Introduction

"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." David Hilbert (1862-1943)

This is an introduction to topics in Fourier analysis and complex analysis. These notes have been class tested several times since 2005.

This book is based on a course in applied mathematics originally taught at the University of North Carolina Wilmington in 2004 and set to book form in 2005. The notes were used and modified in several times since 2005. The course is an introduction to topics in Fourier analysis and complex analysis. Students are introduced to Fourier series, Fourier transforms, and a basic complex analysis. As motivation for these topics, we aim for an elementary understanding of how analog and digital signals are related through the spectral analysis of time series. There are many applications using spectral analysis. These course is aimed at students majoring in mathematics and science who are at least at their junior level of mathematical maturity.

At the root of these studies is the belief that continuous waveforms are composed of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean le Rond d'Alembert (1717-1783).

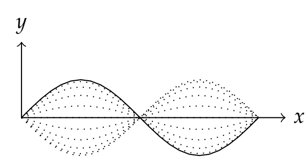

Figure 1: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led himself and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x, 0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

Solutions of the wave equation, such as the one shown, are solved using the Method of Separation of Variables. Such solutions are studies in courses in partial differential equations and mathematical physics.

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extend over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

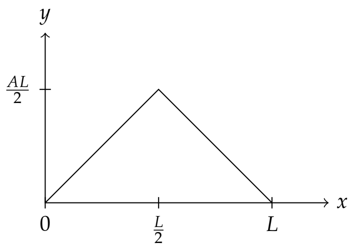

Figure 2: Plot of an initial condition for a plucked string.

However, the initial conditions for such superpositions are

$$y(x,0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2, \\ A(L-x), & L/2 \le x \le L. \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of such Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one-dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

This is another example leading to an infinite series of trigonometric functions.

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems in which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. One example comes from the study of ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun (and many other forces). These periodic forces lead to the tides, which in turn have their own periods of motion. In an analysis of ocean wave heights, one can separate out the tidal components by making use of Fourier analysis. Typically, we views the tide height $y(t)$ as a continuous function. One sits at a specific location and measures the movement of the ocean surface as a function of time. Such a function, or time series, is called an analog function. Another common analog signal is an audio signal, giving the amplitude of a sound (musical note, noise, speech, etc.) as a function of time (or space). However, in both of these cases, we actually observe a part of the signal. This is because we can only sample a finite amount of data over a finite time interval. Thus, we have only the values $y_n = y(t_n)$. However, we are still interested in the spectral (frequency) content of our signals even if the signal is not continuous in time.

For example, for the case of ocean waves we would like to use the discrete signal (the sampled heights) to determine the tidal components. For the case of audio signals, we may want to save a finite amount of discretized information as an audio file to play back later on our computer.

So, how are the analog and discrete signals related? We sample an analog signal, obtaining a discrete version of the signal. By sampling an analog signal, we might wonder how the sampling affects the spectral content of the original analog signal. What mathematics do we need to understand these processes? That is what we will study in this course. We will look at Fourier trigonometric series, integral transforms, and discrete transforms. However, we will actually begin with a review of infinite series. We will recall what infinite series are and when they do, or do not, converge. Then we will be ready to talk about the convergence of series of sinusoidal functions, which occur in Fourier series.

We will see how Fourier series are related to analog signals. A true representation of an analog signal comes from an infinite interval and not a finite interval, such as that the vibrating string lives on. This will lead to Fourier Transforms. In order to work with continuous transforms, we will need a little complex analysis. So, we will spend a few sections on an introduction to complex analysis. This consists of the introduction of complex function, their derivatives, series representations, and integration in the complex plane.

Having represented continuous signals and their spectral content by Fourier transforms, we will then see what needs to be done to represent discrete signals. We end the course by investigating the connection between these two types of signals and some of the consequences of processing analog data through real measurement and/or storage devices.

However, the theory of Fourier analysis is much deeper than just looking at sampling time series. The idea of representing functions as an expansion of oscillatory functions extends far intro both physics and mathematics. In physics, oscillatory and wave motion are crucial in electromagnetism, optics and even quantum mechanics. In mathematics, the concepts of expansion of functions in sinusoidal functions is the basis of expanding functions over an infinite dimensional basis. These ideas can be expanded beyond the sinusoidal basis, as we will see later in the book. Thus, the background to much of what we are doing involves delving into infinite dimensional vector spaces. Hopefully, the basics presented here will be useful in your future studies.

The topics to be studied in the book are laid out as follows:

1. Sequences and Infinite Series
2. Fourier Trigonometric Series
3. Vector Spaces
4. Generalized Fourier Series
5. Complex Analysis
6. Integral Transforms
7. Analog vs Discrete Signals
8. Signal Analysis

At this point I should note that most of the examples and ideas in this book are not original. These notes are based upon mostly standard examples from assorted calculus texts, like Thomas and Finney[1], advanced calculus texts like Kaplan's *Advanced Calculus*, [2], texts in mathematical physics[3], and other areas[4]. A collection of some of these well known sources are given in the bibliography and on occasion specific references will be given for somewhat hard to find ideas.

[1] G. B. Thomas and R. L. Finney. *Calculus and Analytic Geometry*. Addison-Wesley Press, Cambridge, MA, ninth edition, 1995

[2] K. Kaplan. *Advanced Calculus*. Addison Wesley Publishing Company, fourth edition, 1991

[3] G. Arfken. *Mathematical Methods for Physicists*. Academic Press, second edition, 1970

[4] A. J. Jerri. *Integral and Discrete Transforms with Applications and Error Analysis*. Marcal Dekker, Inc, 1992