Chapter 2

Sequences and Series

In this chapter we will review and extend some of the concepts and definitions related to infinite series that you might have seen previously in your calculus class. Working with infinite series can be a little tricky and we need to understand some of the basics before moving on to the study of series of trigonometric functions in the next chapter.

For example, one can show that the infinite series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$. However, the terms can be rearranged to give

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots = \frac{3}{2}\ln 2.$$

In fact, other rearrangements can be made to give any desired sum!

Other problems with infinite series can occur. Try to sum the following infinite series to find that

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2} \sim 0.937548\ldots.$$

A sum of even as many as 10^7 terms only gives convergence to four or five decimal places.

The series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots, \quad x > 0$$

diverges for all x. So, you might think this divergent series is useless. However, truncation of this divergent series leads to an approximation of the integral

$$\int_0^\infty \frac{e^{-t}}{x+t}\,dt, \quad x > 0.$$

So, can we make sense out of any of these series, or any other manipulations of infinite series? We will not answer all of these questions now, but we will go back and review what you have seen in your calculus classes.

2.1 Sequences of Real Numbers

We first begin with the definitions for sequences and series of numbers.

Definition A *sequence* is a function whose domain is the set of positive integers.

Examples are

1. $a(n) = n$ yields the sequence $\{1, 2, 3, 4, 5, \ldots\}$
2. $a(n) = 3n$ yields the sequence $\{3, 6, 9, 12, \ldots\}$

However, one typically uses subscript notation and not functional notation: $a_n = a(n)$. We then call a_n the nth term of the sequence.

Another way to define a particular sequence is recursively.

Definition A *recursive sequence* is defined in two steps:

1. The value of first term (or first few terms) is given.
2. A rule, or recursion formula, to determine later terms from earlier ones is given.

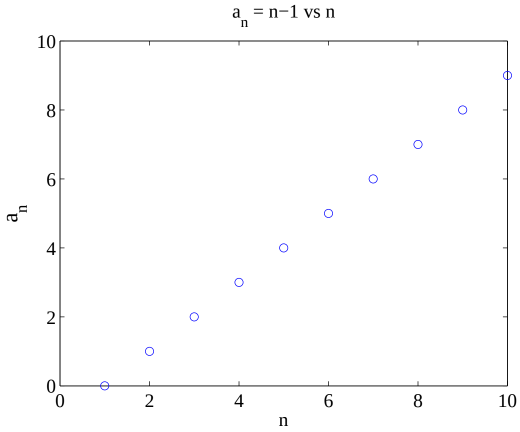

Figure 2.1: Plot of $a_n = n - 1$ for $n = 1 \ldots 10$.

A typical example is given by the Fibonacci sequence. It can be defined by the recursion formula $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ and the starting values of $a_1 = 0$ and $a_1 = 1$. The resulting sequence is $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$. Writing the general expression for the nth term is possible, but it is not as simply stated. Recursive definitions are often useful in doing computations for large values of n.

2.2 Convergence of Sequences

Next we are interested in the behavior of the sequence as n gets large. For the sequence defined by $a_n = n - 1$, we find the behavior as shown in Figure 2.1. Notice that as n gets large, a_n also gets large. This sequence is said to be divergent.

On the other hand, the sequence defined by $a_n = \frac{1}{2^n}$ approaches a limit as n gets large. This is depicted in Figure 2.2. Another related series, $a_n = \frac{(-1)^n}{2^n}$, is shown in Figure 2.3. This sequence is the alternating sequence $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\}$.

Definition The sequence a_n *converges* to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a - L| < \epsilon.$$

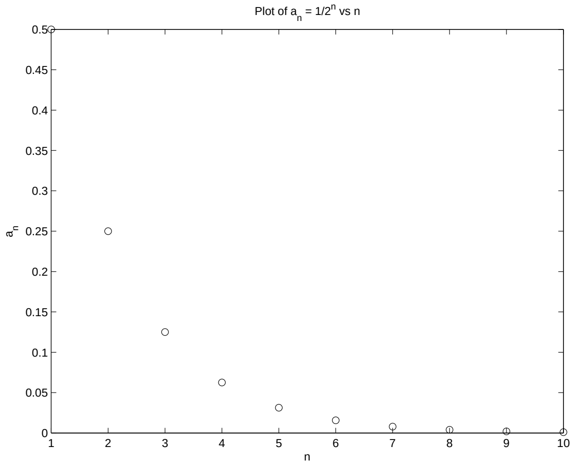

Figure 2.2: Plot of $a_n = \frac{1}{2^n}$ for $n = 1 \ldots 10$.

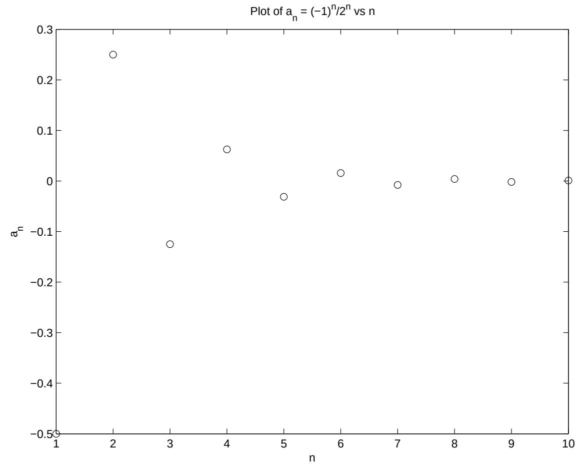

Figure 2.3: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$.

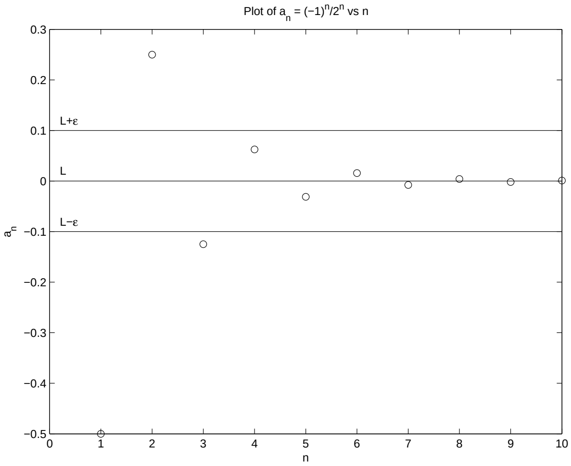

Figure 2.4: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.1$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 3$.

If no such number exists, then the sequence is said to *diverge*.

In Figures 2.4-2.5 we see what this means. For the given sequence, we see that $L = 0$. Given an $\epsilon > 0$, we ask for what value of N the nth terms ($n > N$) lie in the interval $[L - \epsilon, L + \epsilon]$. In these figures this interval is depicted by a horizontal band. We see that for convergence, sooner, or later, the tail of the sequence ends up entirely within this band.

If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit L, then we write either $a_n \to \infty$ or $\lim_{n\to\infty} a_n = L$. For example, we have already seen that $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$.

2.3 Limit Theorems

Once we have defined the notion of convergence of a sequence to some limit, then we can investigate certain properties of limits of sequences. Here we list a few results on limits theorems and some special limits, which arise often.

Theorem Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ and a

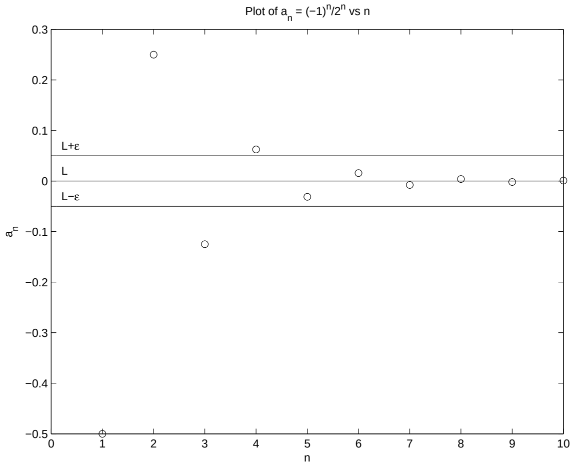

Figure 2.5: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.05$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 4$.

number k. Assume that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then we have

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$.
2. $\lim_{n\to\infty}(kb_n) = kB$.
3. $\lim_{n\to\infty}(a_n b_n) = AB$.
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0$.

Some special limits are given next. These are generally first encountered in a second course in calculus.

Theorem The following are special cases:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.
2. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0$.
4. $\lim_{n\to\infty} x^n = 0, \quad |x| < 1$.

5. $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$.
6. $\lim_{n\to\infty}\frac{x^n}{n!} = 0$.

The proofs generally are straight forward, depending upon L'Hopital's Rule and other manipulations. For example, one can prove the first limit by first realizing that $\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x}$. This limit is indeterminate as $x \to \infty$ in its current form since the numerator and the denominator get large for large x. In such cases one employs L'Hopital's Rule:

Theorem Let c be a finite number or $c = \infty$. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Thus, $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$.

The second limit in the list can be proven by first looking at

$$\lim_{n\to\infty}\ln n^{1/n} = \lim_{n\to\infty}\frac{\ln n}{n} = 0.$$

Now, if $\lim_{n\to\infty}\ln f(n) = 0$, then $\lim_{n\to\infty} f(n) = e^0 = 1$. Thus proving the second limit.

The third limit can be done similarly. The reader is left to confirm the other limits.

Example 1. $\lim_{n\to\infty}\frac{n^2+2n+3}{n^3+n}$

Divide the numerator and denominator by n^2. Then $\lim_{n\to\infty}\frac{n^2+2n+3}{n^3+n} = \lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{3}{n^2}}{n+\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n} = 0$.

Example 2. $\lim_{n\to\infty}\frac{\ln(n^2)}{n}$

Rewriting $\frac{\ln(n^2)}{n} = \frac{2\ln(n)}{n}$, we find from identity 1 of the theorem that $\lim_{n\to\infty}\frac{\ln(n^2)}{n} = 2\lim_{n\to\infty}\frac{\ln(n)}{n} = 0$.

Example 3. $\lim_{n\to\infty}(n^2)^{\frac{1}{n}}$

To compute this limit, we rewrite
$\lim_{n\to\infty}(n^2)^{\frac{1}{n}} = \lim_{n\to\infty}(n)^{\frac{1}{n}}(n)^{\frac{1}{n}} = 1$, using identity 2.

Example 4. $\lim_{n\to\infty}(\frac{(n-2)}{n})^n$

This limit can be written as
$\lim_{n\to\infty}(\frac{(n-2)}{n})^n = \lim_{n\to\infty}(1+\frac{(-2)}{n})^n = e^{-2}$. Here we have used identity 5.

2.4 Infinite Series

In this section we investigate the meaning of series, which are infinite sums of the form

$$a_1 + a_2 + a_2 + \dots. \tag{2.1}$$

A typical example is the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots. \tag{2.2}$$

How would one evaluate this sum? We begin by just adding the terms. For example,

$$\begin{aligned}
1 + \frac{1}{2} &= \frac{3}{2},\\
1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4},\\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8},\\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{31}{16},
\end{aligned} \tag{2.3}$$

etc. The values tend to a limit. We can see this graphically in Figure 2.6.

In general, we want to make sense out of Equation (2.1). As with the example, we look at a sequence of partial sums. Thus, we consider the sums

$$\begin{aligned}
s_1 &= a_1,\\
s_2 &= a_1 + a_2,\\
s_3 &= a_1 + a_2 + a_3,\\
s_4 &= a_1 + a_2 + a_3 + a_4,
\end{aligned} \tag{2.4}$$

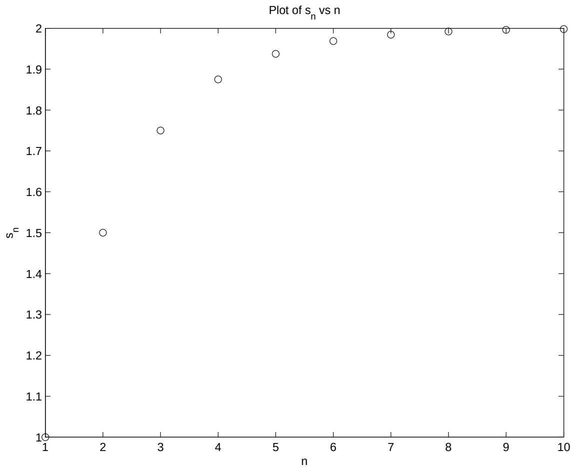

Figure 2.6: Plot of $s_n = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$ for $n = 1 \ldots 10$.

etc. In general, we define the nth partial sum as

$$s_n = a_1 + a_2 + \ldots + a_n.$$

If the infinite series (2.1) is to make any sense, then the sequence of partial sums should converge to some limit. We define this limit to be the sum of the infinite series, $S = \lim_{n\to\infty} s_n$.

Definition If the sequence of partial sums converges to the limit L as n gets large, then the infinite series is said to have the sum L.

We will use the compact summation notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots.$$

Here n will be referred to as the index and it may start at values other than $n = 1$.

2.5 Geometric Series

Example (2.2) is an example of what is known as a geometric series. A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^2 + \ldots + ar^n + \ldots . \tag{2.5}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is $r = \frac{ar^{n+1}}{ar^n}$.

The sum of a geometric series, when it converges, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{2.6}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{2.7}$$

Subtracting these two equations, while noting the many cancellations, we have

$$(1 - r)s_n = a - ar^n. \tag{2.8}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1 - r^n)}{1 - r}. \tag{2.9}$$

Recall that the sum, if it exists, is given by $S = \lim_{n \to} s_n$. Letting n get large in the partial sum (2.9), we need only evaluate $\lim_{n \to \infty} r^n$. From our special limits we know that this limit is zero for $|r| < 1$. Thus, we have the sum of the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}. \tag{2.10}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$
In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure 2.6.

Example 2. $\sum_{k=2}^{\infty} \frac{4}{3^k}$
In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example 3. $\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n})$
Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed. Thus, we have

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

2.6 Convergence Tests

Given a general infinite series, it would be nice to know if it converges, or not. Often, we are only interested in the convergence and not the actual sum. It is often difficult to determine the sum, when the series does converge. In this section we will review some of the standard tests for convergence.

First, we have the nth term divergence test. This can be motivated by two examples:

1. $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \dots.$
2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots.$

In the first example it is easy to see that each term is getting larger and larger, and thus the partial sums will grow without bound. In the second case, each term is bigger than one. Thus, the series will be bigger than adding the same number of ones as there are terms in the sum. Obviously, this series will also diverge.

This leads to the nth term divergence test:

Theorem If $\lim a_n \neq 0$ or if this limit does not exist, then $\sum_n a_a$ diverges.

This theorem does not imply that just because the terms are getting smaller, the series will converge. Otherwise, we would not need any other convergence theorems.

For the next theorems, we will assume that the series has nonnegative terms.

1. **Comparison Test**

   The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ such that $a_n \leq c_n$ for all $n > N$ for some N. The series $\sum a_n$ diverges if there is a divergent series $\sum d_n$ such that $d_n \leq a_n$ for all $n > N$ for some N.

   For this test one has to dream up a second series for comparison. Typically, this requires some experience with convergent series. Often it is better to use other tests first if possible.

2. **Limit Comparison Test**

   If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite then $\sum a_n$ and $\sum b_n$ converge together or diverge together.

   For example, consider the infinite series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{(n+1)^2} = 2$. Thus, these two series both converge, or both diverge. If we knew the behavior of the second series, then we could draw a conclusion. Using the next test, we will prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ diverges.

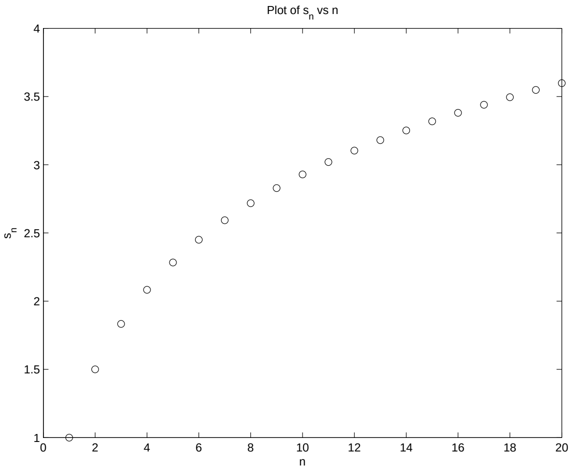

Figure 2.7: Plot of the partial sums for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

3. **Integral Test**

   Consider the infinite series $\sum_{n=1}^{\infty} a_n$. Let $f(n) = a_n$. Then, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge. Here we mean that the integral converges or diverges as an improper integral.

   We are interested in the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we saw in the Limit Comparison Test example. This infinite series is famous and is called the harmonic series. The plot of the partial sums is given in Figure 2.7. It appears that the series could possibly converge or diverge. It is hard to tell graphically.

   In this case we can use the Integral Test. In Figure 2.8 we plot $f(x) = \frac{1}{x}$ and at each integer n we plot a box from n to $n+1$ of height $\frac{1}{n}$. We can see from the figure that the total area of the boxes is greater than the area under the curve. Since the area of each box is $\frac{1}{n}$, then we have that

$$\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

   But, we can compute the integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{x\to\infty} (\ln x) = \infty.$$

   Thus, the integral diverges and the infinite series is larger than this! So, the harmonic series diverges.

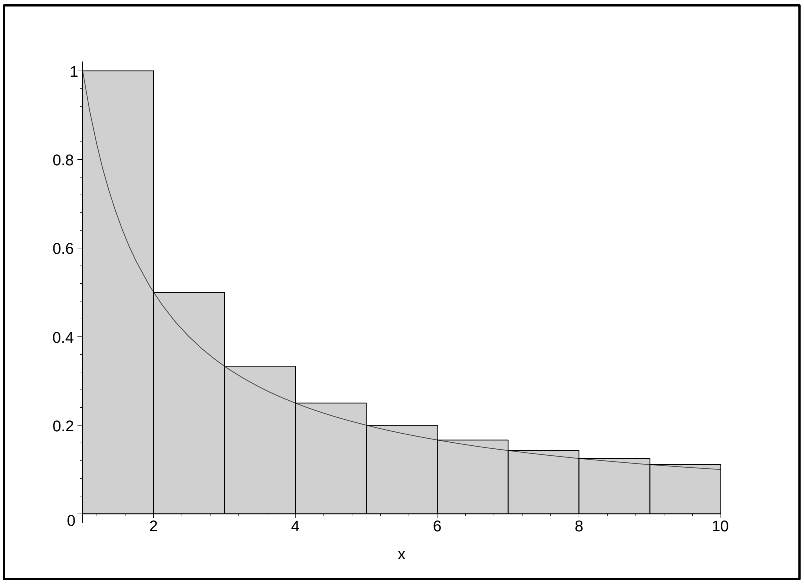

Figure 2.8: Plot of $f(x) = x$ and boxes of height $\frac{1}{n}$ and width 1.

The Integral Test provides us with the convergence behavior for a class of infinite series called p-series. These series are of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Recalling that the improper integrals $\int_1^{\infty} \frac{dx}{x^p}$ converge for $p > 1$ and diverge otherwise, we have the p-test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1$$

and diverges otherwise.

> **Example** $\sum_{n=1}^{\infty} \frac{n+1}{n^3-2}$.
> We first note that as n gets large, the general term behaves like $\frac{1}{n^2}$ since the numerator behaves like n and the denominator behaves like n^3. So, we expect that this series behaves like the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus, by the limit comparison test, $\lim_{n\to\infty} \frac{n+1}{n^3-2}(n^2) = 1$, these series both converge, or both diverge. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test since $p = 2$. Therefore, the original series converges.

4. **Ratio Test**

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the behavior of the infinite series can be determined from from the conditions

$$\begin{aligned} \rho < 1, &\quad \text{converges} \\ \rho > 1, &\quad \text{diverges} \end{aligned}$$

Example 1. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$.
We compute

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \frac{10^n}{10^{n+1}} \\ &= \lim_{n\to\infty} (1 + \frac{1}{n})^{10} \frac{1}{10} \\ &= \frac{1}{10} < 1. \end{aligned} \tag{2.11}$$

Therefore, the series is said to converge by the ratio test.

Example 2. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.
In this case we make use of the fact that $(n+1)! = (n+1)n!$.
We compute

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1 \end{aligned} \tag{2.12}$$

This series also converges by the ratio test.

5. **Root Test**

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} {a_n}^{1/n}$. Then the behavior of the infinite series can be determined from

$$\begin{aligned} \rho < 1, &\quad \text{converges} \\ \rho > 1, &\quad \text{diverges} \end{aligned}$$

Example 1. $\sum_{n=0}^{\infty} e^{-n}$.
We use the nth root test:
$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$. Thus, this series converges by the nth root test.

Example 2. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$.
This series also converges by the nth root test:
$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2^n} = 0 < 1$.

We next turn to series which have both positive and negative terms. We can toss out the signs by taking absolute values of each of the terms. We then note that since $a_n \leq |a_n|$ we have

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|.$$

If the sum $\sum_{n=1}^{\infty} |a_n|$converges, then the original series converges. This type of convergence is useful, because we can use the previous tests to establish convergence of such series. Thus, we say that a series *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series converges, but does not converge absolutely, then it is said to *converge conditionally.*

Example $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$.
This series converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$.

Finally, there is one last test that we recall from your introductory calculus class. We consider the alternating series, given by $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. The convergence of alternating series is determined from Leibniz's Theorem.

Theorem The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

1. a_n's are positive.
2. $a_n \geq a_{n+1}$ for all n.
3. $a_n \to 0$.

The first condition guarantees that we have alternating signs in the series. The next conditions say that the magnitude if the terms gets smaller and approaches zero.

> **Example 1.** The alternating harmonic series converges. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
>
> **Example 2.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ also passes the conditions of Leibniz's Theorem

Note that in Example 2 we can show that the series is absolutely convergent. However, the series of absolute values for Example 1 is the harmonic series, so it is not absolutely convergent. Therefore, we have an example of a series that is conditionally convergent.

2.7 The Order of Sequences and Functions

Often we are interested in comparing the magnitudes of sequences or functions. This is useful in approximation theory. We begin with the comparison of sequences and introduce *big-Oh* notation. We will then extend this to functions of continuous variables.

Definition Let a_n and b_n be two sequences. Then if there are numbers N and K, independent of N, such that

$$\left|\frac{a_n}{b_n}\right| < K \quad \text{whenever} \quad n > N,$$

then we say that a_n is *of the order of* b_n. We write this as

$$a_n = O(b_n) \quad \text{as} \quad n \to \infty$$

and may also say that a_n is *big-Oh* of b_n.

For example, consider the series given by $a_n = \frac{2n+1}{3n^2+2}$ and $b_n = \frac{1}{n}$. Then,

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{2n+1}{3n^2+2}}{\frac{1}{n}}\right| = \left|\frac{2n^2+n}{3n^2+2}\right|.$$

We want to find a bound on the last expression. We divide the numerator and denominator by n^2 and find that

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{2+1/n}{3+2/n^2}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right|.$$

The last term is largest for $n = 1$. This gives

$$\left|\frac{a_n}{b_n}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right| \leq \frac{2}{3}\left|\frac{1+1/2}{1+2/3}\right| = \frac{9}{10}.$$

Thus, for $n > 1$, we have that

$$\left|\frac{a_n}{b_n}\right| < \frac{9}{10}.$$

We then conclude that

$$a_n = O(b_n) = O(\frac{1}{n}).$$

In practice one is given a sequence like a_n, but the second sequence needs to be found by looking at the large n behavior of a_n. Referring to the last example, we are given $a_n = \frac{2n+1}{3n^2+2}$. We look at the large n behavior. The numerator behaves like $2n$ and the denominator behaves like $3n^2$. Thus,

$$a_n = \frac{2n+1}{3n^2+2} \sim \frac{2n}{3n^2} = \frac{2}{3n} \quad \text{for large } n.$$

Therefore, we say that $a_n = O(\frac{1}{n})$. Note that we are only interested in the n-dependence and not the multiplicative constant since $\frac{1}{n}$ and $\frac{2}{3n}$ have the same growth rate.

In a similar way, we can compare functions. We modify our definition of big-Oh for functions of a continuous variable.

Definition $f(x)$ is *of the order of* $g(x)$, or $f(x) = O(g(x))$ as $x \to x_0$ if

$$\lim_{x - x_0}\left|\frac{f(x)}{g(x)}\right| < K$$

for some K independent of x_0.

For example, we recall the Taylor series expansion for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots .$$

(We will review Taylor series in the last section of this chapter.) We want to show that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \text{ as } x \to 0.$$

This should look obvious from the above series expansion. However, we want to verify this using our definition.

We need to compute

$$\lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| .$$

The numerator and denominator separately go to zero, so we have an indeterminate form. This suggests that we need to apply L'Hopital's Rule. In fact, we apply it several times to find that

$$\begin{aligned} \lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| &= \lim_{x\to 0} \left| \frac{-\sin x + x}{4x^3} \right| \\ &= \lim_{x\to 0} \left| \frac{-\cos x + 1}{12x^2} \right| \\ &= \lim_{x\to 0} \left| \frac{\sin x}{24x} \right| = \frac{1}{24}. \end{aligned} \tag{2.13}$$

Thus, for any number $K > \frac{1}{24}$ we have that

$$\lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| > K.$$

We conclude that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \text{ as } x \to 0.$$

Similarly, we can make use of the binomial expansion to determine the behavior of functions such as $f(x) = (a + x)^b - a^b$. Recall that the first terms of the binomial expansion can be written

$$(1 + x)^b = 1 + bx + O(x^2)$$

as $x \to \infty$. We will review the binomial expansion in the next section. It can also be obtained using Taylor series expansions, which we will also cover later in this chapter.

Inserting the binomial expansion into $f(x)$, we have as $x \to 0$ that

$$\begin{aligned} f(x) &= (a+x)^b - a^b \\ &= a^b\left[(1+\frac{x}{a})^b - 1\right] \\ &= a^b\left[\frac{bx}{a} + O\left(\left(\frac{x}{a}\right)^2\right)\right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } x \to 0. \qquad (2.14) \end{aligned}$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^b - a^b = a^{b-1}bx + O\left(\left(\frac{x}{a}\right)^2\right) \qquad \text{as } x \to 0.$$

As an example, we could compute $\sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. Inserting into a calculator, one finds that

$$\sqrt{6378164^2+1} - 6378164 = 1 \times 10^{-7}.$$

How accurate is this? We could use our approximation with $a = R^2$, $x = 1$ and $b = \frac{1}{2}$. Then, our approximation would be of order

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2+1} - 6378164 \approx a^{b-1}bx = (6378164^2)^{-1/2}(0.5)1 = 7.83925 \times 10^{-8}.$$

This is a better approximation. Of course, you should verify how many digits should be kept in reporting the result.