MAT 463/563 Boundary Value Problems Review

I. Preliminary Material
   a. Constant Coefficient Equations
   b. Cauchy-Euler Equations

II. Boundary Value Problems
   a. Direct Solution
   b. Separation of Variables – Heat Equation
   c. Solution of Eigenvalue Equations with Homogeneous BCs

III. Vector Spaces and Function Spaces
   a. Bases/Eigenfunctions
   b. Scalar Product $< f, g >= \int_a^b f(x)g(x)\sigma(x)\, dx$
   c. Orthogonal, Orthonormal, Normalization
   d. Determination of Expansion Coefficients

IV. Fourier Series
   a. Trigonometric Fourier Series Expansion on $[-L, L]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

   b. Fourier Coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx$$

   c. Orthogonality of $\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots\}$ on $[-L, L]$, etc.
   d. Half Range Expansions on $[0, L]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{2n\pi x}{L}\, dx, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{2n\pi x}{L}\, dx$$

   e. Fourier Sine and Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

   f. Even Periodic Extension, Odd Periodic Extension, Periodic Extension

V. Sturm-Liouville Eigenvalue Problems
   a. Converting Second Order Linear ODEs to Self-Adjoint Form,

$p(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)} dx\right)$ Leads to Sturm-Liouville Operator:

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x).$$

   b. Types of Boundary Conditions:
      i. Regular BCs $\alpha u(a) + \beta u'(a) = 0, \delta u(b) + \gamma u'(b) = 0$
      ii. Dirichlet: $u(a) = u(b) = 0.$

iii. Neumann: $u'(a) = u'(b) = 0.$

iv. Periodic: $u(a) = u(b), u'(a) = u'(b).$

c. Sturm-Liouville Eigenvalue Problems:

$$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x) = -\lambda\sigma(x)u(x)$$ plus BCs

d. Lagrange's Identity: $uLv - vLu = \frac{d}{dx}\left[p(uv' - vu')\right]$

e. Green's Identity: $\int_a^b (uLv - vLu)\, dx == \left[p(uv' - vu')\right]_a^b.$

f. Adjoint Operators $\langle v, Lu\rangle = \left\langle L^{\dagger}v, u\right\rangle$

g. Proof that eigenvalues are real and eigenfunctions are orthogonal for self-adjoint problems. – Use Green’s Identity

h. Eigenfunction Expansion Method:
Assume $u(x) = \sum c_n \phi_n(x)$ and find coefficients.

VI. Special Functions

a. Classical Orthogonal Polynomials

b. Gram-Schmidt Orthogonalization Process

c. Legendre Polynomials

i. Rodrigues Formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$

ii. Three Term Recursion Formlua:
$nP_n(x) = (2n+1)xP_{n-1}(x) + (n-1)P_{n-2}(x).$

iii. Generating Function: $g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$,
$|t| \le 1, |x| \le 1.$

iv. Binomial expansion: $(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$

d. Gamma Function: $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\, dt, \quad x > 0.$

e. Factorials and double factorials.

f. Be able to iterate recursion (finite difference) formulae, like $I_{n+1} = I_n$, given I_0.

g. Bessel Functions – General Properties

VII. Green's Functions

a. Variation of Parameters $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

$$c_1' = \frac{1}{W}\begin{vmatrix} 0 & y_2 \\ f/p & y_2' \end{vmatrix}, \quad c_2' = \frac{1}{W}\begin{vmatrix} y_1 & 0 \\ y_1' & f/p \end{vmatrix}$$

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b. $Lu = f \Rightarrow u = \int G(x,\xi) f(\xi)\, d\xi.$

c. Initial Value Green's Function

$$u(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau, \qquad G(t,\tau) = \frac{y_1(\xi) y_2(x) - y_1(x) y_2(\xi)}{pW}$$

d. Boundary Value Green's Function

$$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi \qquad G(x,\xi) = \begin{cases} C y_1(\xi) y_2(x), & \xi \le x \\ C y_1(x) y_2(\xi), & x \le \xi \end{cases}$$

e. Properties of BVP Green's Functions

i. $LG(x,\xi) = \delta(x-\xi)$

ii. Satisfies Homogeneous BCs

iii. Symmetry: $G(x,\xi) = G(\xi, x)$

iv. Continuity: $G(x^+,\xi) = G(x^-,\xi)$

v. Jump Discontinuity of Derivative: $\dfrac{dG(x^+,\xi)}{dx} - \dfrac{dG(x^-,\xi)}{dx} = \dfrac{1}{p(\xi)}$

f. Dirac Delta Function

i. $\delta(x) = 0,\ x \ne 0$ and $\int_{-\infty}^{\infty} \delta(x)\, dx = 0$

ii. $\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$

iii. For the case that a function has multiple simple roots, $f(x_i) = 0,\ f'(x_i) \ne 0,$ for $i = 1,\ldots,n,$ one has that that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

g. Green's Functions from Eigenfunction Expansions

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n \|\phi_n\|^2}$$