Topics for Exam II – MAT 367

1. Infinite Dimensional Spaces
   a. Inner Product $< f, g >= \int_a^b f(x)g(x)\sigma(x)\,dx$, Orthogonality
   b. Generalized Fourier Coefficients: $f(x) \sim \sum_n c_n\phi_n(x),\ c_n = \dfrac{\langle \phi_n, f\rangle}{\langle \phi_n, \phi_n\rangle}$
   c. Gram Schmidt Orthogonalization, $\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \dfrac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j,\ n \geq 2$
   d. Legendre Polynomials – use of Rodrigues formula, three term recursion formula, generating function, and normalization - $\int_{-1}^{1} P_n^2(x)\,dx = \dfrac{2}{2n+1}$
   e. Gamma Function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$,

      $\Gamma(1) = 1,\ \Gamma(x+1) = x\Gamma(x),\ \Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$
2. Complex Numbers
   a. Know how to use polar forms $z = re^{i\theta}$, $x = r\cos\theta,\ y = r\sin\theta$ and $r = \sqrt{x^2+y^2}, \tan\theta = \dfrac{y}{x}$
   b. $e^{i\pi} = -1, e^{2\pi ik} = 1$ for k an integer
   c. Complex Modulus and complex conjugate
   d. nth roots $z^{1/n} = r^{1/n}(\cos\dfrac{\theta+2k\pi}{n} + i\sin\dfrac{\theta+2k\pi}{n})$ for $k = 0,1,\ldots,n-1$
   e. Roots of Unity
3. Complex Functions
   a. Determine real and imaginary parts of functions: $f(z) = u(x,y) + iv(x,y)$
   b. Complex functions $\sin(z), \sinh(z), \ln(z)$, etc.
4. Differentiation
   a. Compute Derivative $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$
   b. Differentiability and CR Equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$
   c. Harmonic Functions: CR => $\nabla^2 u = 0$ and harmonic conjugate.
   d. Terms - Holomorphic, Analytic, Entire, ….
5. Integration $\int_C f(z)\,dz$
   a. Complex Path Integrals – parametrized over line segment, arcs, etc.

      $\int_C f(z)\,dz = \int_a^b f(x(t)+iy(t))\left(\dfrac{dx}{dt} + i\dfrac{dy}{dt}\right)dt$
   b. Path Independence, When can one deform contours?
   c. Cauchy's Theorem $\oint_C f(z)\,dz = 0$ if $f(z)$ is differentiable

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d. Cauchy Integral Formula $f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$

e. Computing Residues

i. $\text{Res}[f(z); z=z_0]=\lim_{z\to z_0}(z-z_0)f(z)$ - simple poles

ii. $\text{Res}[f(z); z=z_0]=\lim_{z\to z_0}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^k f(z)]$ - poles of order k

f. Residue Theorem $\int_C f(z)\,dz=2\pi i \sum_{\text{Poles inside C}}$ Residues

g. $\cos\theta=\frac{z+z^{-1}}{2}$, $\sin\theta=\frac{z-z^{-1}}{2i}$

h. Going from integrals over R to complex integrals

6. Series Expansions

a. Power series, Laurent series, Taylor series

b. Circle of convergence

c. Using geometric series

i. $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n, |z|<1,$

ii. $\frac{1}{1-z}=-\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)=-\sum_{n=1}^{\infty} z^{-n}, |z|>1$