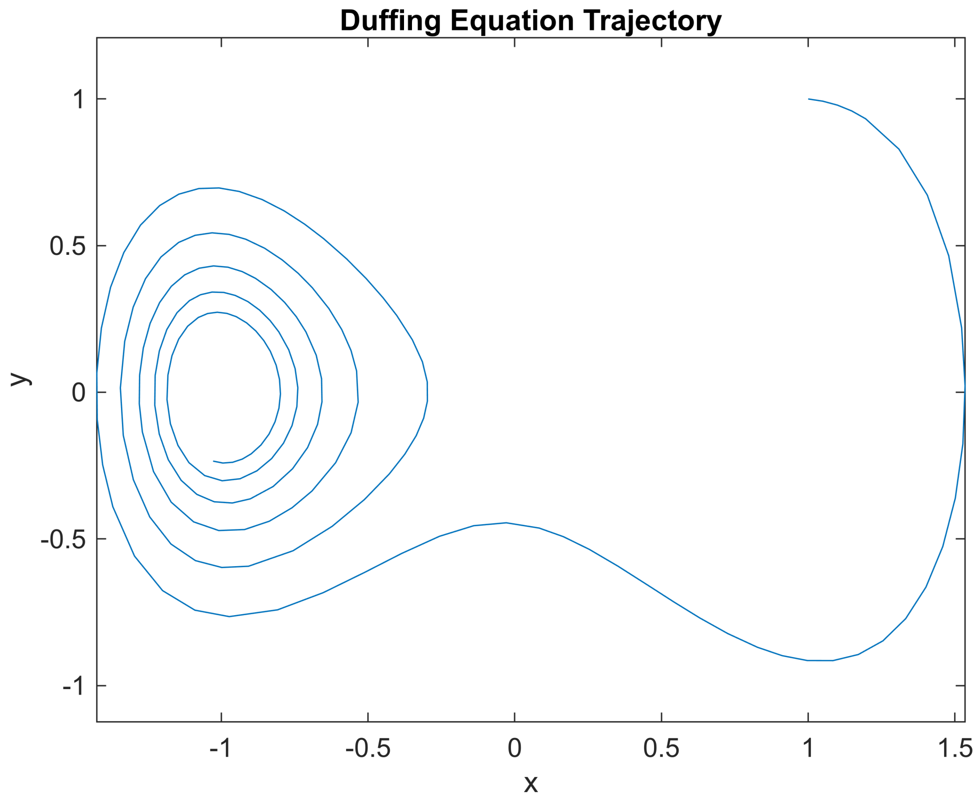


Now, we add back some damping. Changing the initial condition, y0 = [1 1]'; will reveal different types of behavior.

```
clear
alpha = -1; delta = 0.1; beta = 1;  % Damped Duffing Equation
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3];
tspan = [0 30];
y0 = [1 1]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Damped Duffing Equation')
```

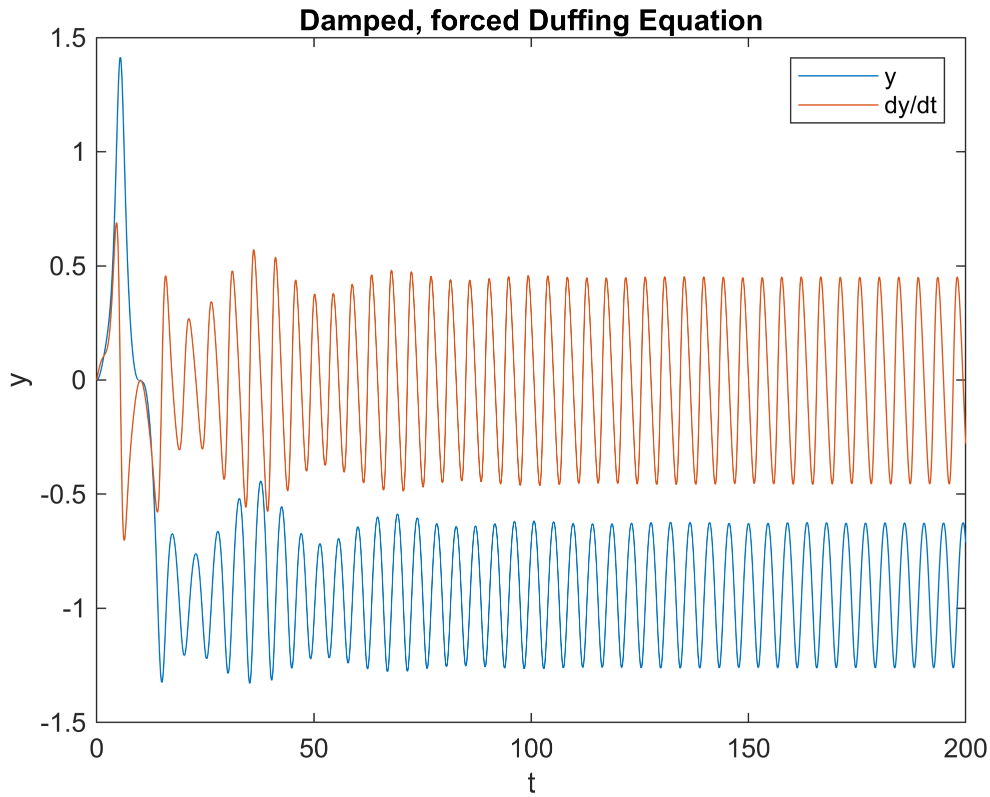



```
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
title('Duffing Equation Trajectory')
axis equal
```

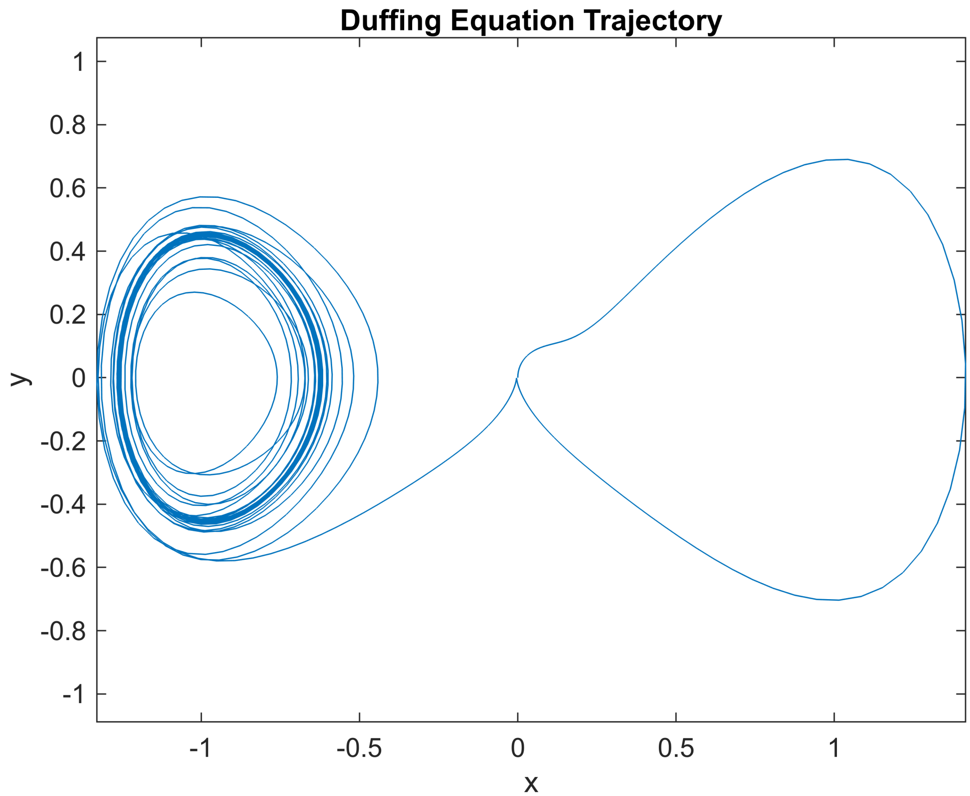


Here we add the forcing.

```
clear
alpha = -1; delta = 0.1; beta = 1; gamma = .1; omega=1.4; % Forced DE
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
tspan = 0:0.1:200;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Damped, forced Duffing Equation')
```

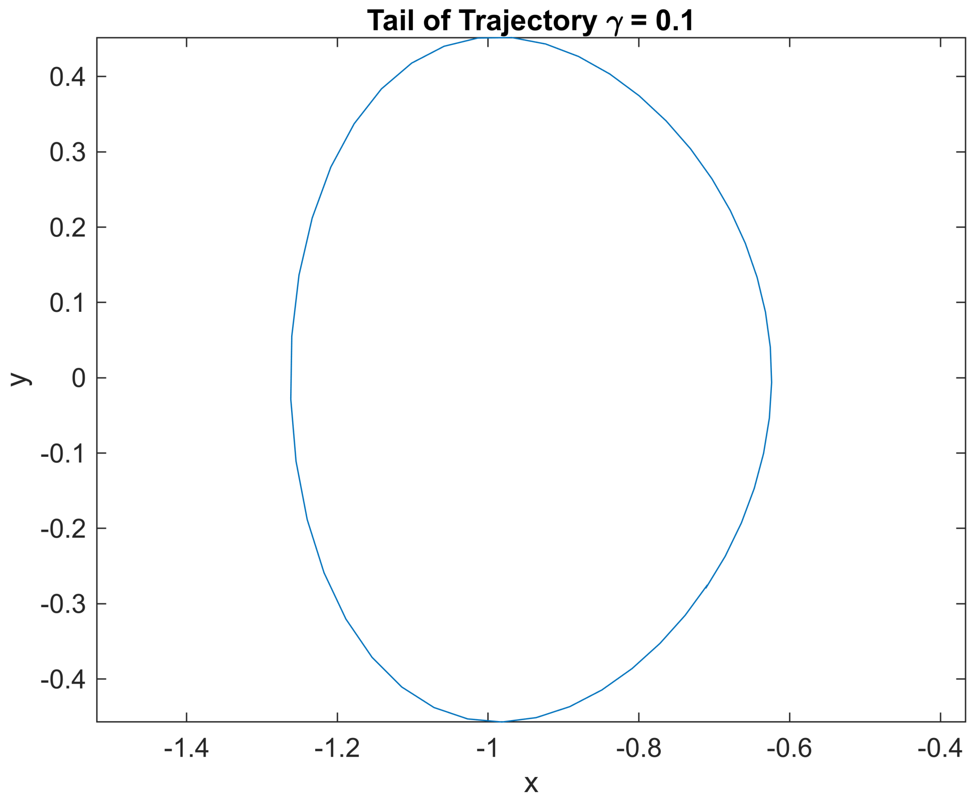



```
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
title('Duffing Equation Trajectory')
axis equal
```



We can examine the tail of the trajectory and notice a periodic behavior with period $T = \frac{2\pi}{\omega} = 4.488.$

```
% Plot tail of trajectory
figure
L=length(y(:,1));
Npts=45;          % Npts+1 points of the tail
plot(y(L-Npts:L,1),y(L-Npts:L,2))
%plot(y(L-44:L,1),y(L-44:L,2))  % roughly 45*.1=period
xlabel('x')
ylabel('y')
title(['Tail of Trajectory \gamma = ',num2str(gamma)])
axis equal
```

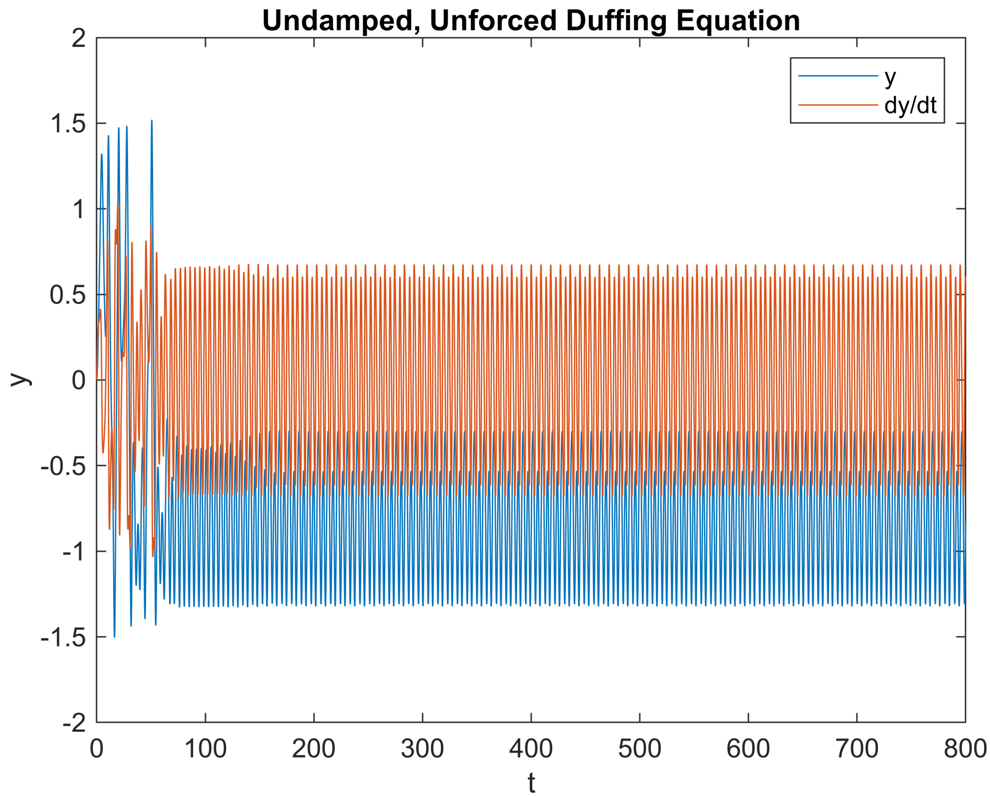



```
T=2*pi/omega
```

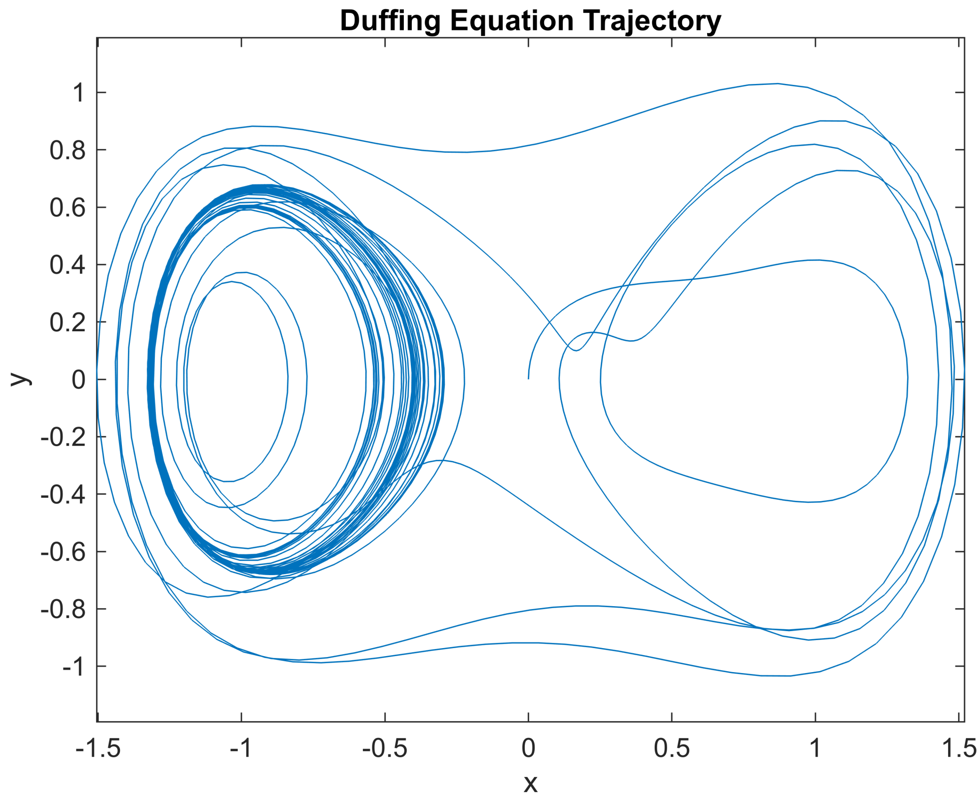
```
T = 4.4880
```

If we change the amplitude to $\gamma = 0.318$, then we notice period doubling.

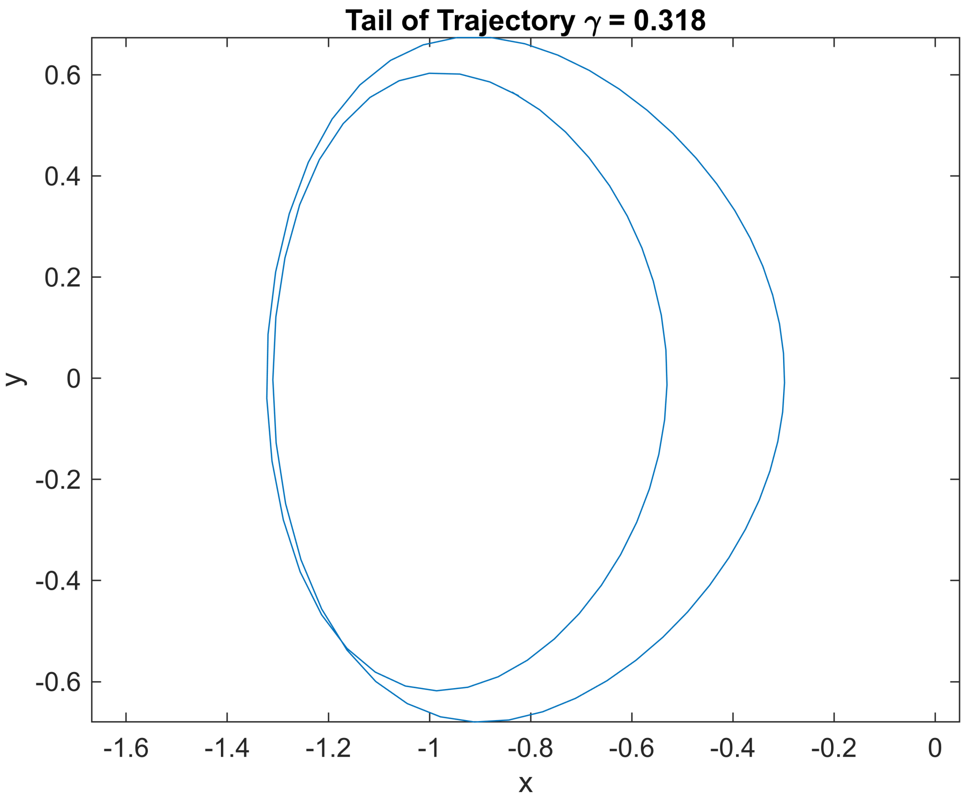
```
clear   % Change gamma to 0.318 - Period Doubling to 2 cycles
alpha = -1; delta = 0.1; beta = 1; gamma = .318; omega=1.4; % Forced DE
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 800;
tspan = 0:0.1:NT;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);
plot(t,y)
legend({'y','dy/dt'})
xlabel('t')
ylabel('y')
title('Undamped, Unforced Duffing Equation')
```




```
figure
plot(y(:,1),y(:,2))
xlabel('x')
ylabel('y')
title('Duffing Equation Trajectory')
axis equal
```



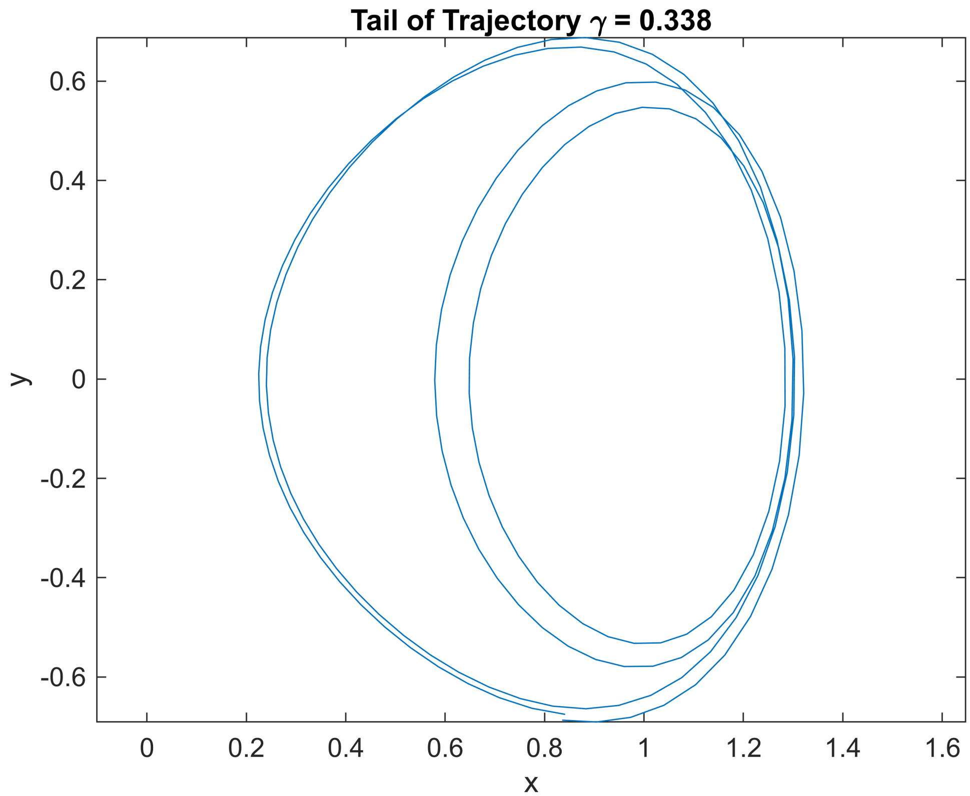

```
% Plot tail of trajectory
figure
L=length(y(:,1));
Npts=180;         % Npts+1 points of the tail
%plot(y(L-Npts:L,1),y(L-Npts:L,2))
plot(y(L-90:L,1),y(L-90:L,2))  % 90*.1=period - double before
xlabel('x')
ylabel('y')
title(['Tail of Trajectory \gamma = ',num2str(gamma)])
axis equal
```




```
% Try gamma = 0.338 but 0.35 is chaotic region
```

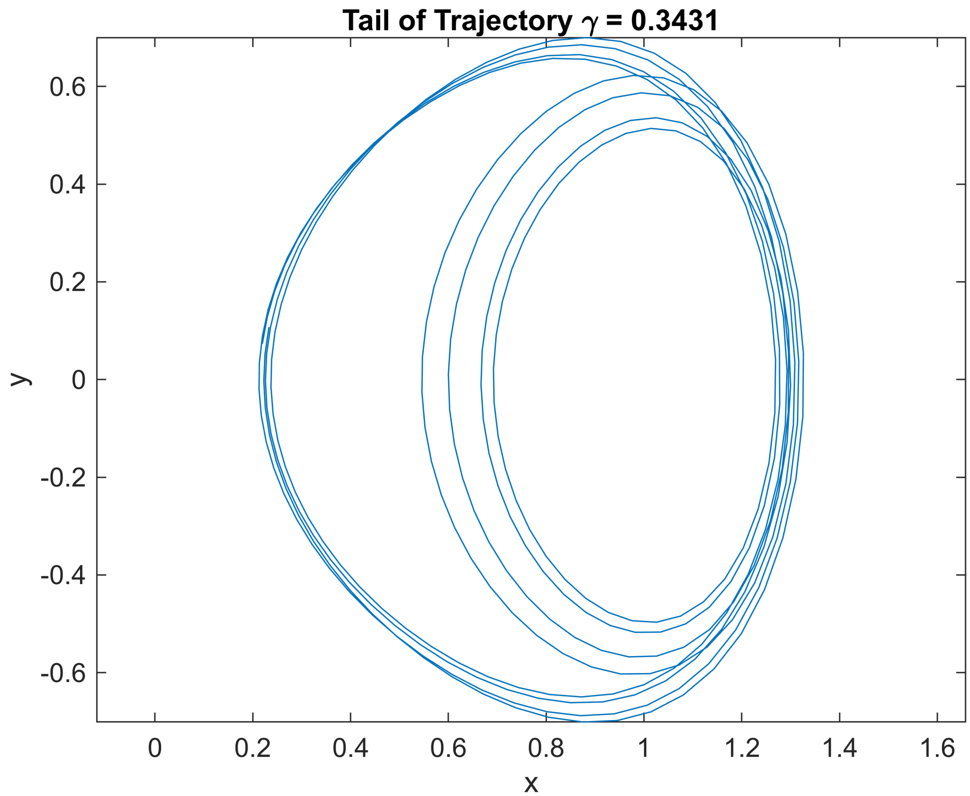
For $\gamma = 0.338,$ we get four cycles, or period 4.

```
clear   % Change gamma to 0.338 - Period Doubling to 4 cycles
alpha = -1; delta = 0.1; beta = 1; gamma = .338; omega=1.4; % Forced DE
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 2000;
tspan = 0:0.1:NT;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);

figure
L=length(y(:,1));
Npts=180;          % Npts+1 points of the tail
plot(y(L-Npts:L,1),y(L-Npts:L,2))
xlabel('x')
ylabel('y')
title(['Tail of Trajectory \gamma = ',num2str(gamma)])
axis equal
```



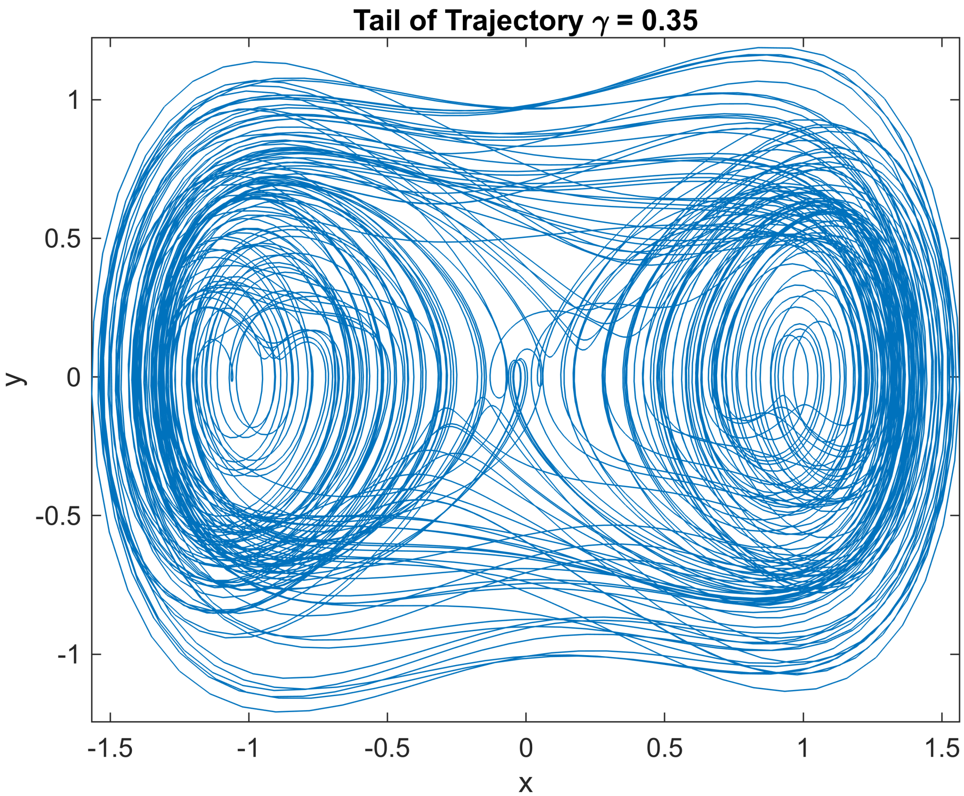
Can we find period 8? Maybe near $\gamma = 0.3431$.

```
clear   % - Period Doubling to more cycles?
alpha = -1; delta = 0.1; beta = 1; gamma = .3431; omega=1.4; % Forced DE
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 6400;
tspan = 0:0.1:NT;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);

figure
L=length(y(:,1));
Npts=360;         % Npts+1 points of the tail
plot(y(L-Npts:L,1),y(L-Npts:L,2))
xlabel('x')
ylabel('y')
title(['Tail of Trajectory \gamma = ',num2str(gamma)])
axis equal
```



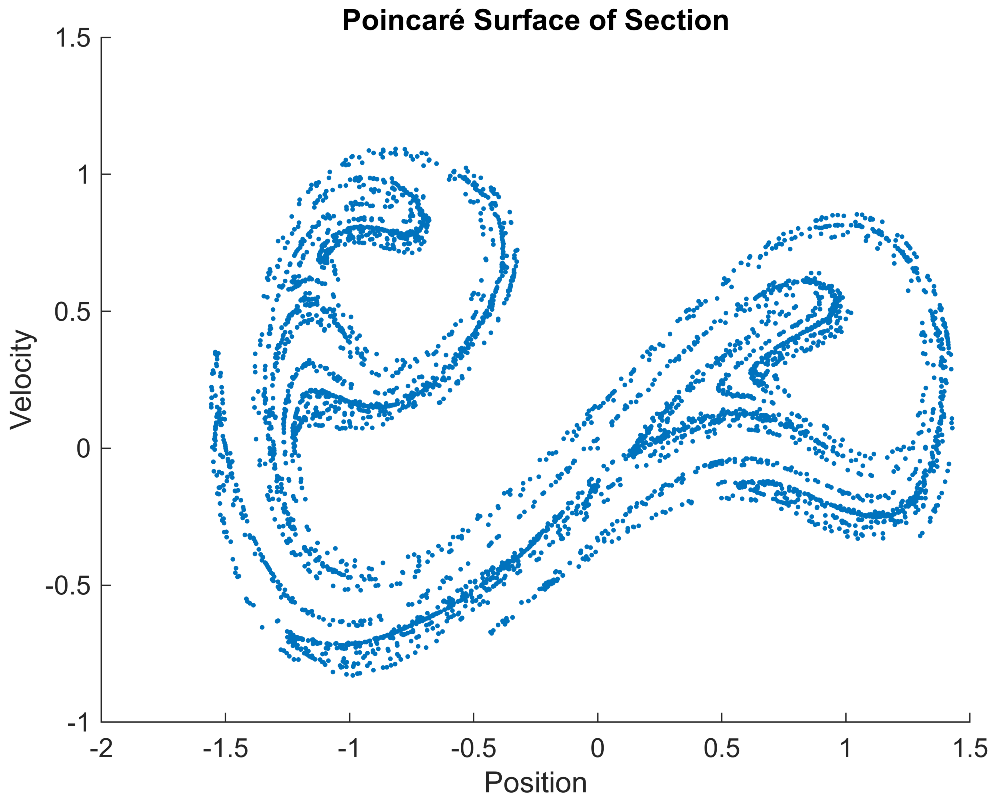
Possibly for $\gamma = 0.35$ there is only chaos.

```
clear   % - Chaos?
alpha = -1; delta = 0.1; beta = 1; gamma = .35; omega=1.4; % Forced DE
duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 10000;
tspan = 0:0.1:NT;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);

figure
L=length(y(:,1));
Npts=NT;         % Npts+1 points of the tail
plot(y(L-Npts:L,1),y(L-Npts:L,2))
xlabel('x')
ylabel('y')
title(['Tail of Trajectory \gamma = ',num2str(gamma)])
axis equal
```



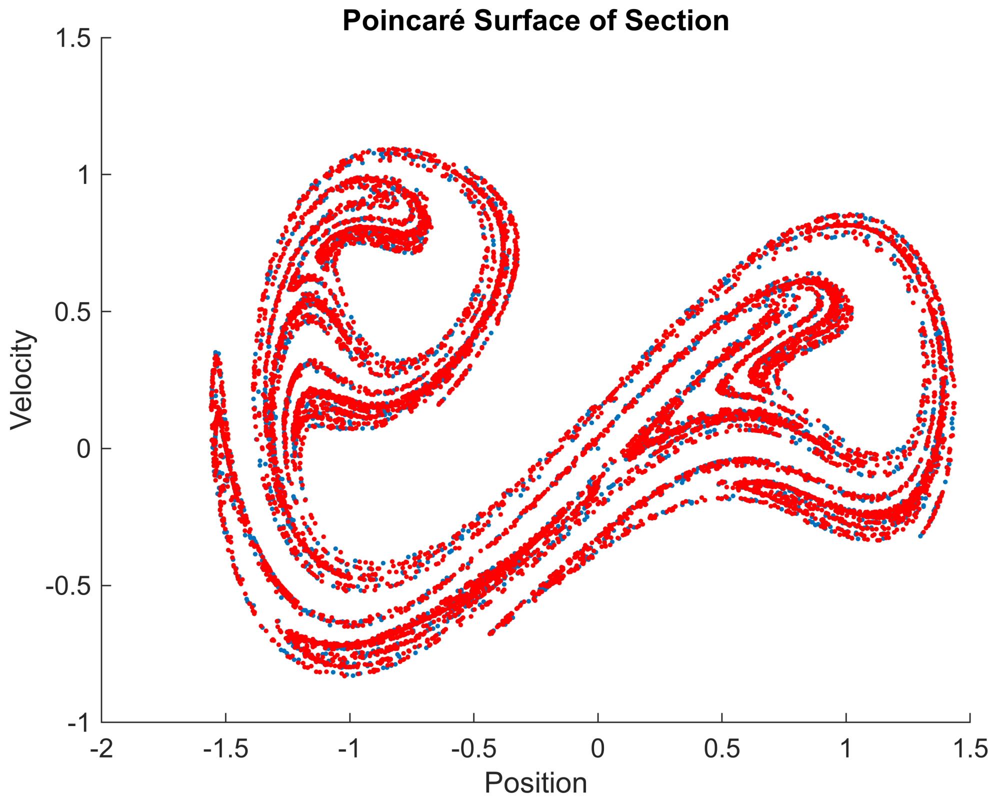
Let's look at a Poincaré Surface of Section for $\gamma = 0.35$.

```
% Strange Attractor
clear
alpha = -1; delta = 0.1; beta = 1; gamma = .35; omega=1.4;
%alpha = -1; delta = 0.25; beta = 1; gamma = .40; omega=1; % Standard Shape
%alpha = -1; delta = 0.2; beta = 1; gamma = .30; omega=1; % Standard Shape

duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 6000;  % or 2000 for gamma = 0.338
period=2*pi/omega;
dt=period/10;
tspan = 0:dt:20000;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);


P_x = y(mod(t, period) < 1e-6, 1);
P_y = y(mod(t, period) < 1e-6, 2);

figure
scatter(P_x,P_y,'.')
xlabel('Position');
ylabel('Velocity');
title('Poincaré Surface of Section');
```



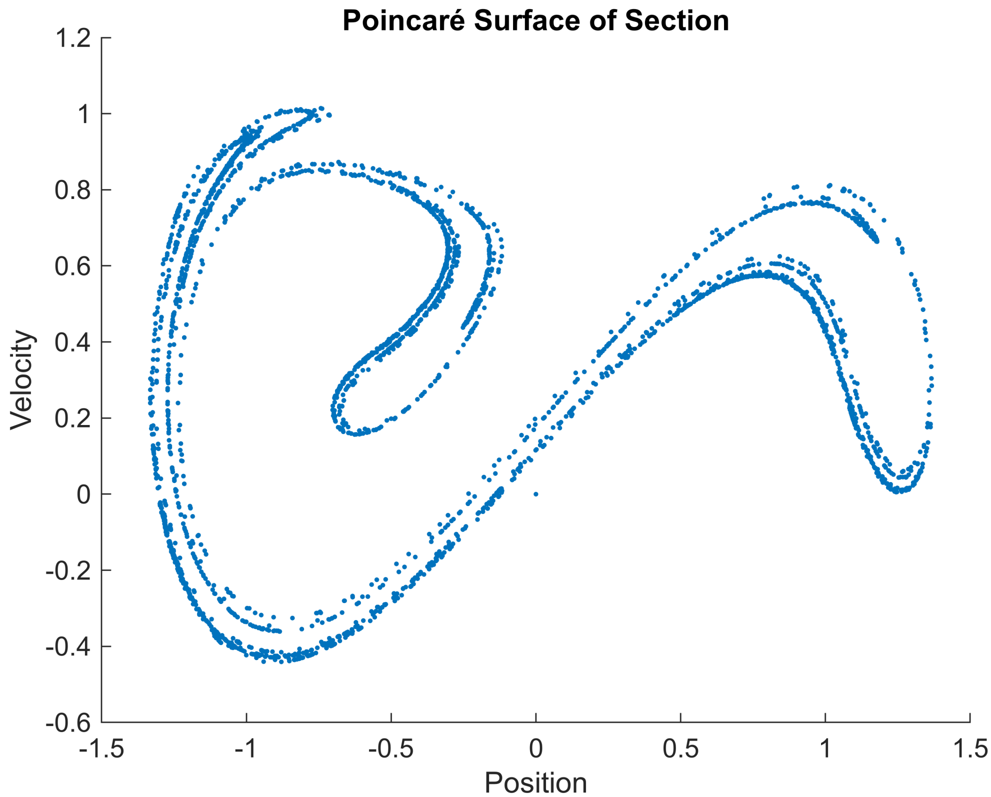
Here we add more initial conditions and find the same attractor. In fact, it is a strange attractor.

```
% Add more ICs
figure
scatter(P_x,P_y,'.')
xlabel('Position');
ylabel('Velocity');
title('Poincaré Surface of Section');
hold on
for j=1:50
    y0 = [j/50,0]';
    [t,y] = ode45(duffing,tspan,y0);
    P_x = y(mod(t, period) < 1e-6, 1);
    P_y = y(mod(t, period) < 1e-6, 2);
    L=length(P_y);
    Npts=180;
   % scatter(P_x,P_y,'.r')
    scatter(P_x(L-Npts:L),P_y(L-Npts:L),'.r')
end
hold off
```



A more standard strange attractor for the Duffing equation is found using a unit amplitude. Here are two possibles cases with varying parameters.

```
% Strange Attractor
clear
alpha = -1; delta = 0.25; beta = 1; gamma = .40; omega=1; % Standard Shape
%alpha = -1; delta = 0.2; beta = 1; gamma = .30; omega=1; % Standard Shape

duffing = @(t,y) [y(2); -delta*y(2)-alpha*y(1)-beta*y(1).^3+gamma*cos(omega*t)];
NT = 6000;  % or 2000 for gamma = 0.338
period=2*pi/omega;
dt=period/10;
tspan = 0:dt:20000;
y0 = [0,0]';
[t,y] = ode45(duffing,tspan,y0);


P_x = y(mod(t, period) < 1e-6, 1);
P_y = y(mod(t, period) < 1e-6, 2);

figure
scatter(P_x,P_y,'.')
xlabel('Position');
ylabel('Velocity');
title('Poincaré Surface of Section');
```


Poincaré Surface of Section
Velocity
Position
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1.5
-1
-0.5
0
0.5
1
1.5