2.11 Power Series

A typical example of a series of functions that the student has encountered in previous courses is the power series, with examples being provided by Taylor and MacLaurin series.

Definition A *power series* expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n(x-a)^n$.

For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

An example of such a power series is the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots. \tag{2.29}$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to what our infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (2.10), we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide various approximations to the function $(1-x)^{-1}$. If x is small, we could write

$$(1-x)^{-1} = 1 + x + O(x^2).$$

In Figure 2.12 we see that for small values of x these functions do agree. Of course, if we want better agreement, we select more terms. In Figure 2.13 we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure 2.14 shows that keeping only quadratic terms may not be good enough. Keeping the cubic terms is

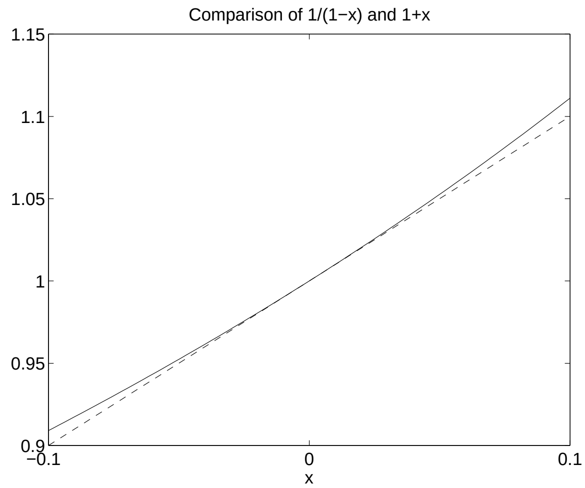

Figure 2.12: Comparison of $\frac{1}{1-x}$ (solid) to $1+x$ (dashed) for $x \in [-0.1, 0.1]$.

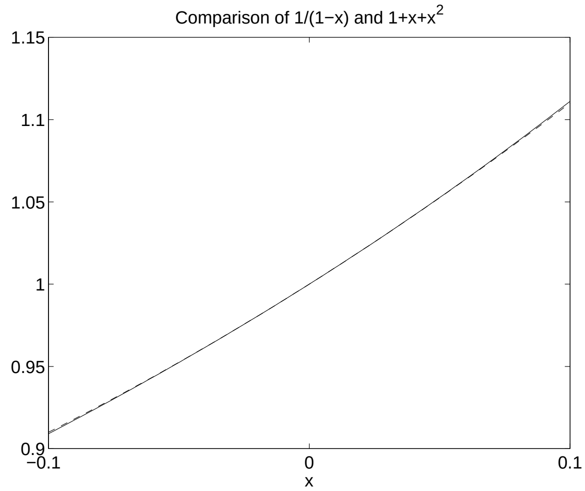

Figure 2.13: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) for $x \in [-0.1, 0.1]$.

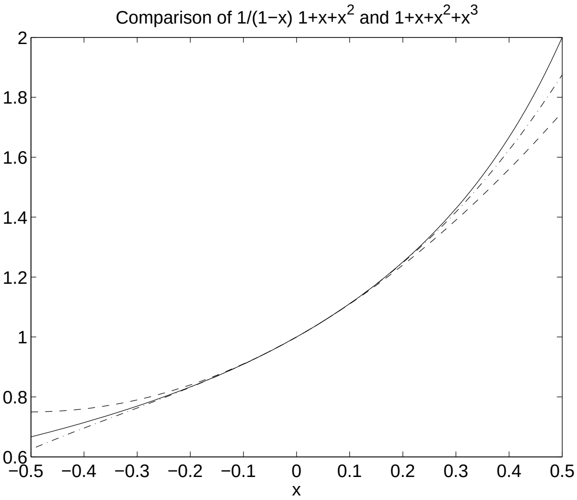

Figure 2.14: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) and $1 + x + x^2 + x^3$ (dash-dot) for $x \in [-0.5, 0.5]$.

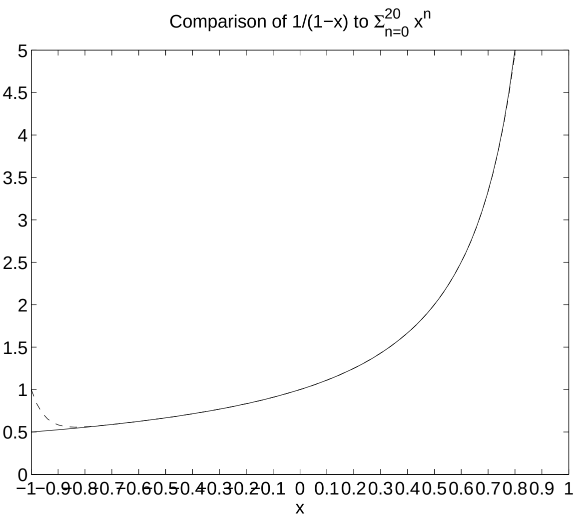

Figure 2.15: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{\infty} x^n$ for $x \in [-1, 1]$.

better. Finally, in Figure 2.15 we show the sum of the first 21 terms over the entire interval $[-1,1]$.

With this example we see how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with the function and then find a series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and MacLaurin series.

Definition A *Taylor series* expansion of $f(x)$ about $x=a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{2.30}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{2.31}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function.

A special class of series are those for which the expansion is about $x=0$.

Definition A *MacLaurin series* expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x=0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{2.32}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{2.33}$$

Example Expand $f(x)=e^x$ about $x=0$.

We begin by creating a table. We will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then one only needs to evaluate the second column at $x=0$ and divide by $n!$.

n	$f^{(n)}(x)$	c_n
0	e^x	$\frac{e^0}{0!} = 1$
1	e^x	$\frac{e^0}{1!} = 1$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$

Next, one looks at the last column and tries to determine some pattern so as to write down the general term of the series. If there is only a need to get an approximation, then the first few terms may be sufficient. In this case, we have that the pattern is obvious: $c_n = \frac{1}{n!}$. So,

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example Expand $f(x) = e^x$ about $x = 1$.

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. We could create a table like the last example. In fact, the last column would have values of the form $\frac{e}{n!}$. (You should confirm this.)

However, we could make use of the MacLaurin series expansion for e^x and get the result quicker. Note that $e^x = e^{x-1+1} = ee^{x-1}$. Now, apply the known expansion for e^x :

$$e^x = ee^{x-1} = e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \dots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

Example Expand $f(x) = \frac{1}{1-x}$ about $x = 0$.

This is the example with which we started our discussion. We set up a table again. We see from the last column that we get back our geometric series (2.29). Namely, we have that $c_n = 1$. Thus,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n.$$

n	$f^{(n)}(x)$	c_n
0	$\frac{1}{1-x}$	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$\frac{3!}{3!} = 1$

Common series expansions are

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.34}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{2.35}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{2.36}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \tag{2.37}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{2.38}$$

$$\ln(1+x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} \tag{2.39}$$

What is still left to be determined is for what values do such power series converge. The first three of the above expansions converge for all reals, but the other three only converge for $|x| < 1$.

We consider the convergence of a general power series expansion, $\sum_{n=0}^{\infty} c_n (x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

Theorem If $\sum_{n=0}^{\infty} c_n (b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n (x-a)^n$

converges absolutely for all x satisfying $|x-a| < |b-a|$.

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may only converge at $x = a$.
2. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.

The number R is called the *radius of convergence* of the power series and $|x-a| < R$ is called the *interval of convergence.* This interval can also be written as $(a-R, a+R)$. Convergence at the endpoints of this interval has to be tested for each power series.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty} |c_n(x-a)^n|$. This is easily done using either the ratio test or the nth root test. We first identify our nonnegative terms $a_n = |c_n||x-a|^n$. Then we apply our tests.

For example, the nth root test gives the convergence condition

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{|c_n|}|x-a| < 1.$$

Thus,

$$|x-a| < \left(\lim_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1} \equiv R.$$

Similarly, we can apply the ratio test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}|x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \equiv R.$$

Example 1 $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Since there is a factorial, we will use the ratio test with $a = 0$..

$$\rho = \lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}|x| = \lim_{n\to\infty} \frac{1}{n+1}|x| = 0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x. We also say that the radius of convergence is infinite.

Example 2 $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

In this example we will use the nth root test with $a = 0$.

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.

Example 3 $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.

In this example, we have an expansion about $x = 2$. Using the nth root test we find that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $(2-\frac{1}{3}, 2+\frac{1}{3}) = (\frac{5}{3}, \frac{7}{3})$.

We also need to test the end points of the interval. First we consider $x = \frac{7}{3}$: The series becomes $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series. This series does not converge. For $x = \frac{5}{3}$, we get the negative of the alternating harmonic series,$\sum_{n=1}^{\infty} \frac{3^n(-\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which does converge. So, we have convergence on $[\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the endpoint at $x = \frac{5}{3}$.

Example 4 Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We will first write $f(x)$ as a function of $x-1$. This is done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. However, this is not in terms of a known function. Namely, we would like to use the expansion for $g(z) = \frac{1}{1+z} = 1 - z + z^2 + \ldots$. Factoring a 3 from the denominator, we have

$$f(x) = \frac{1}{x+2} = \frac{1}{(x-1)+3} = \frac{1}{3}\frac{1}{[1+\frac{1}{3}(x-1)]}.$$

Now we note that $f(x) = \frac{1}{3}g(\frac{1}{3}(x-1))$. So,

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + (\frac{1}{3}(x-1))^2 - (\frac{1}{3}(x-1))^3 + \ldots\right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \ldots.$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $|-\frac{1}{3}(x-1)| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for endpoints, $x = -2$ and $x = 4$. You should verify this!

As a final application, we can derive Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots.$$

Next we note that each term has a power of i. The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \ldots\}$. See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n: $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$. We also have $e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$. Equating these two expressions, we are lead to DeMoivre's Formula

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n .$$

This formula is useful for deriving needed identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$, we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have $e^{-i\theta} = \cos\theta - i\sin\theta$. Combining this with Euler's Formula, we have that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.