2.9 Series of Functions

Our immediate goal is to provide a preparation useful for studying Fourier series, which are series whose terms are functions. So, in this section we begin to discuss series of functions and the convergence of such series. Once more we will need to resort to the convergence of the sequence of partial sums. This means we really need to start with sequences of functions.

A *sequence of functions* is simply a set of functions $f_n(x)$, $n = 1, 2, \ldots$ defined on a common domain D. A frequently used example is the sequence of functions $\{1, x, x^2, \ldots\}$.

An infinite *series of functions* is given by $\sum_{n=1}^{\infty} f_n(x), x \in D$. Using powers of x again, an example would be $\sum_{n=1}^{\infty} x^n, x \in [-1, 1]$. In order to investigate the convergence of this series, we really mean that we should substitute values for x in the series and then determine if the resulting real series of number converges. This means that we would need to consider the Nth partial sums

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Does this sequence of functions converge?

Definition We say that a sequence of functions f_n *converges pointwise* on D to a limit g if

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for each $x \in D$. More formally, we write that

$$\lim_{n\to\infty} f_n = g \text{ (pointwise on } D)$$

if given $x \in D$ and $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N.$$

Example Consider the sequence of functions $f(x) = \frac{1}{1+nx}, \quad |x| < \infty, \quad n = 1, 2, 3, \ldots$. The limits depends on the value of x.

(a) $x = 0$. Here $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1$.

(b) $x \neq 0$. Here $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0$.

Therefore, we can say that $f_n \to g$ pointwise for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2.27}$$

We also note that in general N depends on both x and ϵ.

Example We consider the functions $f_n(x) = x^n$, $x \in [0, 1]$, for $n = 1, 2, \ldots$. We recall that the above definition suggests that for each x we seek an N such that $|f_n(x) - g(x)| < \epsilon, \forall n \geq N$. Here are two examples:

(a) $x = 0$. Here we have $f_n(0) = 0$ for all n. So, given $\epsilon > 0$ we seek an N such that $|f_n(0) - 0| < \epsilon, \forall n \geq N$, or $0 < \epsilon$. But all n work, so we can pick $N = 1$.

(b) $x = \frac{1}{2}$. In this case we have $f_n(\frac{1}{2}) = \frac{1}{2^n}$, for $n = 1, 2, \ldots$. As n gets large, $f_n \to 0$. So, given $\epsilon > 0$, we seek N such that $|\frac{1}{2^n} - 0| < \epsilon, \forall n \geq N$. This means that $\frac{1}{2^n} < \epsilon$, or $n > -\frac{\ln \epsilon}{\ln 2} \geq [-\frac{\ln \epsilon}{\ln 2}] \equiv N$. (Here we use $[x]$ to mean the greatest integer less than or equal to x.) Thus, our choice of N depends on ϵ.

There are other questions that can be asked about sequences of functions. Let the sequence of functions f_n be continuous on D. If the sequence of functions converges pointwise to g on D then we can ask the following.

1. Is g continuous on D?
2. If each f_n is integrable on $[a, b]$, then does
$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b g(x)\, dx?$$
3. If each f_n is differentiable at c, then does $\lim_{n\to\infty} f_n'(c) = g'(c)$?

It turns out that pointwise convergence is not enough to provide an affirmative answer to any of these questions. Though we will not prove it here, what we will need is uniform convergence.

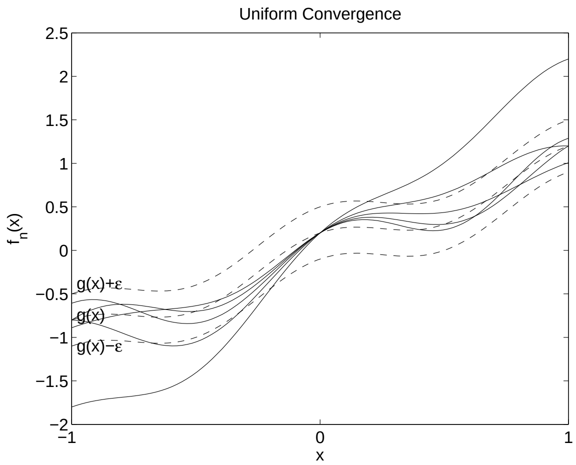

Figure 2.9: For uniform convergence, as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.

Definition Consider a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on D. Let $g(x)$ be defined for $x \in D$. Then the sequence *converges uniformly* on D, or

$$\lim_{n\to\infty} f_n = g \text{ uniformly on } D,$$

if given $\epsilon > 0$, there exists an N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N \text{ and } \forall x \in D.$$

This definition almost looks like the definition for pointwise convergence. However, the seemingly subtle difference lies in the fact that N does not depend upon x. The sought N works for all x in the domain. As seen in Figure 2.9 as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.

Example $f_n(x) = x^n$, for $x \in [-1, 1]$. Note that in this case as n gets large, $f_n(x)$ does not lie in the band $(g(x) - \epsilon, g(x) - \epsilon)$. This is displayed in Figure 2.10.

Example $f_n(x) = \cos(nx)/n^2$ on $[-\pi, \pi]$. For this example we plot the first several members of the sequence in Figure 2.11. We can see that eventually $(n \geq N)$ members of this sequence do lie inside a band of width ϵ about the limit $g(x) = 0$ for all values of x. Thus, this sequence of functions will converge uniformly to the limit.

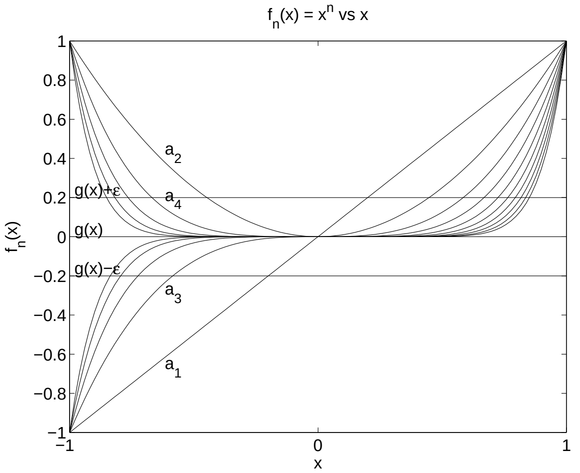

Figure 2.10: Plot of $a_n = x^n$ on $[-1, 1]$ for $n = 1 \ldots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

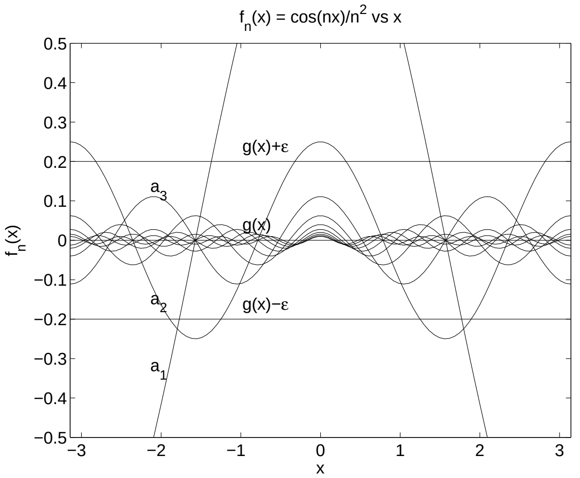

Figure 2.11: Plot of $a_n = \cos(nx)/n^2$ on $[-\pi, \pi]$ for $n = 1 \ldots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

Finally, we should note that if a sequence of functions is uniformly convergent then it converges pointwise. However, the examples should bear out that the converse is not true.

2.10 Infinite Series of Functions

We now turn our attention to infinite series of functions, which will form the basis of our study of Fourier series. Recall that we are interested in the convergence of the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n(x)$ for $x \in D$. But the sequence of partial sums is just a sequence of functions So, it is natural to define the convergence of sequences of functions in terms of pointwise and uniform convergence.

We define the sequence of partial sums

$$s_n(x) = \sum_{j=1}^{n} f_j(x).$$

Then the definitions of pointwise and uniform convergence are as follows:

Definition $\sum f_j(x)$ *converges pointwise* to $f(x)$ on D if given $x \in D$, and $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$.

Definition $\sum f_j(x)$ *converges uniformly* to $f(x)$ on D given $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Again, we state without proof the following:

1. Uniform convergence implies pointwise convergence.

2. If f_n is continuous on D, and $\sum_n^\infty f_n$ converges uniformly to f on D, then f is continuous on D.

3. If f_n is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges uniformly on D, and $\int_a^b f_n(x)\,dx$ exists, then

$$\sum_n^\infty \int_a^b f_n(x)\,dx = \int_a^b \sum_n^\infty f_n(x)\,dx = \int_a^b g(x)\,dx.$$

4. If f_n' is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges pointwise to g on D, and $\sum_n^\infty f_n'$ converges uniformly on D,, then $\sum_n^\infty f_n'(x) = \frac{d}{dx}(\sum_n^\infty f_n(x)) = g'(x)$ for $x \in (a, b)$.

Since uniform convergence of series gives so much, like term by term integration and differentiation, we would like to be able to recognize when we have a uniformly convergent series. One test for such convergence is the Weierstrass M-Test.

Theorem Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions on D. If $|f_n(x)| \leq M_n$, for $x \in D$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly of D.

Proof First, we note that for $x \in D$

$$\sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty M_n.$$

Thus, since by the assumption that $\sum_{n=1}^\infty M_n$ converges, we have that $\sum_{n=1}^\infty f_n$ converges absolutely on D. Therefore, $\sum_{n=1}^\infty f_n$ converges pointwise on D. So, let $\sum_{n=1}^\infty f_n = g$.

We now want to prove that this convergence is in fact uniform. So, given $\epsilon > 0$ we need to find an N such that

$$|g(x) - \sum_{j=1}^n f_j(x)| < \epsilon$$

if $n \geq N$ for all $x \in D$.

So, for any $x \in D$,

$$\begin{aligned} |g(x) - \sum_{j=1}^{n} f_j(x)| &= |\sum_{j=1}^{\infty} f_j(x) - \sum_{j=1}^{n} f_j(x)| \\ &= |\sum_{j=n+1}^{\infty} f_j(x)| \\ &\leq \sum_{j=n+1}^{\infty} |f_j(x)|, \quad \text{by the triangle inequality} \\ &\leq \sum_{j=n+1}^{\infty} M_j. \end{aligned} \tag{2.28}$$

Now, the sum over the M_j's is convergent, so we can choose our N such that

$$\sum_{j=n+1}^{\infty} M_j < \epsilon, \quad n \geq N.$$

Then, we have from above that

$$|g(x) - \sum_{j=1}^{n} f_j(x)| \leq \sum_{j=n+1}^{\infty} M_j < \epsilon$$

for all $n \geq N$ and $x \in D$. Thus, $\sum f_j \to g$ uniformly on D. **QED**

We now give an example of how to use the M-Test.

> **Example** We consider the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ defined on $[-\pi, \pi]$. Each term is bounded by $\left|\frac{\cos nx}{n^2}\right| = \frac{1}{n^2} \equiv M_n$. We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Thus, we can conclude that the original series converges uniformly, as it satisfies the conditions of the Weierstrass M-Test.