Chapter 7

Analog and Discrete Signals

As you may recall, the goal of this course was to see the connection between analog and discrete signals. We have gathered the tools needed to discuss analog signals, but will need the Discrete Fourier Transform (DFT) in order to study discrete signals. However, before we go into the DFT analysis, we pause to point out a few connections to what we have covered to date.

We consider an analog function and its transform as shown in Figure 7.1. We see that analog signals can be described as a piecewise continuous function defined over an infinite time interval. The resulting Fourier transform is also piecewise continuous and defined over an infinite interval of frequencies. We can represent the signal and its transform using the Fourier transform and the inverse transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega t}\,d\omega, \tag{7.1}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt. \tag{7.2}$$

Note that the figures in this section are drawn as if the transform is real. However, in general they are not and we will investigate how this can be handeled in the last chapter.

Real signals cannot be studied on an infinite interval. One usually only records data for a finite time interval. Let's assume that the recording starts at $t = 0$. Then the interval can be written as $[0, T]$, where T will be

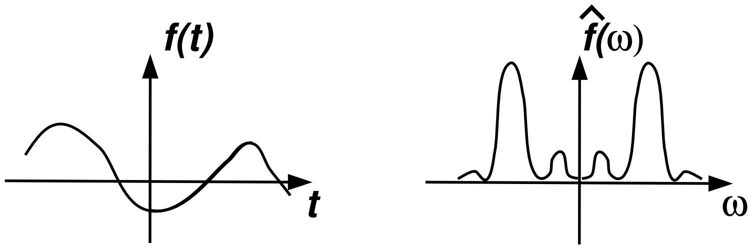

Figure 7.1: Plot of an analog signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

called the record length.

The natural representation would be to extend the signal to a periodic signal, knowing that the physical signal is only defined on $[0, T]$. This can be modeled by a Fourier series. Recall that this is given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t} \tag{7.3}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{i\omega_n t}\, dt. \tag{7.4}$$

Here we have defined $\omega_n = \frac{2\pi}{T}$.

Given that $f(t)$ is a periodic function, we would like to relate the above Fourier series to the Fourier transform of $f(t)$. This is easily done by simply computing the Fourier transform. Thus,

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}\right) e^{i\omega t}\, dt \\ &= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} e^{i(\omega-\omega_n)t}\, dt. \end{aligned} \tag{7.5}$$

We recall that

$$\int_{-\infty}^{\infty} e^{i\omega x}\, dx = 2\pi\delta(\omega).$$

Using this result, we have that

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n 2\pi\delta(\omega - \omega_n). \tag{7.6}$$

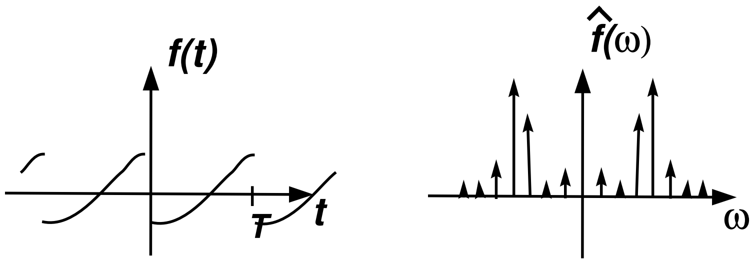

Figure 7.2: A periodic signal leads to a discrete number of frequencies.

Thus, the Fourier transform of a periodic function is a series of spikes at discrete frequencies. This is represented in Figure 7.2.

A function that often occurs in signal analysis is the comb function defined by

$$\text{comb}_a(t) = \sum_{n=-\infty}^{\infty} \delta(t - na). \tag{7.7}$$

This function is simply a set of translated delta function spikes. It is a distribution and care needs to be taken in studying the properties of this function. *These properties will be expanded later in this section in a later version of the notes.* One can show that the Fourier transform of a comb function is a comb function.

$$F[\text{comb}_a(t)] = \frac{1}{a}\text{comb}_{\frac{1}{a}}(\omega). \tag{7.8}$$

The convolution of a function with a comb function leads to a periodic function. We can show this by first considering the convolution of a function $f(t)$ with a shifted Dirac delta function, $\delta(t - a)$. This convolution is easily computed as

$$(f * \delta_a)(t) = \int_{-\infty}^{\infty} f(t - \tau)\delta(\tau - a)\, d\tau = f(t - a).$$

Therefore, we have found that the convolution of a function with one shifted delta function is a copy of $f(t)$ that is shifted by a.

The convolution of a function $f(t)$ with a comb function is then the sum of copies of shifted copies of $f(t)$, as shown in Figure 7.4. If the function has

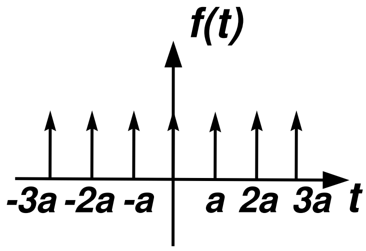

Figure 7.3: The comb function, $\text{comb}_a(t)$.

compact support on $[-a/2, a2]$, i.e., the function is zero for $|t| > 1/a$, then the convolution with the comb function will be periodic.

Returning to the Fourier transform of our periodic function, we found that the result was a series of spikes. This series of spikes is a convolution in frequency space of the Fourier transform of $f(t)$ and a comb function. Thus, the inverse transform is a periodic function.

Finally, we would like to sample our signal at a discrete set of times. This is how one normally records signals. One samples the signals every so many time steps, like every tenth of a second, or more. We can model sampling at discrete time points by multiplying $f(t)$ by a comb function. The Fourier transform will yield a convolution of the Fourier transform of $f(t)$ with the Fourier transform of the comb function. But this is a convolution of $\hat{f}(\omega)$ with another comb function, since the transform of a comb function is a comb function. Therefore, we will obtain a periodic representation in Fourier space.

In collecting data, we not only sample at a discrete set of points, but we also sample for a finite length of time. By sampling like this, we will not gather enough information to obtain the high frequencies in a signal. Thus, there will be a natural cutoff in the spectrum of the signal. This is represented in Figure 7.5. This process will lead to the discrete transform, the topic in the next chapter.

In Figure 7.6 we summarize the steps for going from analog signals to discrete signals.

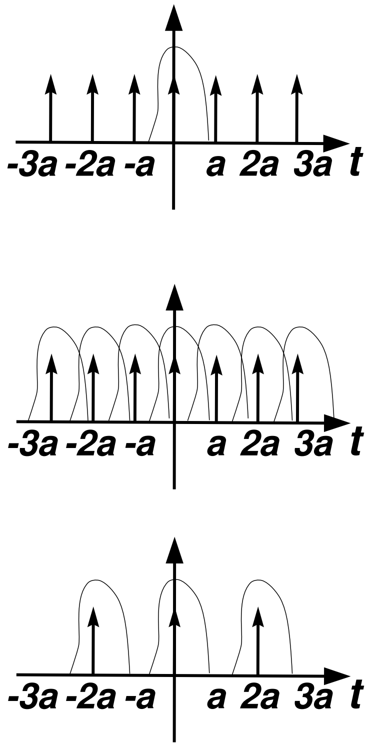

Figure 7.4: The convolution of $f(t)$ with the comb function, $\mathrm{comb}_a(t)$. In the second of this plots the result is the sum of several translations of $f(t)$. Incorrect sampling will lead to overlap in the translates and cause problems like aliasing. In the last of these plots, one has no overlap and the periodicity is evident.

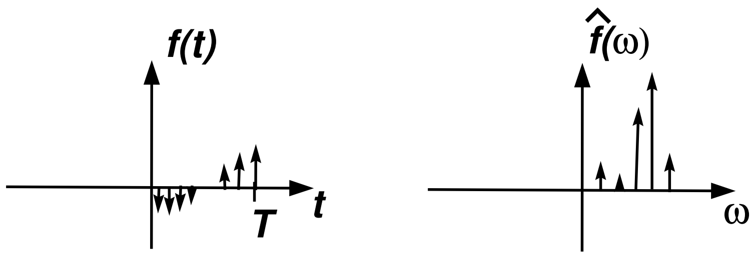

Figure 7.5: Sampling the original signal at a discrete set of times defined on a finite interval leads to a discrete set of frequencies in the transform that are restricted to a finite interval of frequencies.

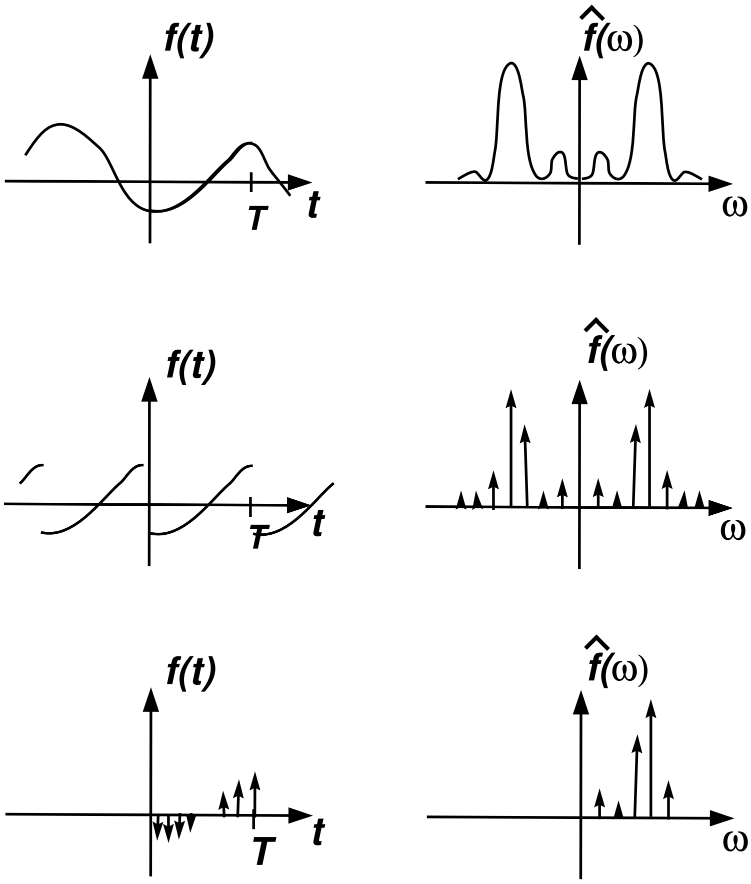

Figure 7.6: Summary of transforming analog to discrete signals. One starts with a continuous signal $f(t)$ defined on $(-\infty, \infty)$ and a continuous spectrum. By only recording the signal over a finite interval $[0, T]$, the recorded signal can be represented by its periodic extension. This in turn forces a discretization of the transform as shown in the second row of figures. Finally, by restricting the range of the sampled, as shown in the last row, the original signal appears as a discrete signal. This is also interpreted as the sampling of an analog signal leads to a restricted set of frequencies in the transform.