Genesis of Differential Equations

Spring 2025 - R. L. Herman





In the beginning there was Calculus

- Eudoxus: Method of exhaustion.
- Bonaventura Cavalieri (1598–1647), follower of Galileo Galilei (1564–1642), 1635, *Geometria indivisibilibus continuorum* - method of indivisibles for integration.
- Fermat had described in 1629, *Methodus ad disquirendam maximam et minimam,* method to find maxima and minima.

  Also found method to obtain the area under the curve by dividing it into an infinite number of rectangles
- John Wallis (1616–1703), published in 1656 *Arithmetica Infinitorum,* area under x^k is $\dfrac{x^{k+1}}{k+1}$.

  So, one can find the quadrature of any function represented as a power series expansion.
- Proofs of the fundamental theorem of calculus: James Gregory (1638–1675), Isaac Barrow (1630–1677),
- Integral as the antiderivative had to wait for Newton and Leibniz's calculus.

Early contributors: Newton, Leibniz and the Bernoulli brothers

- Isaac Newton (1643–1727)

  Developed the fundamentals of differential calculus in the second half around 1666,

  Wrote *Methodus fluxionum et serierum infinitarum* of 1671 (published 1736).

  He would also write the *De analysi per aequationes numero terminorum infinitas* in 1669 (published in 1711), and

  A geometrical form of his differential calculus in section I of book I of the Principia, 1687.

- In *Methodus fluxionum ...* - fluxion equations using infinte series.

- Gottfried Leibniz (1646–1716)

  His works were based on those of Descartes, Blaise Pascal and Fermat.

  Derived 1674-1676, but published 1684 in *Acta Eruditorum* titled *Nova Methodus pro Maximis et Minimis.*

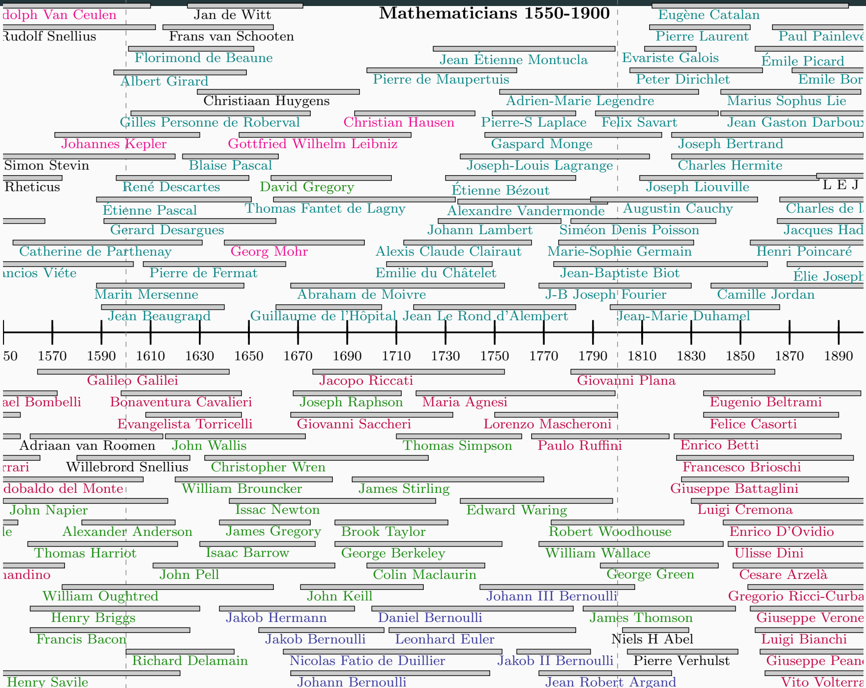
  October and November 1675: introduced d and $\int$.

- First text, *De constructione Aequationum Differentialium Primi Gradus*, 1707 by Gabriele Manfredi (1681–1761).

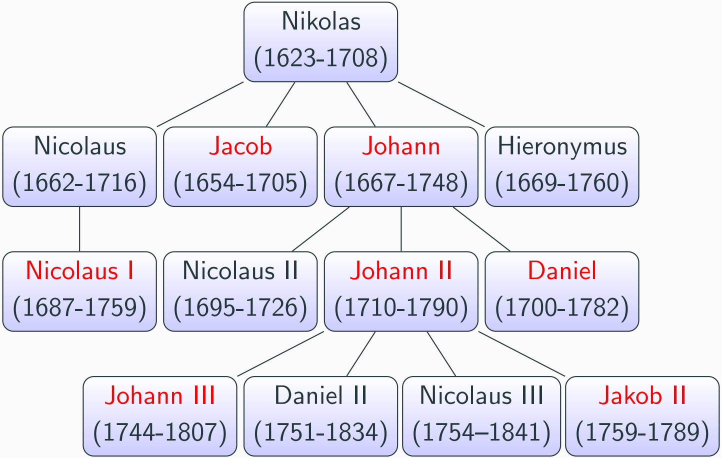
- Later developers: Euler, D. Bernoulli, Lagrange and Laplace, ...

- Euler's book *Institutionum Calculi Integralis*, 1768–70.

Mathematicians - 1550-1900



The Bernoulli Family



Jakob Bernoulli (aka James or Jacques or Jacob)

Johann Bernoulli (aka Jean or John)

Newton's Classification

Newton gives three types of problems in 1736 translation of 1671 work on fluxions, *The method of fluxions and infinite series*:

> *13. But in respect of this Problem Equations may be distinguish'd into three Orders.*
>
> *14. First: In which two Fluxions of Quantities, and only one of their flowing Quantities are involved.*
>
> *15. Second: In which the two flowing Quantities are involved, together with their Fluxions.*
>
> *16. Third: In which the Fluxions of more than two Quantities are involved.*

These are often cited as Newton giving three classifications of differential equations in terms of fluxions and fluents. Here Newton referred to a flowing quantity as a fluent and to its instantaneous rate of change as a fluxion. Thus, he wrote $\dot{x}$ and $\dot{y}$.

Newton's Problems

When Newton needed the slope of the curve $y = y(x)$, he sought $\frac{\dot{y}}{\dot{x}}$. The classification is often summarized as [see Krishnachandran.]

$dy/dx = f(x)$:

$dy/dx = f(y)$:

$dy/dx = f(x, y)$:

However, in his work on fluxions, he listed a number of problem types and examples. He relied on infinite series to obtain solutions to the various differential equations he considered.

Note: Newton gave a 'geometrical form' of his differential calculus in section I of book I of the *Principia* of 1687. Online 1846 translation.

The method of fluxions and infinite series, pg 29.

19. So propofing the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$; I fuppofe x to be the Correlate Quantity, and the Equation being accordingly reduced, we fhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6$, &c. Now I multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and there arifes $x + x^3 - x^5 + 2x^7$, &c. which Terms I divide feverally by their number of Dimenfions, and the Refult $x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7$, &c. I put $=y$. And by

Newton solves $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$, or $\left(\frac{\dot{y}}{\dot{x}}\right)^2 = \frac{\dot{y}}{\dot{x}} + x^2$, for $\frac{\dot{y}}{\dot{x}}$ as a series:

$$\frac{\dot{y}}{\dot{x}} = \frac{1}{2} + \frac{\sqrt{4x^2+1}}{2} = 1 + x^2 - x^4 + 2x^6 + \mathcal{O}(x^8).$$

"Integrate," $y(0) = 0$:

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 + \mathcal{O}(x^9).$$

Maple:

$$\frac{x}{2} \pm \frac{x\sqrt{4x^2+1}}{4} \pm \frac{\operatorname{arcsinh}(2x)}{8}$$

Footnotes

Newton discovered binomial series in 1665.

Hyperbolic functions introduced 1757 by Vincenzo Riccati (1707-1775). 1768 - Johann Heinrich Lambert (1728-1777)

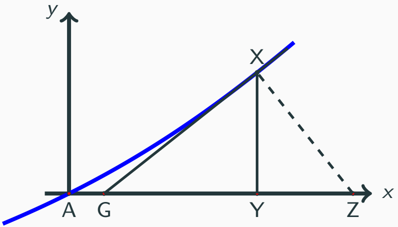
Now, we go to continental Europe starting with Descartes.

La Géométrie - 1637, online
- On algebraic operations and geometric constructions

- René Descartes (1596-1650)
- Book I: *Problems Which Can Be Constructed by Means of Circles and Straight Lines Only.*
  He introduced algebraic notation: x, y, z, etc. denote unknown variables, a, b, c, etc. denote constants.
- Book II: *On the Nature of Curved Lines*,
  Descartes described two kinds of curves, called by him geometrical and mechanical.
  Algebraic method for finding the normal at any point of a curve whose equation is known. The construction of the tangents to the curve follows.
- Book III. *On the Construction of Solid and Supersolid Problems*
  Introduction to nature of equations and their solution.

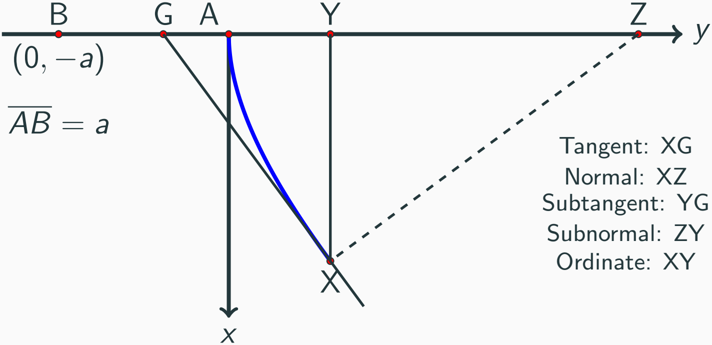
de Beaune's Second Problem

- Erasmus Bartholin (1625-1698) edited *Introduction to the geometry of Descartes* by Frans van Schooten (1615 - 1660).
- During the years 1664- 1674 he produced several volumes of a book *Dissertatio de problematibus geometricis* that consisted of theses he had proposed for his students.
- In 1672 he gave a proof of the **second problem of de Beaune**.
- Originally posed in a letter from Florimond de Beaune (1601 - 1652) to Marin Mersenne (1588 - 1648) in 1638. **1st. inverse tangent problem.**



Inverse Tangent Problem

The curve AX with vertex A and axis AY is determined as follows: From the arbitrary point X (with the axis intersections G and Z) and specify a fixed distance AB. Then there is always the ratio ZY: XY = AB: (XY - AY). [Note: ZY:XY = XY: YG.]



Tangent: XG
Normal: XZ
Subtangent: YG
Subnormal: ZY
Ordinate: XY

The First Differential Equation?

Inverse tangent problem: Find a curve given properties of its tangents.

In modern notation, we seek the solution to a differential equation.

Debeaune's second problem can be written as

$$\frac{dy}{dx} = \frac{x - y}{a}, \tag{1}$$

for some constant a.

We would now solve this as

$$y = x + a\left(e^{-x/a} - 1\right). \tag{2}$$

K. M. Pedersen [*Centaurus* 22 (2) (1978), 99-107] suggests Bartholin's geometric proof may be Debeaune's original proof which he sent to Descartes. Debeaune sent Bartholin papers for safe keeping shortly before his death in 1652.

Leibniz's Solution

René Descartes, fond of Debeaune, supposedly solved Debeaune's problem in 1639.

Gottfried Wilhelm Leibniz discusses Debeaune's problem in his first calculus publication (1684), "A New Method for Finding Minima and Maxima," in *Acta Eruditorum*, in the form [See Leibniz's Figure on next page. See paper online and a translation and next slides.]

> *to find a line WW of such a nature that, drawn to the axis tangent to WC, XC is always equal to the same straight line constant a.*

quisquam pari facilitate tractabit. Appendicis loco placet adjicere ſolutionem Problematis, quod Carteſius a Beaunio ſibi propoſitum, Tom.3. Epiſt. tentavit, ſed non ſolvit. Lineam invenire WW talis naturæ, ut ducta ad axem tangente WC, ſit XC ſemper æqualis eidem rectæ conſtanti, a. Jam. XW ſeu w ad XC ſeu a, ut d w ad d x: Ergo ſi dx (quæ aſſumi poteſt pro arbitrio) aſſumatur conſtans ſive ſemper eadem nempe b, ſeu ſi ipſæ x ſive AX creſcant uniformiter, fiet W æqu. $\frac{a}{b}$ dw, quæ erunt ipſæ W ordinatæ, ipſis dw, ſuis incrementis ſive differētiis, proportionales, hoc eſt ſi x ſint progreſſionis arithmeticæ, erunt w progreſſionis Geometricæ, ſeu ſi w ſint numeri, x erunt logarithmi: linea ergo WW logarithmica eſt.

Leibniz (1684) *Nova methodus pro maximis et minimis, ...*

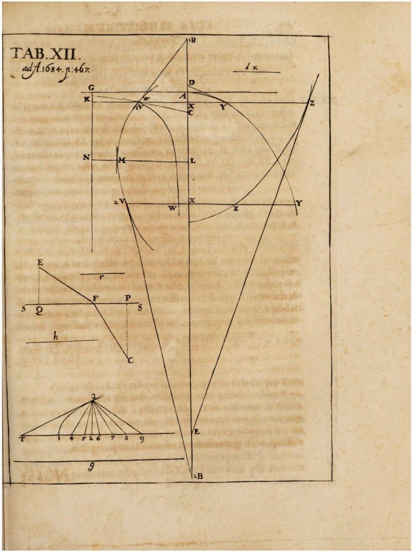
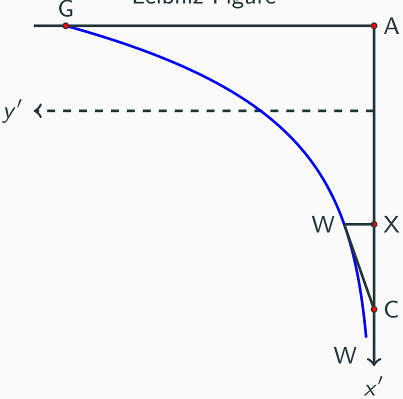


Figure 1: The curve WW is used to discuss the de Beaune problem.

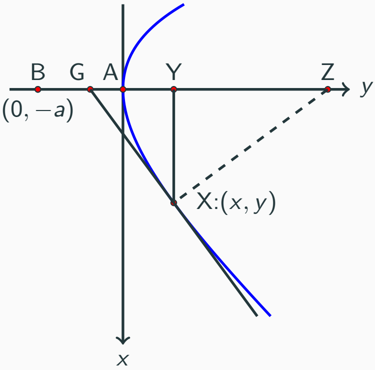
Problem Comparison

Leibniz solves De Beaune's inverse-tangent problem: What kind of curve will have a constant subtangent? He said that it is "logarithmica."

Leibniz Figure



de Beaune Figure



Leibniz says (in Latin)

Appendicis loco placet adjicere solutionem Problematis, quod Cartesius a Beaunio sibi propositum Tom. 3. Epist. tentavit, sed non solvit : Lineam invenire WW talis naturae, ut ducta ad axem tangente WC, sit XC semper aequalis eidem rectae constanti a. Jam XW seu w ad XC seu a, ut dw ad dx; ergo si dx (quae assumi potest pro arbitrio) assumatur constans sive semper eadem, nempe b, seu si ipsae x sive AX crescant uniformiter, fiet w aequ. a/b dw , quae erunt ipsae w ordinatae ipsis dw, suis incrementis sive differentiis proportionales, hoc est si x sint progressionis arithmeticae, erunt w progressionis Geometricae, seu si w sint numeri, x erunt logarithmi: linea ergo WW logarithmica est.

Ian Bruce (2014) translates

It pleases to add the solution of a problem as an appendix, which De Baune proposed to Decartes to attempt himself, in Vol. 3 of his letters, but which he did not solve: To find the line of such a kind WW, [adapted from the first figure] so that with the tangent WC drawn to the axis, XC shall always be equal to the same constant right line a. Now XW or w shall be to XC or a, as dw to dx; therefore if dx (which can be taken by choice) may be assumed constant or always the same, truly b, or if x itself or if AX may increase uniformly, w will be made equal to a/b dw, and b the ordinates w themselves which will proportional to their increments, or differentials, from dw, that is if the x shall be in an arithmetic progression, the w shall be in a geometric progression, or if w shall be numbers, x will be their logarithms: therefore the line WW is logarithmic.

The Differential Equation

For Leibniz's formulation of the problem,

$$\frac{dw}{dx} = -\frac{w}{a}, \tag{3}$$

whose solution is readily found as

$$w = Ae^{-x/a}. \tag{4}$$

However, back in the late 1600s it was not known how to "integrate" $\frac{1}{x}$. This was noticeable in Bernoulli's solution of the problem.

Often referred to *logarithmica.* [We had to wait for Euler to give e in 1736.]

This solution in Equation (2) can be written as $y = x + a\left(\frac{w}{A} - 1\right)$. We can map the diagrams in onto each other.

Leibniz - 1686

pendent a sectione generali rationis, seu a Lo-
one generali anguli, ſeu ab arcubus circuli, aliæ
uæſtionibus magis compoſitis) ideo præter li-
dhuc tertiam ut v, quæ tranſcendentem quan-
x. his tribus formo æquationem generalem ad
qua lineæ tangentem quæro, ſecundum meam
m in Actis Octobr. 84 publicatam, quæ nec
tur. Deinde id quod invenio comparans cum
gentium curvæ, reperio non tantum literas aſ-
d & ſpecialem tranſcendentis naturam. Quan-
ſo fieri poſſit, ut plures adhibendæ ſint tranſ-
andoque inter ſe diverſæ, & dentur tranſcen-
um, & omnino talia procedant in infinitum, ta-
tilioribus contenti eſſe poſſumus; & plerumq;
uti licet ad calculum contrahendum, proble-
d terminos ſimplices revocandum, quæ non ſunt
em methodo ad Tetragoniſmos applicata, ſeu
rum quadratricium (in quibus utique ſemper
as data eſt) patet non tantum, quomodo inve-
indefinita ſit Algebraice impoſſibilis, ſed &
tate hac deprehenſa reperiri poſſit quadratrix
actenus traditum non fuit. Adeo ut videar
Geometriam hac methodo ultra terminos a
tos in immenſum promoveri. Cum hac ra-
& generalis ad ea porrigatur problemata, quæ
lus, atque adeo Algebraicis æquationibus non

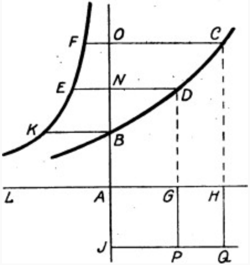
ad problemata Tranſcendentia, ubicunque di-
ue occurrunt, calculo tractanda, vix quicquam

res in axe ſumtorum, & ad axem applicatorum, æquetur ſemiquadra-
to ordinatæ ultimæ) in cujus executione tamen nonnihil a ſcopo deflexit, quod in nova methodo non miror; ideo gratiſſimum ipſi aliisq; fore arbitror, *ſi hoc loco aditum rei, cujus tam late patet utilitas, patefecero.* Nam inde omnia hujusmodi theoremata ac problemata, quæ admirationi merito fuere, ea facilitate fluunt, ut jam non magis ea diſci teneriq; neceſſe ſit, quam plurima vulgaris Geometriæ theoremata illi ediſcenda ſunt, qui ſpecioſam tenet. Sic ergo in caſu prædicto procedo. Sit ordinata x, abſciſſa y, intervallum inter perpendicularem & ordinatam quod dixi ſit p, patet ſtatim methodo mea fore $p\,dy = x\,dx$ quod & Dn. Craigius ex ea obſervavit; qua æquatione differentiali verſa in ſummatricem, fit $\int p\,dy = \int x\,dx$. Sed ex iis quæ in methodo tangentium expoſui, patet eſſe $d, \frac{1}{2}xx = x\,dx$; ergo contra $\frac{1}{2}xx = \int x\,dx$ (ut enim poteſtates & radices in vulgaribus calculis, ſic nobis ſummæ & differentiæ ſeu ∫ & d, reciprocæ ſunt). Habemus ergo $\int p\,dy = \frac{1}{2}xx$. Quod erat dem. Malo autem dx & ſimilia adhibere, quam literas pro illis, quia iſtud dx eſt modificatio quædam ipſius x, & ita ope ejus fit, ut ſola quando id fieri opus eſt litera x, cum ſuis ſcilicet poteſtatibus & differentialibus calculum ingrediatur, & relationes tranſcendentes inter x & aliud exprimantur. Qua ratione etiam lineas tranſcendentes æquatione explicare licet, verbi grat. Sit arcus a, ſinus verſus x, fiet $a = \int dx : \sqrt{2x - xx}$, & ſi cycloidis ordinata ſit y, fiet $y = \sqrt{2x - xx} + \int dx : \sqrt{2x - xx}$, quæ æquatio perfecte exprimit relationem inter ordinatam y & abſciſſam x, & ex ea omnes cycloidis proprietates demonſtrari poſſunt; promotusque eſt hoc modo calculus analyticus ad eas lineas, quæ non aliam magis ob cauſam hactenus excluſæ ſunt, quam quod ejus incapaces crederentur: Interpolationes quoque Walliſianæ & alia innumera hinc deri-

Other Problems

- Tautochrone - Time independent of starting point, Huygens, 1659.
- Brachistochrone - Curve of fastest descent, Johann Bernoulli, 1696.
- Isochrone - Connects points of equal time travel, Leibniz 1687, Jacob Bernoulli 1690.
- Cycloid - Huygens pendulum driven clock, 1656.
- Catenary - hanging chain.
- 1691 - Jacob Bernoulli - parabolic spiral.
- Elastica and lemniscate, Sep 1694, Jacob, Oct 1694, Johann.

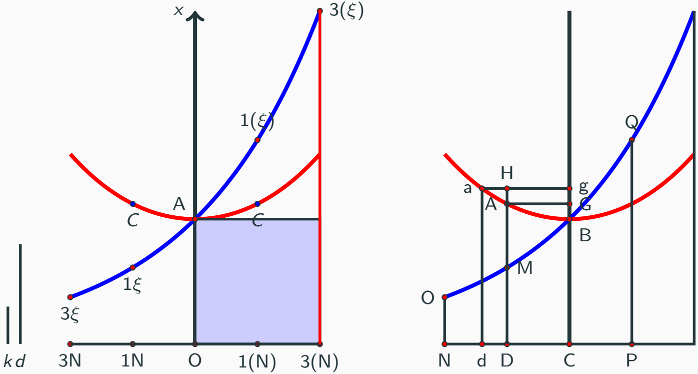
Problems involving *quadrature* and *rectification.* Solution "by quadratures" of the differential equation $a\,dx = a^2\,dy/y$ [Johann Bernoulli, 1692] Lines EN and PG are chosen so that the areas KBNE and AJPG are equal. Intersection D is on the sought curve. Bernoulli was aware that the solution curve was the "Logarithmica" but he considered the geometrical construction more fundamental



The Catenary - Find the curve of a hanging chain.

- Leonardo Da Vinci was the first to consider it.
- Followed by Galileo and others - believed the curve was a parabola.
- Christian Huygens claimed at 17 the curve was not algebraic without any proof.
- May 1690, Jacob Bernoulli posed challenge in the *Acta Eruditorum.*
- Gottfried Leibniz immediately responded. Leibniz,Huygens and Johann Bernoulli separately found the equation for catenaryand published their solutions on 1691.
- While in Paris, Johann Bernoulli discussed catenary in his Lectures on the Integral Calculus, Lecture Thirty-Six, Lecture Twelve, and Lecture Thirty-Seven. The lectures were written out for Guillaume Marquis de L'Hôpital in 1691-1692.
- Compared Leibniz's solution to his own.

The Catenary Solutions of Leibniz (left) and Bernoulli (right)



CB = subtangent = a BG= x, GA=CD=y, DM=z Gg= dx, Dd=Ha=dy

CD=CP $\Rightarrow$ DM:CB = CB:PQ

Therefore, PQ $= a^2/z$ and DA $= \frac{1}{2}$(DM+PQ) $= \dfrac{a^2+z^2}{2z}$ = CB + BG $= a + x$

Solve for z : $z = a + x + \sqrt{2ax + x^2}$, find dz and use $z\,dy = a\,dz$

$$\frac{a\,dx}{\sqrt{2ax+x^2}} = dy$$

Leonhard Euler (1707-1783)

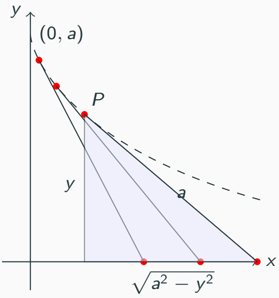
- Connection with Bernoulli family.
- 1748 *Introductio in analysin infinitorum, on precalculus,*
- 1755 *Institutiones calculi differentialis,* on differential calculus,
- 1768 *Institutionum Calculi Integralis* on integral calculus, Euler Archives
  - Vol. I Integration of first order differential equations, E342.
  - Vol II. Resolution of higher order ODEs, E366.
  - Vol III. Resolution of partial differential equations and includes calculus of variations, E385.

- Separable eq.
- Homogeneous eq.
- Reducible to separable eq.
- Linear first order eq
- Bernoulli eq.
- Riccati eq.
- Exact eq.
- Integrating factor
- Particular/General solutions
- Resolution of first order ODEs in Power Series
- First order non-explicit eq.
- Calculus of variations

Tractrix

French physician Claude Perrault (1613-1688), brother of Charles Perrault, who published *Cinderella* and *Little Red Riding Hood,* placed his watch in the middle of the table and pulled the end of the watch chain along the edge of the table. "What is the shape of the curve traced by the watch?" First studied by Christiaan Huygens, gave it the name tractrix (1692).

- Another inverse tangent problem: Find a curve whose tangent has a constant length, a.
- $\dfrac{dy}{dx} = -\dfrac{y}{\sqrt{a^2 - y^2}}$.
- Huygens, 1693, tractional motion and mechanical devices.
- Vincenzo Riccatti (1676-1775) (son of Jacobo) proved all 1st order differential equations could be constructed using tractional motion, 1752. Too late!



Summary

- Descartes and Fermat - geometric vs algebraic curves.
- Newton - fluxional equations.
- Leibniz (Huygens, Bernoulli's) - inverse tangent problems.
- Euler - sought general theory, no figures, introduction of methods.
- First textbooks
    - 1671 Newton, published 1736.
    - 1694 David Gregory, 1st systematic presentation of the method of fluxions Manuscript 'Isaaci Neutoni Methodus Fluxionum'.
    - 1696 Marquis de Hôpital, 1st differential calculus text. Online version and online translation.
    - 1700 Louis Carré, 1st French book on the integral calculus. online.
    - 1704 Charles Hayes, online, 1st English fluxions text.
    - 1707 Gabriele Manfredi, 1st differential equations text. online.
    - 1737 Thomas Simpson, *A New Treatise of Fluxions* online
    - 1742 Colin Maclaurin, *A Treatise of Fluxions,* online.
    - 1748 Maria Agnesi, *Foundations of Analysis for the Use of Italian Youth* with 200 pages on solving differential equations. online