Chapter 3

Fourier Series

3.1 Introduction

In this chapter we will look at trigonometric series. Previously, we saw that many functions could have series representations as expansions in powers of $x - a$. This lead to Taylor and MacLaurin series. These series did not necessarily converge, even pointwise, for some values of x to the given function. In this chapter we will replace the powers of $x - a$ with expansions involving sines and cosines. Given a function, when will it have such a representation as a trigonometric series? For what values of x will it converge? First we motivate such expansions.

A natural appearance of such sums over sinusoidal functions is in music. A pure note can be represented as

$$y(t) = A \sin 2\pi f t,$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 3.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility. However, you should be cautious. The plots you get might not be what you expect, even for a simple sine function. In

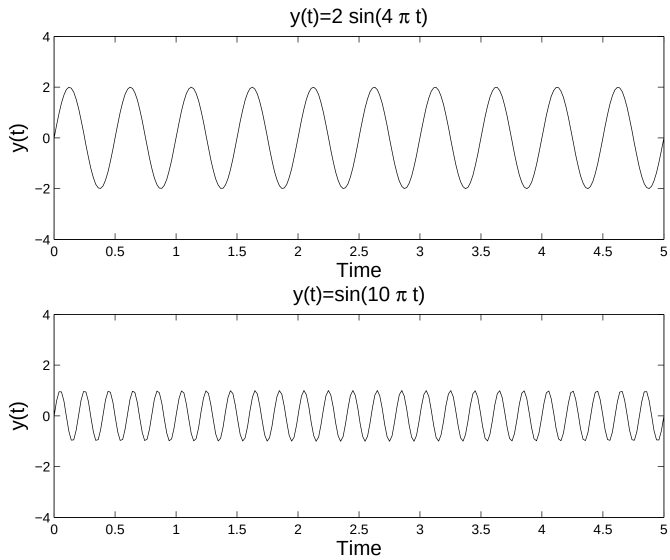

Figure 3.1: Plots of $y(t) = A \sin 2\pi f t$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

Figure 3.2 we show four plots of the function $y(t) = 2 \sin 4\pi t$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Why are there such differences? How can we tell ahead of time if we are using the right number of points?

These are questions that we will answer later in the course. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process for which you might attempt to analyze a continuous signal.

Next, we want to consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 3.3 we see what happens when we add several sinusoids. One notes that as one adds more and more tones with different characteristics, the resulting signal gets

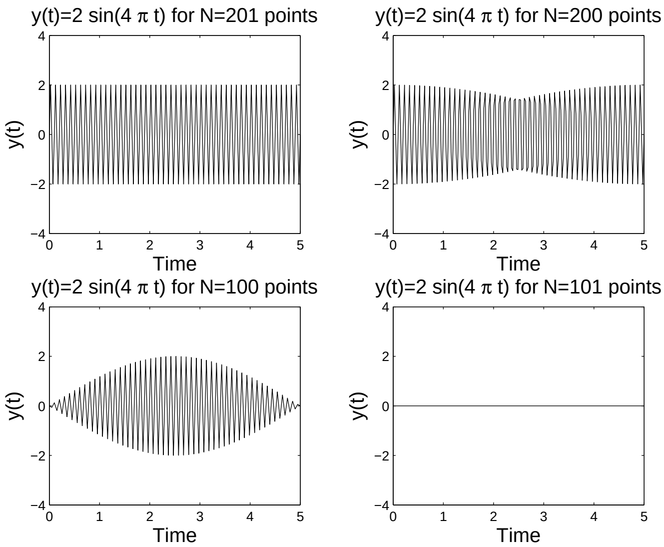

Figure 3.2: Problems can occur while plotting signals. Here we plot the function $y(t) = 2\sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

more complicated. Looking at the superpositions in Figure 3.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely.

Definition A function is said to be *periodic with period* T if $f(x+T) = f(x)$ for all x and the smallest such positive number T is called the *period.*

For example, we consider the functions used in Figure 3.3. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 3.1 we can verify this result. (You might

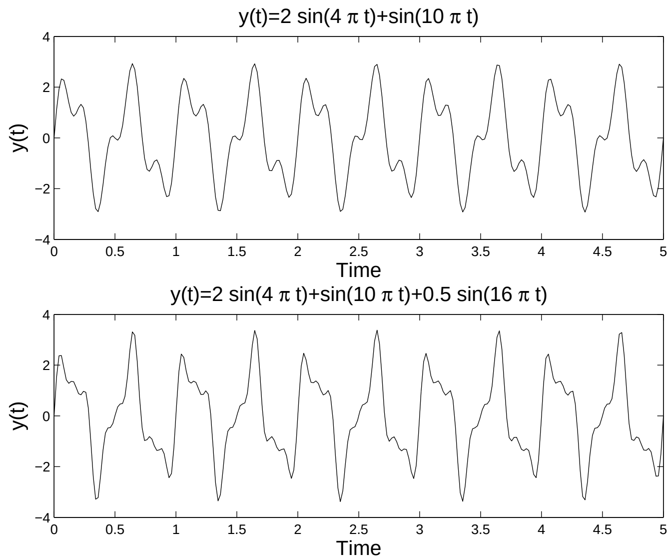

Figure 3.3: Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and and $f = 8$ Hz.

want to count the full number of cycles in the graph and divide this into the total time to get a more accurate value.)

In general, if $y(t) = A \sin 2\pi f t$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to the superpositions in Figure 3.3, $y(t) = \sin 10\pi t$ has a period of 0.2 Hz and $y(t) = \sin 16\pi t$ has a period of 0.125 Hz. The two superpositions retain the largest period of the three signals added, which is 0.5 Hz.

Our goal will be to start with a function and then determine both the amplitudes and the frequencies of the simple sinusoids needed to sum to that function. First of all, we will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of trigonometric functions. Secondly, we will find that using just sine functions may not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 3.4 we show the functions $y(t) = 2\sin 4\pi t$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be can written as $y(t) = 2\sin 4\pi(t + 7/32)$. Thus, this corresponds to a time shift of $-7/8$.

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine for all frequencies the unknown time shifts as well as the amplitudes of the sinusoidal functions that make up our signal, $y(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach in which each y can be written in terms of both sines and cosines.

Consider the general shifted function

$$y(t) = A\sin(2\pi ft + \phi).$$

Note that $2\pi ft + \phi$ is called the *phase* of our sine function and ϕ is called the *phase shift.* We can use the trigonometric identity for the sine of the sum of two angles to obtain

$$y(t) = A\sin(2\pi ft + \phi) = A\sin\phi\cos 2\pi ft + A\cos\phi\sin 2\pi ft.$$

Defining $a = A\sin\phi$ and $b = A\cos\phi$, we can rewrite this as

$$y(t) = a\cos 2\pi ft + b\sin 2\pi ft.$$

Thus, we see that our signal is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2} \quad \tan\phi = \frac{b}{a}.$$

We are now in a position to state our goals.

Goal: Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function.

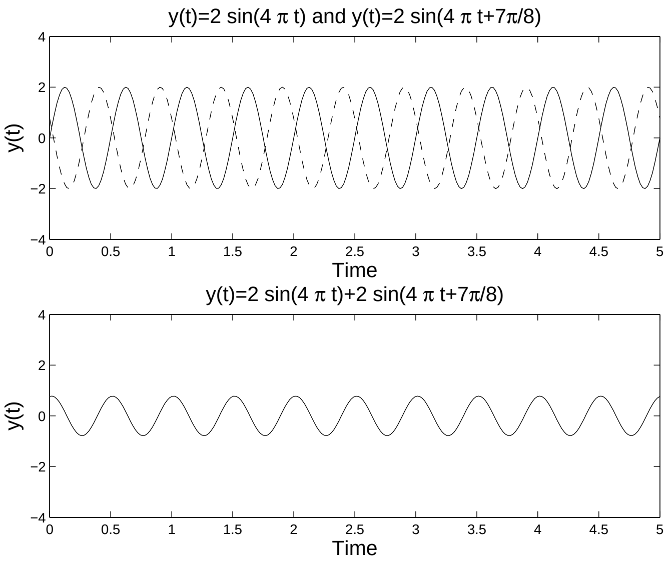

Figure 3.4: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

There are still a few problems in practice. First of all, there are a continuum of frequencies to add over. This typically means our sum is an integral! In practice, our measuring devices do not sample frequently enough to get at the high frequency content. So, we have to settle for a discrete set of frequencies. The resulting sum would be an infinite series, the subject of this chapter. If we only have frequencies $f_n = nf_s$, $n = 1, 2, ...$ for f_s the sampling rate of our device, then our phases in the trigonometric series are

$$2\pi ft = 2\pi nf_s t.$$

Defining $x = 2\pi f_s t$, we then are lead to sums over the functions

$$y_n(t) = a_n \cos nx + b_n \sin nx.$$

Actually, we will also need a zero frequency term, corresponding to $n = 0$, to account for any vertical shift in our function.

This is enough to get started. However, our discrete sampling will also mean that we do not gather data for a continuous range of times or for all times. This will be discussed further in a later section of the book. When considering that both t and f are continuous, we are led to Fourier transforms as the appropriate way to represent analog signals. When the time and frequency variables are both discrete, we will be led to discrete transforms. However, for now we will only consider the case that t is continuous and the frequencies are discrete.

3.2 Trigonometric Series

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{3.1}$$

Notice that we have opted to drop reference to the frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications involving time series later in the course.

The series representation in Equation (3.1) is called a *Fourier series.* The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients.* The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0 instead of $a_0/2$. Our goal is to find the Fourier series representation for a given $f(x)$. Later we will be interested in knowing what functions have such a representation, when the resulting Fourier series converges, and to what function the Fourier series converges.

From our discussion in the last section, we see that the infinite series is periodic. The largest period of the terms comes from the $n = 1$ terms. The periods of $\cos x$ and $\sin x$ are $T = 2\pi$. Thus, the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 3.5 we show a function defined on $[0, 2\pi]$. In the same figure, we should see its periodic extension and copies of the original function shifted by the period and glued together. Such an extension can be represented by a Fourier series. Restricting the Fourier

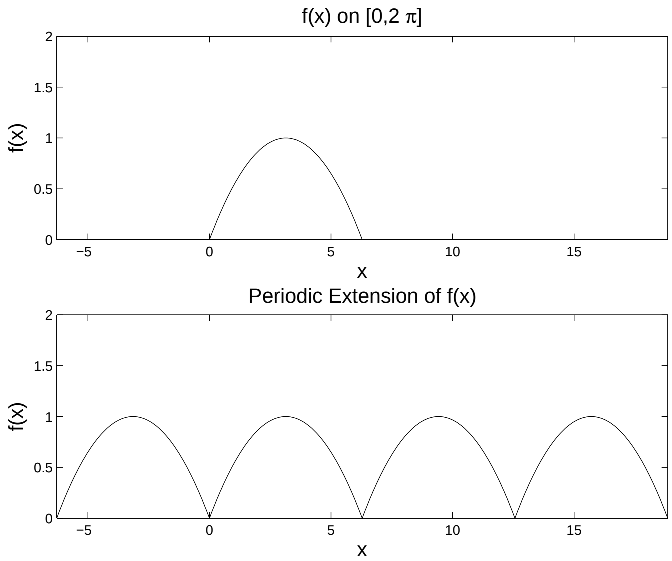

Figure 3.5: Plot of the function $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

series to $[0, 2\pi]$ leads to a representation of the original function. Therefore, we will consider Fourier series representations of functions defined on this interval. Note that we could just as easily consider a function defined on $[-\pi, \pi]$ or any interval of length 2π.

Theorem The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$ when it exists, is given by (3.1) with Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{3.2}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will first look at the derivations of the a_n's.

We begin by integrating the Fourier series in Equation (3.1).

$$\int_0^{2\pi} f(x)\, dx = \int_0^{2\pi} \frac{a_0}{2}\, dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\, dx. \tag{3.3}$$

We assume that we can integrate the infinite sum term by term. In order to do this we will need to compute the integrals

$$\begin{aligned}\int_0^{2\pi} \frac{a_0}{2}\, dx = \frac{a_0}{2}(2\pi) = \pi a_0,\\ \int_0^{2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0,\\ \int_0^{2\pi} \sin nx\, dx = \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0.\end{aligned} \tag{3.4}$$

From these results we see that only one term in the integrated sum does not vanish, leaving

$$\int_0^{2\pi} f(x)\, dx = \pi a_0.$$

This confirms the value for a_0.

Next, we need to find a_n for $n \neq 0$. We will multiply the Fourier series (3.1) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned}\int_0^{2\pi} f(x)\cos mx\, dx &= \int_0^{2\pi} \frac{a_0}{2}\cos mx\, dx\\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \cos mx\, dx.\end{aligned} \tag{3.5}$$

Integrating term by term, the right side becomes

$$\frac{a_0}{2}\int_0^{2\pi} \cos mx\, dx \quad + \quad \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos mx\, dx + b_n \int_0^{2\pi} \sin nx \cos mx\, dx\right]. \tag{3.6}$$

We have already established that $\int_0^{2\pi} \cos mx\, dx = 0$.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals.

We first want to evaluate $\int_0^{2\pi} \cos nx \cos mx\, dx$. We do this by using the product identity. Recall the addition formulae for cosines:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Adding these equations gives

$$2\cos A \cos B = \cos(A+B) + \cos(A-B).$$

We can use this identity with $A = mx$ and $B = nx$ to compute our integral. We have

$$\begin{aligned} \int_0^{2\pi} \cos nx \cos mx\, dx &= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{3.7}$$

There is a caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of our integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to carefully compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

So, for $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx\, dx$. This can also be handled using a trigonometric identity. We had found in the last chapter that

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta.)$$

Inserting this into our integral, we have

$$\begin{aligned} \int_0^{2\pi} \cos^2 mx\, dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos^2 2mx)\, dx \\ &= \frac{1}{2}[x + \frac{1}{2m}\sin 2mx]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{3.8}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{3.9}$$

This holds true for $m, n = 0, 1, \ldots$. When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition A set of (real) functions $\{\phi_n(x)\}$ is said to be *orthogonal* on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)\, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)\, dx = 1$, these functions are called *orthonormal.*

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\int_0^{2\pi} \cos^2 nx\, dx = \sqrt{\pi}$.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)g(x)\, dx$ generalizes the dot product, and is called the scalar (or, inner) product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. We can *normalize* such vectors by dividing by the length, $\int_a^b f^2(x)\, dx$:

$$\hat{f}(x) = \frac{\int_a^b f(x)g(x)\, dx}{\int_a^b f^2(x)\, dx}.$$

Note that this is not the most general definition of the inner product. However, this digression into infinite dimensional vector spaces is a topic for a later section.

Returning to the evaluation of the integrals in equation (3.6), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. This is also be evaluated using trigonometric identities. In this case, we need an identity involving products of sines and cosines. Such products occur in the addition formulae for sine functions:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A.$$

Adding these equations, we find that

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

Setting $A=nx$ and $B=mx$, we find that

$$\begin{aligned}\int_0^{2\pi}\sin nx\cos mx\,dx &= \frac{1}{2}\int_0^{2\pi}[\sin(n+m)x+\sin(n-m)x]\,dx\\ &= \frac{1}{2}\left[\frac{-\cos(n+m)x}{n+m}+\frac{-\cos(n-m)x}{n-m}\right]_0^{2\pi}\\ &= (-1+1)+(-1+1)=0. \qquad (3.10)\end{aligned}$$

For these integrals we also should be careful about setting $n=m$. In this special case, we have the integrals

$$\int_0^{2\pi}\sin mx\cos mx\,dx=\frac{1}{2}\int_0^{2\pi}\sin 2mx\,dx=\frac{1}{2}[\frac{-\cos 2mx}{2m}]_0^{2\pi}=0.$$

Finally, we can finish our evaluation of (3.6). We have determined that all but one integral vanishes. In that case, $n=m$. This leaves us with

$$\int_0^{2\pi}f(x)\cos mx\,dx=a_m\pi.$$

Solving for a_m gives

$$a_m=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos mx\,dx.$$

Since this is true for all $m=1,2,\ldots,$ we have proven this part of the theorem. The only part left is finding the b_n's. This will be left as an exercise for the reader.

We now consider some examples of finding Fourier coefficients. In all of these cases we consider functions $f(x)$ defined on $[0,2\pi]$. We will look at Fourier expansions of functions defined on other intervals in the next two sections.

Example 1. $f(x)=3\cos 2x$.

We first compute the integrals for the Fourier coefficients.

$$a_n=\frac{1}{\pi}\int_0^{2\pi}3\cos 2x\cos nx\,dx=0,\quad n\neq 2.$$

$$a_2 = \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\,dx = 3.$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \sin nx\,dx = 0, \forall n.$$

Therefore, we have the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$. Well, we should have known this before doing all of these integrals. If we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 2. $f(x) = \sin^2 x$.

We could integrate as in the last example, but it is easier to use identities. We know that

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, so $a_0 = 0$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$.

Example 3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \end{aligned} \tag{3.11}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\, dx - \int_{\pi}^{2\pi} \cos nx\, dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\
&= 0. \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\, dx - \int_{\pi}^{2\pi} \sin nx\, dx\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\
&= \frac{2}{n\pi}(1-\cos n\pi). \qquad (3.13)
\end{aligned}$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (3.1), we note that $\cos n\pi = (-1)^n$. Therefore,

$$1-\cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \qquad (3.14)$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,\ \text{odd}}^{\infty} \frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ and write

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin(2k-1)x,$$

But does this series converge? Does it converge to $f(x)$? We will answer this question in a couple of sections.

3.3 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L. Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to our Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

Define $x \in [0, 2\pi]$ and $t \in [0, L]$. A transformation relating these intervals is simply $x = \frac{2\pi t}{L}$. Given $f(x)$, this transformation would result in a new function $g(t) = f(x(t))$. This function is defined on $[0, L]$. We would like to determine the Fourier series representation of this function. We begin with the representation for $f(x)$.

Recall the form of the Fourier representation for $f(x)$ in Equation (3.1):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{3.15}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L}\right]. \tag{3.16}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for our coefficient is

$$a_n = \frac{2}{L} \int_0^{L} g(t) \cos \frac{2n\pi t}{L} \, dt. \tag{3.17}$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^{L} g(t) \sin \frac{2n\pi t}{L} \, dt. \tag{3.18}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L.

In Table 3.1 we collect some of the more common Fourier series expansions and their corresponding Fourier coefficients. In the next section we present the derivation of the Fourier series representation for a general interval for the interested reader. We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

Example: Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} |x| \, dx = \pi \end{aligned} \tag{3.25}$$

In this evaluation we made use of the fact that the integrand is an even function. Recall the definition of an even function:

Definition $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis.

If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x) \, dx = 2 \int_0^{a} f(x) \, dx. \tag{3.26}$$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{3.19}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.20}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{3.21}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.22}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{3.23}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.24}$$

Table 3.1: Fourier series on some standard intervals

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(-y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx. \qquad (3.27)\end{aligned}$$

A similar computation could be done for odd functions.

Definition $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin.

If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 0. \qquad (3.28)$$

We now continue with our computation of the Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \qquad (3.29)$$

Here we have made use of the fact that $|x|\cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts,

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du,$$

by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$. Continuing with the computation, we have

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^\pi \\
&= -\frac{2}{\pi n^2}(1-(-1)^n). \qquad (3.30)
\end{aligned}
$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor of $(1-(-1)^n)$. This factor can be simplified as

$$
1-(-1)^n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad (3.31)
$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x|\sin nx$ from $x=-\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x)=|x|$ on $[-\pi,\pi]$ is given as

$$
f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\ \text{odd}}^{\infty} \frac{\cos nx}{n^2}. \qquad (3.32)
$$

While this is correct, there is a way to write the sum over only odd n by reindexing. We let $n = 2k-1$ for $k = 1,2,3,\ldots$. Then we only get the odd integers. The series can then be written as

$$
f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \qquad (3.33)
$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. What does this series sum to? We show in Figure 3.6 the first few partial sums. They appear to be converging to $f(x)=|x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi,\pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 3.7, we plot the series on a larger domain. We see that the representation agrees with $f(x)$ on the interval $[-\pi,\pi]$. Outside this interval we see that we have a periodic extension of $f(x)$ with period 2π.

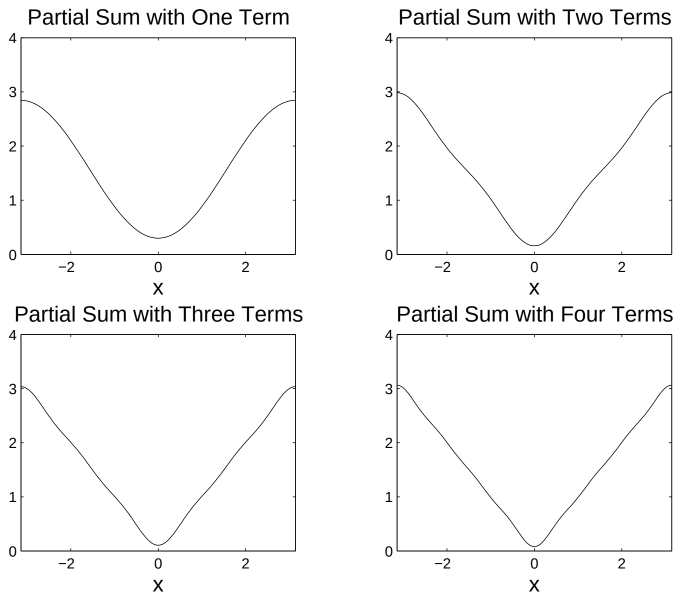

Figure 3.6: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

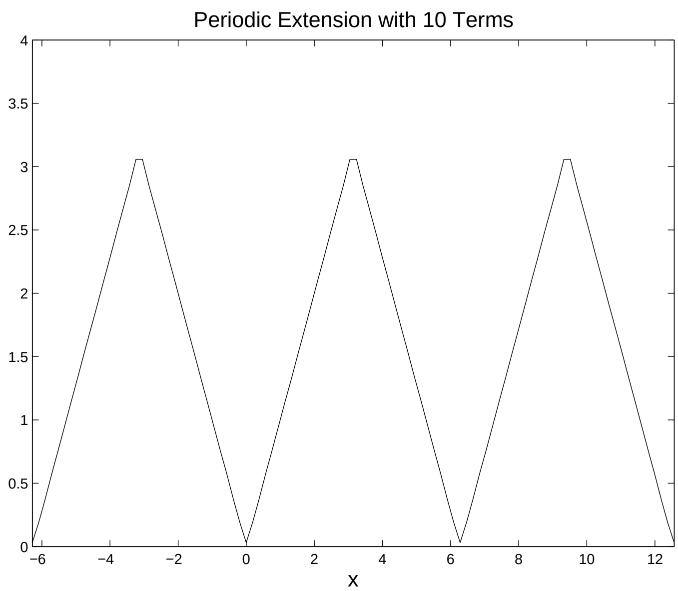

Figure 3.7: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

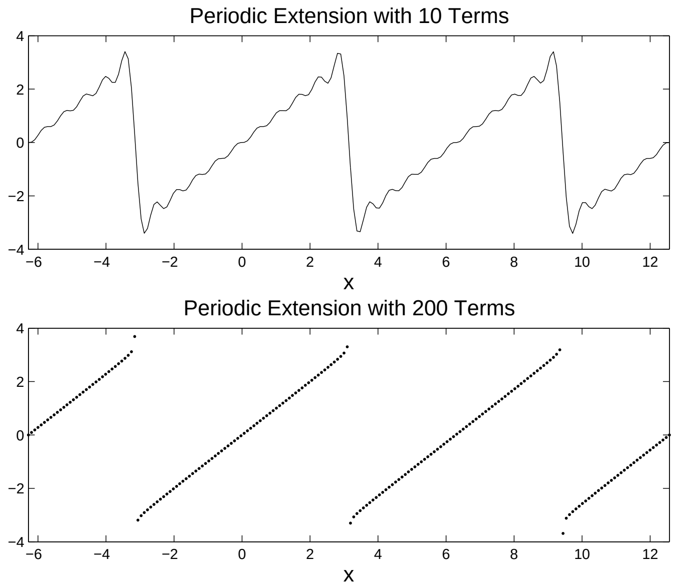

Figure 3.8: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Example: Another example exhibiting the convergence of a trigonometric series is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$. The reader can show that

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{3.34}$$

As seen in Figure 3.8 we again obtain a periodic extension of the given function. In this case we needed many more terms. Also, the vertical parts of the first plot are actually nonexistent. In the second graph we only plot the points and not the typical connected points that most software packages plot as the default style. We see that the extension is discontinuous. This is due to the fact that $f(-\pi) \neq f(\pi)$.

3.4 Fourier Series on $[a, b]$ *Optional*

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. In this section we prove the following general result:

Theorem The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \tag{3.35}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.36}$$

The special cases listed in the last section easily follow from this theorem.

As before, we prove the theorem by transforming the interval $[a,b]$ to $[0,2\pi]$. Letting

$$x = 2\pi \frac{t-a}{b-a},$$

we map $t \in [a,b]$ into $tx \in [0,2\pi]$. Inserting this into the Fourier series (3.1) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right]. \tag{3.37}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos\frac{2n\pi t}{b-a} \left(a_n \cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + \sin\frac{2n\pi t}{b-a} \left(a_n \sin\frac{2n\pi a}{b-a} + b_n \cos\frac{2n\pi a}{b-a} \right) \right]. \qquad (3.38)
\end{aligned}
$$

Defining $A_0 = a_0$ and

$$
\begin{aligned}
A_n &\equiv a_n \cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a} \\
B_n &\equiv a_n \sin\frac{2n\pi a}{b-a} + b_n \cos\frac{2n\pi a}{b-a}, \qquad (3.39)
\end{aligned}
$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$
g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\frac{2n\pi t}{b-a} + B_n \sin\frac{2n\pi t}{b-a} \right]. \qquad (3.40)
$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi\frac{t-a}{b-a}$ we have

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \\
&= \frac{2}{b-a}\int_a^b g(t)\cos\frac{2n\pi(t-a)}{b-a}\, dt, \qquad (3.41)
\end{aligned}
$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \\
&= \frac{2}{b-a}\int_a^b g(t)\sin\frac{2n\pi(t-a)}{b-a}\, dt. \qquad (3.42)
\end{aligned}
$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$
\begin{aligned}
A_n &\equiv a_n \cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a} \\
&= \frac{2}{b-a}\int_a^b g(t)\left[\cos\frac{2n\pi(t-a)}{b-a}\cos\frac{2n\pi a}{b-a} - \sin\frac{2n\pi(t-a)}{b-a}\sin\frac{2n\pi a}{b-a} \right] dt \\
&= \frac{2}{b-a}\int_a^b g(t)\cos\frac{2n\pi t}{b-a}\, dt. \qquad (3.43)
\end{aligned}
$$

A similar computation gives

$$B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a}\, dt. \qquad (3.44)$$

This concludes the proof of the theorem.

3.5 Sine and Cosine Series

In the last two examples we have seen Fourier series representations that contain only sine or only cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called *Fourier sine series* and those involving just cosines (and the constant term) are called *Fourier cosine series.*

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 3.7 and 3.7 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

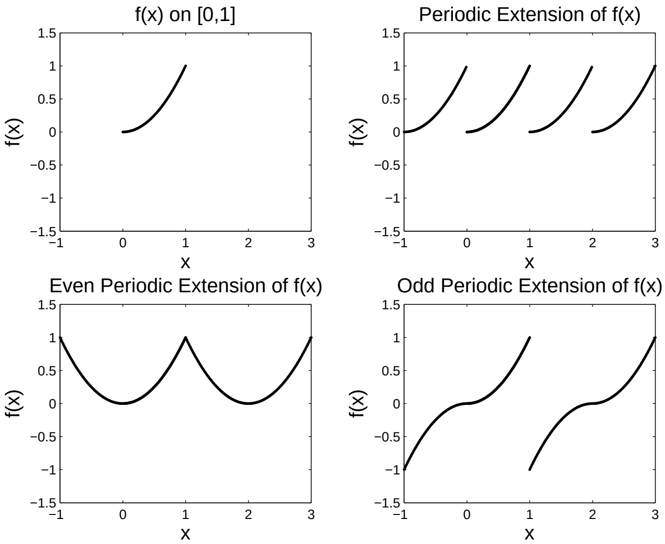

Figure 3.9: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0, 1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In Figure 3.9 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the *periodic, even and odd extensions*. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. We have already seen that the periodic extension of $f(x)$ is obtained through the Fourier series representation in Equation (3.19).

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{3.45}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (3.35) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{3.46}$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{3.47}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{3.48}$$

where

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{3.49}$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{3.50}$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.51}$$

where

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{3.52}$$

1. **Example** In Figure 3.9 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

   For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is as follows:

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);

                              a0 := 2/3

> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);

                                    1
                           an := -------
                                   2   2
                                 n~   Pi

> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);

                                     1
                           bn := - -----
                                   n~ Pi

> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=`Periodic Extension`,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

   (a) **Periodic Extension - Trigonometric Fourier Series** Using the above code, we have that $a_0 = \frac{2}{3}$ $a_n = \frac{1}{n^2\pi^2}$ and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right].$$

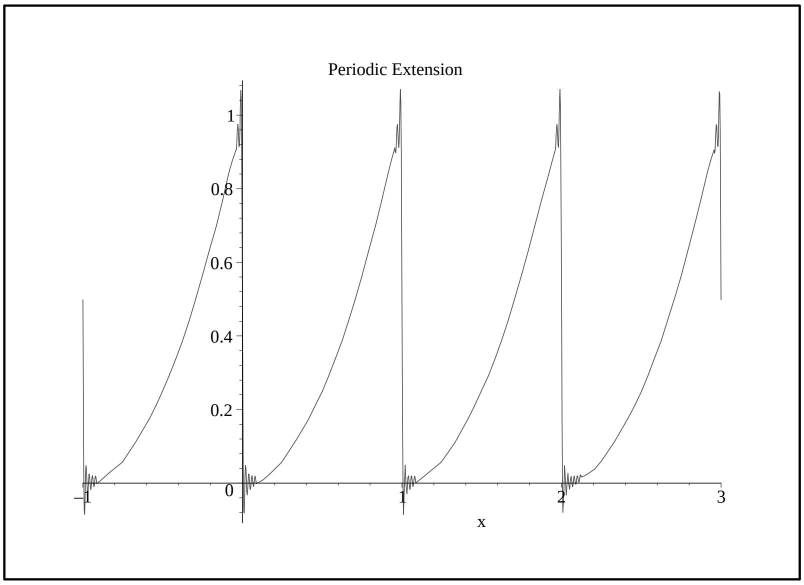

Figure 3.10: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

In Figure 3.10 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension.

(b) **Even Periodic Extension - Cosine Series**

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

In Figure 3.11 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

(c) **Odd Periodic Extension - Sine Series**

Finally, we look at the sine series for this function. We find that

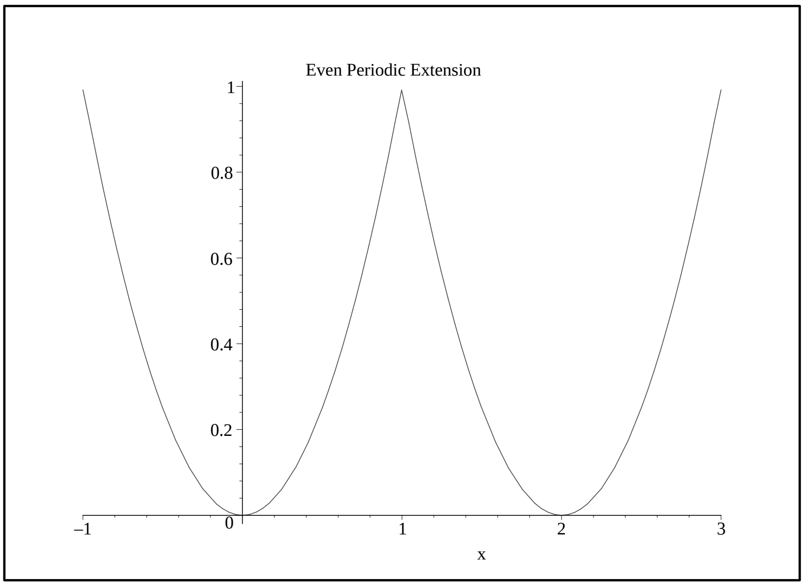

Figure 3.11: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

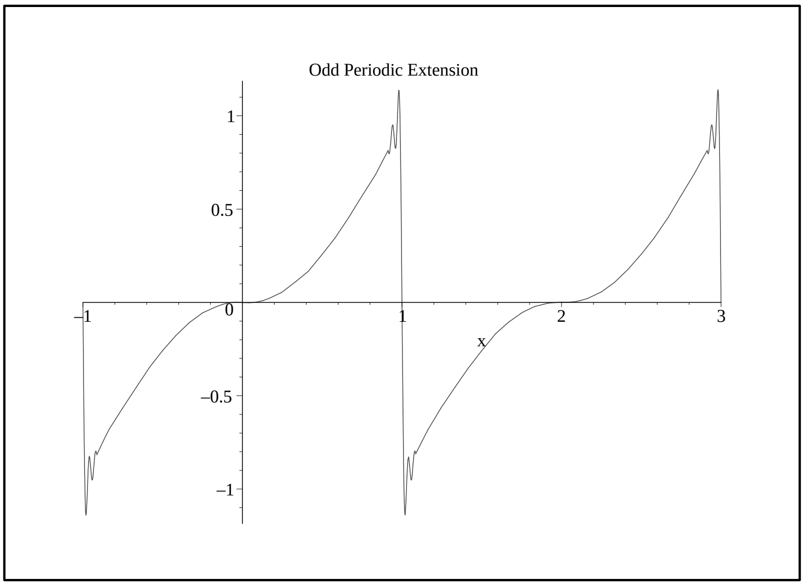

Figure 3.12: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.

$b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. Therefore,

$$f(x) \sim -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)\sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 3.12. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 3.9.

For quick reference, we summarize the Fourier sine and cosine series in 3.5

3.6 Convergence of Fourier Series - *Optional*

The material up to this point provides sufficient background for working with Fourier series. One can use a computer algebra system to visually look at the convergence of the partial sums. However, this is not rigorous and does not shed light on some important questions, such as what functions can one represent as Fourier series expansions and to what do such series converge. In this section we will look into these problems. However, we first need to provide the backdrop for that theory.

We will see that the Fourier series representation of a function can be viewed as the expansion of the given function over a set of basis functions in an infinite dimensional vector space. With this in mind, we will first recall finite dimensional vector spaces and then see how concepts in finite dimensions can be extended to infinite dimensional function spaces.

3.6.1 Vector Spaces

Much of the discussion and terminology that we will use comes from the theory of vector spaces. Up until now you may only have dealt with finite dimensional vector spaces. Even then, you might only be comfortable with vectors in two and three dimensions. We will review a little of what we know about finite dimensional vector spaces. In later sections we will introduce the more general function spaces, which may be useful in later application in the text.

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{3.53}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{3.54}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.55}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \tag{3.56}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{3.57}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots. \tag{3.58}$$

Table 3.2: The Fourier sine and cosine series representations of a function $f(x)$ defined on $[0, L]$.

The notion of a vector space is a generalization of three dimensional vector spaces. In three dimensions, we have things called vectors, which you first visualized as arrows of a specific length and pointing in a given direction. To each vector, we can associate a point in a three dimensional Cartesian system. We just attach the tail of the vector $\mathbf{v}$ to the origin and the head lands at some point, (x, y, z). We then used the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the coordinate axes to write the vector in the form

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Having defined vectors, we then learned how to add vectors and multiply vectors by numbers, or scalars. Under these operations, we expected to get back new vectors. Then we learned that there were two types of multiplication of vectors. We could multiply two vectors to get either a scalar or a vector. This lead to the operations of dot and cross products, respectively. The dot product was useful for determining the length of a vector, the angle between two vectors, or if the vectors were orthogonal.

In physics you first learned about vector products when you defined work. Cross products were useful in describing things like torque, or the force on a moving charge in a magnetic field. We will return to these more complicated vector operations later when reviewing Maxwell's equations of electrodynamics.

The basic concept of a vector can be generalized to spaces of more than three dimensions. You may first have seen this in your linear algebra class. The properties outlined roughly above need to be preserved. So, we have to start with a space of vectors and the operations between them. We also need a set of scalars, which generally come from some *field*. However, in our applications the field will either be the set of real numbers or the set of complex numbers.

A *vector space* V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions: For any $u, v, w \in V$ and $a, b \in F$

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. There exists a 0 such that $0 + v = v$.

4. There exists a $-v$ such that $v + (-v) = 0$.
5. $a(bv) = (ab)v$.
6. $(a + b)v = av + bv$.
7. $a(u + v) = au + bv$.
8. $1(v) = v$.

In three dimensions the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ play an important role. Any vector in the space can be written as a linear combination of these vectors,

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

In fact, given any three non-coplanar vectors, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, all vectors can be written as a linear combination of those vectors,

$$\mathbf{v} = c_1\mathbf{a}_1 + c_2\mathbf{a}_1 + c_3\mathbf{a}_1.$$

Such vectors are said to span the space and are called a basis for the space.

We can generalize these ideas. In an n-dimensional vector space any vector in the space can be represented as the sum over n linearly independent vectors (the equivalent of non-coplanar vectors). Such a linearly independent set of vectors $\{\mathbf{v}_j\}_{j=1}^n$ satisfies the condition

$$\sum_{j=1}^n c_j\mathbf{v}_j = \mathbf{0} \quad \Leftrightarrow \quad c_j = 0.$$

Note that we will often use summation notation instead of writing out all of the terms in the sum.

The standard basis in an n-dimensional vector space is a generalization of the standard basis in three dimensions ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$). We define

$$\mathbf{e}_k = (0, \ldots, 0, \underbrace{1}_{k\text{th space}}, 0, \ldots, 0), \quad k = 1, \ldots, n. \tag{3.59}$$

Then, we can expand any $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{k=1}^n v_k\mathbf{e}_k, \tag{3.60}$$

where the v_k's are called the components of the vector in this basis. Sometimes we will write $\mathbf{v}$ as an n-tuple $(v_1, v_2, \ldots, v_n)$. This is similar to the ambiguous use of (x, y, z) to denote a vector in space as well as to represent points in space.

The only other thing we will need at this point is to generalize the dot product. Recall that there are two forms for the dot product in three dimensions. First, one has that

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta, \tag{3.61}$$

where u and v denote the length of the vectors. The other form is the component form:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k. \tag{3.62}$$

Of course, this form is easier to generalize. So, we define the *scalar product* between two n-dimensional vectors as

$$< \mathbf{u}, \mathbf{v} >= \sum_{k=1}^{n} u_k v_k. \tag{3.63}$$

Actually, there are a number of notations that are used in other texts. One can write the scalar product as $(\mathbf{u}, \mathbf{v})$ or even in the Dirac notation $< \mathbf{u} | \mathbf{v} >$.

While it does not always make sense to talk about angles between general vectors in higher dimensional vector spaces, there is one concept that is useful. It is that of orthogonality, which in three dimensions is another way of saying vectors are perpendicular to each other. So, we also say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if and only if $< \mathbf{u}, \mathbf{v} >= 0$. If $\{\mathbf{a}_k\}_{k=1}^n$, is a set of basis vectors such that

$$< \mathbf{a}_j, \mathbf{a}_k >= 0, \quad k \neq j,$$

then it is called an *orthogonal basis*. If in addition each basis vector is a unit vector, then one has an *orthonormal basis*.

Let $\{\mathbf{a}_k\}_{k=1}^n$, be a set of basis vectors for vector space V. We know that any vector $\mathbf{v}$ can be represented in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. If we know the basis and vector, can we find the components? The answer is yes. We can use the scalar product of $\mathbf{v}$ with each basis element $\mathbf{a}_j$. So, we have for $j = 1, \ldots, n$

$$\begin{aligned} < \mathbf{a}_j, \mathbf{v} > &= < \mathbf{a}_j, \sum_{k=1}^{n} v_k \mathbf{a}_k > \\ &= \sum_{k=1}^{n} v_k < \mathbf{a}_j, \mathbf{a}_k > . \end{aligned} \tag{3.64}$$

Since we know the basis elements, we can easily compute the numbers

$$A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$$

and

$$b_j \equiv < \mathbf{a}_j, \mathbf{v} > .$$

Therefore, the system (3.64) for the v_k's is a linear algebraic system, which takes the form

$$b_j = \sum_{k=1}^{n} A_{jk} v_k. \tag{3.65}$$

We can write this set of equations in a more compact form. The set of numbers A_{jk}, $j, k = 1, \ldots n$ are the elements of an $n \times n$ matrix A with A_{jk} being an element in the jth row and kth column. Also, v_j and b_j can be written as column vectors, $\mathbf{v}$ and $\mathbf{b}$, respectively. Thus, system (3.64) can be written in matrix form as

$$A\mathbf{v} = \mathbf{b}.$$

However, if the basis is orthogonal, then the matrix $A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$ is diagonal and the system is easily solvable. Recall that two vectors are orthogonal if and only if

$$< \mathbf{a}_i, \mathbf{a}_j > = 0, \quad i \neq j. \tag{3.66}$$

Thus, in this case we have that

$$< \mathbf{a}_j, \mathbf{v} > = v_j < \mathbf{a}_j, \mathbf{a}_j >, \quad j = 1, \ldots, n. \tag{3.67}$$

or

$$v_j = \frac{< \mathbf{a}_j, \mathbf{v} >}{< \mathbf{a}_j, \mathbf{a}_j >}. \tag{3.68}$$

In fact, if the basis is orthonormal, i.e., the basis consists of an orthogonal set of unit vectors, then A is the identity matrix and the solution takes on a simpler form:

$$v_j = < \mathbf{a}_j, \mathbf{v} > . \tag{3.69}$$

3.6.2 Inner Product Spaces

We spent some time looking at this simple case of extracting the components of a vector in a finite dimensional space. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are the key ingredients that we will need in the infinite dimensional case. Recall that when we solved the heat equation, we have a function (vector?) that we wanted to expand in a set of eigenfunctions (basis?) and we need to find the expansion coefficients (components?). So, we need to extend our notions from finite dimensional spaces to infinite dimensional spaces and we will have the needed background linear algebra in which to think about all that is to follow. Also, this conceptual framework is very important in other areas, such a quantum mechanics, because this is the basis of solving the eigenvalue problems that come up there so often with the Schrödinger equation.

We will consider the space of functions of a certain type. They could be the space of continuous functions on [0,1], or the space of differentiably continuous functions, or the set of functions integrable from a to b. Later, we will specify the types of functions. However, you can see that there are many types of function spaces. We will further need to be able to add function and multiply them by scalars. So, we can easily obtain a vector space of functions.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. For a real vector space, we define

An inner product $<,>$ on a real vector space V is a mapping from $V \times V$ into R such that for $u, v, w \in V$ and $\alpha \in R$ one has

1. $< u+v, w >=< u, w > + < v, w >$.
2. $< \alpha v, w >= \alpha < v, w >$.
3. $< v, w >=< w, v >$.
4. $< v, v >\geq 0$ and $< v, v >= 0$ iff $v = 0$.

A real vector space equipped with the above inner product leads to a real inner product space. A more general definition with the third item replaced with $< v, w >= < \bar{w, v} >$ is needed for complex inner product spaces.

For the time being, we are dealing just with real valued functions. We need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$. Then, we define the inner product, if the integral exists, as

$$< f, g >= \int_a^b f(x)g(x)\, dx. \tag{3.70}$$

So, we have functions spaces equipped with an inner product. Can we find a basis for the space? For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will think about those things later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^{\infty}$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

How do we find the c_n's? Does this remind you of the problem we had earlier?

Formally, we take the inner product of f with each ϕ_j, to find

$$\begin{aligned} < \phi_j, f > &= < \phi_j, \sum_{n=1}^{\infty} c_n \phi_n > \\ &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > . \end{aligned} \tag{3.71}$$

If our basis is an orthogonal basis, then we have

$$< \phi_j, \phi_n >= N_j \delta_{ij}, \tag{3.72}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{3.73}$$

Thus, we have

$$
\begin{aligned}
< \phi_j, f > &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > \\
&= \sum_{n=1}^{\infty} c_n N_j \delta_{ij} \\
&= c_j N_j. \qquad (3.74)
\end{aligned}
$$

So, the expansion coefficient is

$$
c_j = \frac{< \phi_j, f >}{N_j} = \frac{< \phi_j, f >}{< \phi_j, \phi_j >}.
$$

Earlier in the chapter we had seen that the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[= \pi, \pi]$. In the language of this section, we need to show that $< \phi_n, \phi_m >= 0$ for $n \neq m$. Thus, we have for $n \neq m$

$$
\begin{aligned}
< \phi_n, \phi_m > &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} = 0. \qquad (3.75)
\end{aligned}
$$

So, we have determineds that the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[= \pi, \pi]$. Just as with vectors in three dimensions, we can normalize our basis functions to arrive at an orthonormal basis. This is simply done by dividing by the *length* of the vector. Recall that the length of a vector was obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ In the same way, we define the *norm* of our functions by

$$
\|f\| = \sqrt{< f, f >}.
$$

Note, there are many types of norms, but this will be sufficient for us.

For the above basis of sine functions, we want to first compute the norm of each function. Then we would like to find a new basis from this one such that each basis eigenfunction has unit length and is therefore an orthonormal basis. We first compute

$$\begin{aligned} \|\phi_n\|^2 &= \int_{-\pi}^{\pi} \sin^2 nx\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [1-\cos 2nx]\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi. \end{aligned} \tag{3.76}$$

We have found for our example that

$$< \phi_j, \phi_n >= \pi\delta_{ij} \tag{3.77}$$

and that $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}}\phi_n(x)$, we have normalized the ϕ_n's and have obtained an orthonormal basis of functions on $[-\pi, \pi]$.

3.6.3 Classical Orthogonal Polynomials

In this section we will provide another class of functions which is often used as a mutually orthogonal set of basis functions for an infinite dimensional vector space. These are the well know classical orthogonal polynomials, consisting of such functions as Legendre polynomials, which occur in many physical applications.

We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstrass Approximation Theorem this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. We are familiar with being able to expand functions over this basis, since the expansions are just power series representation of the functions,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis

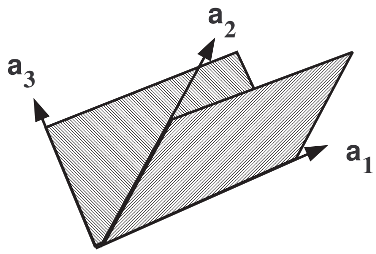

Figure 3.13: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$ considered in the text.

functions with $\sigma(x) = 1$ and (a,b)=$(-1,1)$. For example,

$$\int_{-1}^{1} x^0 x^2\, dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask if it is possible to obtain an orthogonal basis involving these powers of x. Of course, finite combinations of these basis element are just polynomials!

OK, we will ask. "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which most covers finite dimensional vector spaces, that there is a method for carrying this out called the **Gram-Schmidt Orthogonalization Process.** We will recall this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 3.13. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

Of course, we might want to normalize our new basis vectors, so we would denote such a normalized vector with a 'hat':

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

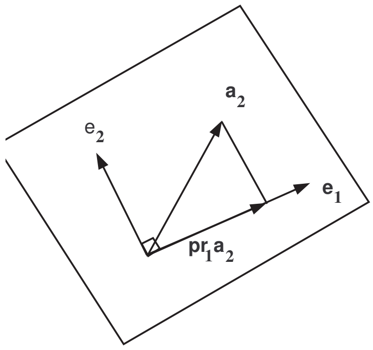

Figure 3.14: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 3.14 we see the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. $\mathbf{a}_2$ can be written as a sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{3.78}$$

We recall the projection of one vector onto another from our vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{3.79}$$

Note that this is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (3.78)-(3.79), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{3.80}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned}\mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{3.81}$$

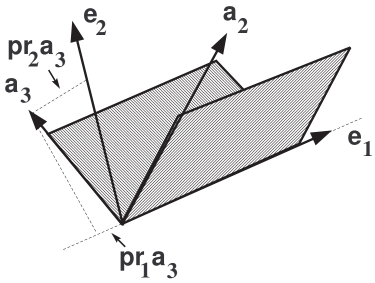

Figure 3.15: A plot of the vectors and their projections for determining $\mathbf{e}_3$.

Now, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the new vector. This is shown in Figure 3.15. Then we have,

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{3.82}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

We can easily generalize the procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and for $n > 1$,

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j. \tag{3.83}$$

Now, we can generalize this idea to function spaces. Let $f_n(x)$, $n \in N$ and $x \in [a, b]$ be a linearly independent sequence of continuous functions. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N$ can be found and is given by $\phi_0(x) = f_0(x)$ and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{< f_n, \phi_j >}{\|\phi_j\|^2}\,\phi_j(x), \quad n = 1, 2, \ldots. \tag{3.84}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\,dx. \tag{3.85}$$

Note the similarity between this expression and the expression for the finite dimensional case in Equation (3.83).

Example Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = \frac{1}{2}.$$

We could use this result to fix the normalization of our new basis, but we will hold off on doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{< f_1, \phi_0 >}{\|\phi_0\|^2}\, \phi_0(x) \\ &= x - \frac{< x, 1 >}{\|1\|^2}\, 1 = x, \end{aligned} \tag{3.86}$$

since $< x, 1 >$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned} \phi_2(x) &= f_2(x) - \frac{< f_2, \phi_0 >}{\|\phi_0\|^2}\, \phi_0(x) - \frac{< f_2, \phi_1 >}{\|\phi_1\|^2}\, \phi_1(x) \\ &= x^2 - \frac{< x^2, 1 >}{\|1\|^2}\, 1 - \frac{< x^2, x >}{\|x\|^2}\, x \\ &= x^2 - \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} dx} \\ &= x^2 - \frac{1}{3}. \end{aligned} \tag{3.87}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x) = \phi_1(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

The set of Legendre polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many had originally

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty,\infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0,\infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

Table 3.3: Common classical orthogonal polynomials with the interval and weight function used to define them.

appeared as solutions of important boundary value problems in physics. They all have similar properties and we list these in Table 3.6.3.

3.6.4 The Least Squares Approximation

This section is incomplete as it needs reformatting and additional explanatory text. The basic idea is that the mean square deviation is obtained for an approximation of functions as a linear combination over an orthogonal basis when the expansion coefficients are the generalized Fourier coefficients.

Goal: To find the best approximation of $f(x)$ on $[a, b]$ by $S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$ for a set of fixed functions $\phi_n(x)$; i.e., to find the c_n's such that $S_N(x)$approximates $f(x)$ in the *least squares sense.*

Definitions:

•Mean Square Deviation: $\int_a^b [f(x) - S_N(x)]^2\rho(x)\,dx$ with weight function $\rho(x) > 0$.

•Converges in the mean: $\int_a^b [f(x) - S_N(x)]^2\rho(x)\,dx \ \to 0$ as $N \to \infty$.

•Inner product: $(\phi, \psi) = \int_a^b \phi(x)\psi(x)\rho(x)\,dx$.

•Orthogonal functions: $(\phi, \psi) = \int_a^b \phi(x)\psi(x)\rho(x)\,dx = 0$.

•Mutually Orthogonal set $\{\phi_n(x)\}_{n=1}^{\infty}$: $(\phi_n, \phi_m) = 0,\ m \neq n$.

Minimization in Least Squares Sense

$$\int_a^b [f(x) - S_N(x)]^2\rho(x)\,dx = \int_a^b [f(x) - \sum_{n=1}^{N} c_n\phi_n(x)]^2\rho(x)\,dx$$
$$= \int_a^b f^2(x)\rho(x)\,dx - 2\int_a^b f(x)\sum_{n=1}^{N} c_n\phi_n(x)\rho(x)\,dx + \int_a^b \sum_{n=1}^{N} c_n\phi_n(x)\sum_{m=1}^{N} c_m\phi_m(x)\rho(x)\,dx$$
$$= (f, f) - 2\sum_{n=1}^{N} c_n(f, \phi_n) + \sum_{n=1}^{N}\sum_{m=1}^{N} c_nc_m(\phi_n, \phi_m)$$
$$= (f, f) - 2\sum_{n=1}^{N} c_n(f, \phi_n) + \sum_{n=1}^{N} c_n^2(\phi_n, \phi_n).$$

Aiming to find coefficients, so complete the square in c_n. Focusing on the last two terms,

$$\begin{aligned}\sum_{n=1}^{N} c_n^2(\phi_n,\phi_n) - 2\sum_{n=1}^{N} c_n(f,\phi_n) &= \sum_{n=1}^{N} (\phi_n,\phi_n)c_n^2 - 2(f,\phi_n)c_n \\ &= \sum_{n=1}^{N} (\phi_n,\phi_n)\left[c_n^2 - \tfrac{2(f,\phi_n)}{(\phi_n,\phi_n)}c_n\right] \\ &= \sum_{n=1}^{N} (\phi_n,\phi_n)\left[\left(c_n - \tfrac{(f,\phi_n)}{(\phi_n,\phi_n)}\right)^2 - \left(\tfrac{(f,\phi_n)}{(\phi_n,\phi_n)}\right)^2\right].\end{aligned}$$

The mean square deviation is minimized by choosing $c_n = \frac{(f,\phi_n)}{(\phi_n,\phi_n)}$. These are the Fourier Coefficients!

Bessel's Inequality

Inserting in the mean square deviation yields

$$\begin{aligned}0 &\le \int_a^b [f(x) - S_N(x)]^2\rho(x)\,dx \\ &= (f,f) - 2\sum_{n=1}^{N} c_n(f,\phi_n) + \sum_{n=1}^{N} c_n^2(\phi_n,\phi_n) \\ &= (f,f) - \sum_{n=1}^{N} c_n^2(\phi_n,\phi_n).\end{aligned}$$

Thus, we obtain Bessel's Inequality: $(f,f) \ge \sum_{n=1}^{N} c_n^2(\phi_n,\phi_n)$.

Parseval's Equality

Let $N \to \infty$. Then $\sum_{n=1}^{N} c_n^2(\phi_n,\phi_n)$ converges if $(f,f) = \int_a^b f^2(x)\rho(x)\,dx < \infty$. The space of all such $f$is denoted $L^2_\rho(a,b)$, the space of square integrable functions on (a,b) with weight $\rho(x)$. Thus, from Calculus II we know that $\sum a_n$converges implies that $a_n \to 0$ as $n \to \infty$. Therefore, in this problem the terms $c_n^2(\phi_n,\phi_n) \to 0$ as $n \to \infty$. This is only possible if $c_n \to 0$ as $n \to \infty$. Thus, if $\sum_{n=1}^{N} c_n\phi_n$ *converges in the mean* to f,then $\int_a^b [f(x) - \sum_{n=1}^{N} c_n\phi_n]^2\rho(x)\,dx$approaches zero as $N \to \infty$. This implies from the above derivation of Bessel's inequality that $(f,f) - \sum_{n=1}^{N} c_n^2(\phi_n,\phi_n) \to 0$. This leads to Parseval's equality:

$$(f, f) = \sum_{n=1}^{\infty} c_n^2 (\phi_n, \phi_n).$$

Parseval's equality holds if and only if $\lim\limits_{N\to\infty} \int\limits_a^b (f(x) - \sum\limits_{n=1}^{N} c_n \phi_n(x))^2 \rho(x)\, dx = 0$. If this is true for every square integrable function in $L^2_\rho(a, b)$, then the set of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ is said to be **complete**. One can view these functions as an infinite dimensional basis for the space of square integrable functions on (a, b) with weight $\rho(x) > 0$.

One can extend the above limit $c_n \to 0$ as $n \to \infty$, by assuming that $\frac{\phi_n(x)}{\|\phi_n\|}$ is uniformly bounded and that $\int\limits_a^b |f(x)|\rho(x)\, dx < \infty$. This is the **Riemann-Lebesque Lemma**, but will not be proven now.

3.6.5 Convergence of Trigonometric Fourier Series

This section is incomplete Currently, it is a list of definitions, lemmas and theorems needed to provide convergence arguments for trigonometric Fourier series.

Definitions

1. For any nonnegative integer k, a function u is C^k if every k-th order partial derivative of u exists and is continuous.

2. For two functions $f$and g defined on an interval $[a, b]$, we will define the **inner product** as $< f, g >= \int_a^b f(x)g(x)\, dx$.

3. A function f is **periodic with period p** if $f(x+p) = f(x)$ for all x.

4. Let f be a function defined on $[-L, L]$ such that $f(-L) = f(L)$. The **periodic extension** $\tilde{f}$of f is the unique periodic function of period $2L$ such that $\tilde{f}(x) = f(x)$for all $x \in [-L, L]$.

5. $D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos\frac{n\pi x}{L}$ is called the ***N*-th Dirichlet Kernel.** [This will be summed later and the sequences of kernels converges to what is called the **Dirac Delta function**.]

6. A sequence of functions $\{s_1(x), s_2(x), \ldots\}$ is said to **converge pointwise** to$f(x)$on the interval $[-L, L]$ if for each fixed$x$ in the interval, $\lim_{N\to\infty} |f(x) - s_N(x)| = 0$.

7. A sequence of functions $\{s_1(x), s_2(x), \ldots\}$ is said to **converge uniformly** to$f(x)$on the interval $[-L, L]$ if $\lim_{N\to\infty}\left(\max_{|x|\le L} |f(x) - s_N(x)|\right) = 0$.

8. One-sided limits: $f(x_0^+) = \lim_{x\downarrow x_0} f(x)$ and $f(x_0^-) = \lim_{x\uparrow x_0} f(x)$.

9. A function f is **piecewise continuous** on $[a, b]$ if the function satisfies

$a.f$is defined and continuous at all but a finite number of points of $[a, b]$.

b. For all $x \in (a, b)$, the limits $f(x^+)$ and $f(x^-)$ exist.

c. $f(a^+)$ and $f(b^-)$ exist.

10. A function is **piecewise** C^1 on $[a,b]$ if $f(x)$ and$f'(x)$are piecewise continuous on $[a,b]$.

Lemmas

1. Bessel's Inequality: Let $f(x)$ be defined on $[-L,L]$ and $\int_{-L}^{L} f^2(x)\,dx < \infty$. If the trigonometric Fourier coefficients exist, then $a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx$. This follows from the previous handout, **Orthogonality and the Least Squares Approximation**.

2. Riemann-Lebesgue Lemma: Under the conditions of Bessel's Inequality, the Fourier coefficients approach zero as $n \to 0$. This is based upon some earlier convergence results seen in Calculus in which one learns for a series of nonnegative terms, $\sum c_n$with $c_n \ge 0$, *if* c_n*does not approach 0 as* $n \to \infty$, *then* $\sum c_n$*does not converge.* Therefore, the contrapositive holds, if $\sum c_n$ *converges, then*$c_n \to 0$ *as* $n \to \infty$. From Bessel's Inequality, we see that when$f$ is square integrable, the series formed by the sums of squares of the Fourier coefficients converges. Therefore, the Fourier coefficients must go to zero as n increases. This is also referred to in the last handout, **Orthogonality and the Least Squares Approximation**. However, an extension to absolutely integrable functions exists, which is called the Riemann-Lebesgue Lemma.

3. Green's Formula: Let $f$and $g$be C^2 functions on $[a,b]$. Then $< f'', g > - < f, g'' >= [f'(x)g(x) - f(x)g'(x)]|_a^b$. [Note: This is just an iteration of integration by parts.]

4. Special Case of Green's Formula: Let $f$and $g$be C^2 functions on $[-L,L]$ and both functions satisfy the conditions $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then $< f'', g >=< f, g'' >$.

5. Lemma 1: If g is a periodic function of period $2L$and c any real number, then $\int_{-L+c}^{L+c} g(x)\,dx = \int_{-L}^{L} g(x)\,dx$.

6. Lemma 2: Let$f$ be a C^2 function on $[-L,L]$ such that $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then for $M = \max_{|x|\le L} |f''(x)|$ and

$n \geq 1, \quad |a_n| = \left| \frac{1}{L} \int\limits_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \right| \leq \frac{2L^2 M}{n^2\pi^2}$ and
$|b_n| = \left| \frac{1}{L} \int\limits_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \right| \leq \frac{2L^2 M}{n^2\pi^2}$.

7. Lemma 3: For any real θ such that $\sin \frac{\theta}{2} \neq 0$,
$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}$

8. Lemma 4: Let $h(x)$be C^1 on $[-L, L]$. Then
$\lim\limits_{n\to\infty} \frac{1}{L} \int\limits_{-L}^{L} D_n(x)h(x)\, dx = h(0)$.

Theorems

1. **Theorem 1.** (Pointwise Convergence) Let $f$be C^1on $[-L, L]$ with $f(-L) = f(L), f'(-L) = f'(L)$.Then $FS\ f(x) = f(x)$ for all x in $[-L, L]$.

2. **Theorem 2.** (Uniform Convergence) Let $f$be C^2on $[-L, L]$ with$f(-L) = f(L), f'(-L) = f'(L)$. Then$FS\ f(x)$ converges uniformly to$f(x)$. In particular, $|f(x) - S_N(x)| \leq \frac{4L^2 M}{\pi^2 N}$ for all x in $[-L, L]$, where $M = \max\limits_{|x|\leq L} |f''(x)|$.

3. **Theorem 3.** (Piecewise C^1 −−Pointwise Convergence) Let $f$be a piecewiseC^1 function on$[-L, L]$. Then $FS\ f(x)$converges to the periodic extension of $f(x) = \begin{cases} \frac{1}{2}[f(x^+) + f(x^-)], & -L < x < L \\ \frac{1}{2}[f(L^+) + f(L^-)], & x = \pm L \end{cases}$ for all x in $[-L, L]$.

4. **Theorem 4.** (Piecewise C^1 −−Uniform Convergence) Let $f$be a piecewiseC^1 function on $[-L, L]$ such that $f(-L) = f(L)$. Then$FS\ f(x)$ converges uniformly to$f(x)$.

Proof of Convergence

We are considering the Fourier series of$f(x)$:

$$FS\ f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

We are first interested in the pointwise convergence of the infinite series. Thus, we need to look at the partial sums for each x. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$S_N(x) = a_0 + \sum_{n=1}^{N}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$$
$$= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L} + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy.\right)\sin\frac{n\pi x}{L}\right]$$
$$= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy$$
$$= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy$$
$$\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy.$$

Here $D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L}$ is called the ***N*-th Dirichlet Kernel**.
What we seek to prove is (**Lemma 4**) that
$\lim_{N\to\infty}\frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy = f(x)$. [Technically, we need the periodic extension of f.] So, we need to consider the Dirichlet kernel. Then pointwise convergence follows, as $\lim_{N\to\infty} S_N(x) = f(x)$.

Proposition: $D_n(x) = \begin{cases} \frac{\sin((n+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0 \\ n+\frac{1}{2}, & \sin\frac{\pi x}{2L} = 0 \end{cases}.$

Proof: Actually, this follows from **Lemma 3**. Let $\theta = \frac{\pi x}{L}$ and multiply $D_n(x)$by $2\sin\frac{\theta}{2}$to obtain:

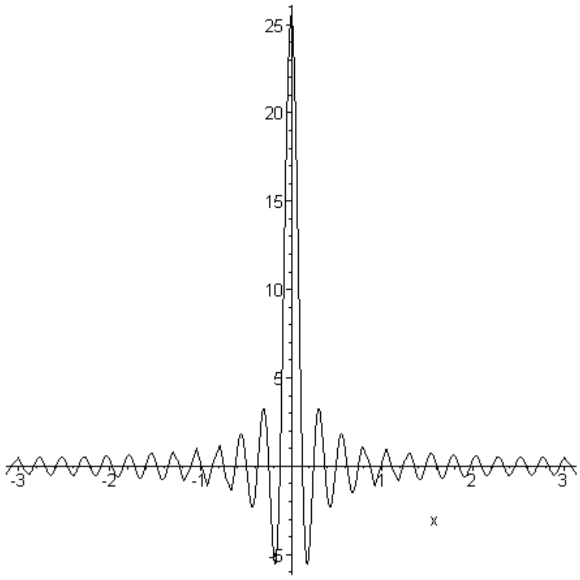

Figure 3.16: N=25.

$$
\begin{aligned}
2\sin\tfrac{\theta}{2} D_n(x) &= 2\sin\tfrac{\theta}{2}\left[\tfrac{1}{2} + \cos\theta + \cdots + \cos n\theta\right] \\
&= \sin\tfrac{\theta}{2} + 2\cos\theta\sin\tfrac{\theta}{2} + 2\cos 2\theta\sin\tfrac{\theta}{2} + \cdots + 2\cos n\theta\sin\tfrac{\theta}{2} \\
&= \sin\tfrac{\theta}{2} + \left(\sin\tfrac{3\theta}{2} - \sin\tfrac{\theta}{2}\right) + \left(\sin\tfrac{5\theta}{2} - \sin\tfrac{3\theta}{2}\right) + \cdots + \left(\sin\left((n+\tfrac{1}{2})\theta\right) - \sin\left((n-\tfrac{1}{2})\theta\right)\right) \\
&= \sin\left((n+\tfrac{1}{2})\theta\right).
\end{aligned}
$$

Thus, $2\sin\frac{\theta}{2} D_n(x) = \sin\left((n+\frac{1}{2})\theta\right)$, or if
$\sin\frac{\theta}{2} \neq 0, \quad D_n(x) = \frac{\sin\left((n+\frac{1}{2})\theta\right)}{2\sin\frac{\theta}{2}}, \ \theta = \frac{\pi x}{L}$.

If $\sin\frac{\theta}{2} = 0$,then one needs to apply L'Hospital's Rule:
$\lim\limits_{\theta\to 2m\pi} \frac{\sin\left((n+\frac{1}{2})\theta\right)}{2\sin\frac{\theta}{2}} = \lim\limits_{\theta\to 2m\pi} \frac{(n+\frac{1}{2})\cos\left((n+\frac{1}{2})\theta\right)}{\cos\frac{\theta}{2}} = \frac{(n+\frac{1}{2})\cos(2mn\pi+m\pi)}{\cos m\pi} = n + \frac{1}{2}$.

As $n \to \infty$, $D_n(x) \to \delta(x)$, the **Dirac delta function,** on the interval $[-L, L]$. The following are some plots for $L = \pi$and $n = 25, 50, 100$.Note how a central peak grows and the values tend towards zero for nonzero x.

The Dirac delta function can be defined as that quantity satisfying (a)

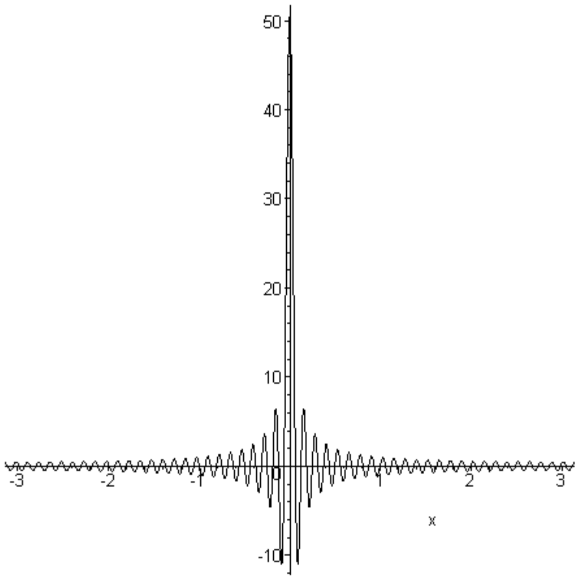

Figure 3.17: N=50.

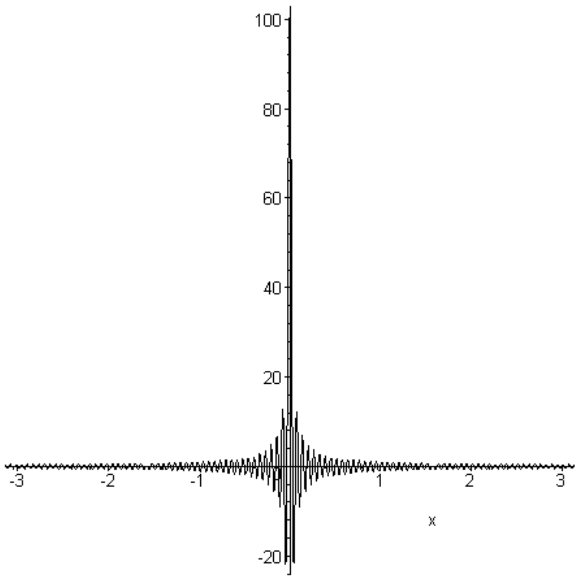

Figure 3.18: N=100.

$\delta(x) = 0,\ x \neq 0$; (b) $\int\limits_{-\infty}^{\infty} \delta(x)\, dx = 1.$

This generalized function, or **distribution**, also has the property: $\int\limits_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$.

Thus, under the appropriate conditions on f, one can show $\lim\limits_{N\to\infty} \frac{1}{L} \int\limits_{-L}^{L} D_N(y-x) f(y)\, dy = f(x)$. We need to prove **Lemma 4** first.

Proof: Since $\frac{1}{L} \int\limits_{-L}^{L} D_N(x)\, dx = \frac{1}{2L} \int\limits_{-L}^{L} dx = 1$, we have that

$$\frac{1}{L} \int\limits_{-L}^{L} D_N(x) h(x)\, dx - h(0) = \frac{1}{L} \int\limits_{-L}^{L} D_N(x) \left[h(x) - h(0)\right]\, dx$$
$$= \frac{1}{2L} \int\limits_{-L}^{L} \left[\cos \tfrac{n\pi x}{L} + \cot \tfrac{\pi x}{L} \sin \tfrac{n\pi x}{L}\right] \left[h(x) - h(0)\right]\, dx.$$

The two terms look like the Fourier coefficients. An application of the Riemann-L:ebesgue Lemma indicates that these coefficients tend to zero as $n \to \infty$, provided the functions being expanded are square integrable and the integrals above exist. The cosine integral follows, but a little work is needed for the sine integral. One can use L'Hospital's Rule with $h \in C^1$.

Now we apply **Lemma 4** to get the convergence from $\lim\limits_{N\to\infty} \frac{1}{L} \int\limits_{-L}^{L} D_N(y-x) f(y)\, dy = f(x)$. Due to periodicity, we have

$$\frac{1}{L} \int\limits_{-L}^{L} D_N(y-x) f(y)\, dy = \frac{1}{L} \int\limits_{-L}^{L} D_N(y-x) \tilde{f}(y)\, dy$$
$$= \frac{1}{L} \int\limits_{-L+x}^{L+x} D_N(y-x) \tilde{f}(y)\, dy$$
$$= \frac{1}{L} \int\limits_{-L}^{L} D_N(z) \tilde{f}(x+z)\, dz.$$

We can apply **Lemma 4** providing $\tilde{f}(z+x)$ is C^1in z, which is true since f is $C^1$and behaves well at $\pm L$.

To prove **Theorem 2** on uniform convergence, we need only combine **Theorem 1** with **Lemma 2**. Then we have,

$$
\begin{aligned}
|f(x) - S_N(x)| &= |f(x) - S_N(x)| \\
&\leq \sum_{n=N+1}^{\infty} \left[\left|a_n \cos \tfrac{n\pi x}{L}\right| + \left|b_n \sin \tfrac{n\pi x}{L}\right| \right] \\
&\leq \sum_{n=N+1}^{\infty} \left[|a_n| + |b_n| \right] \\
&\leq \tfrac{4L^2 M}{\pi^2} \sum_{n=N+1}^{\infty} \tfrac{1}{n^2} \\
&\leq \tfrac{4L^2 M}{\pi^2 N}.
\end{aligned}
$$

This gives the uniform convergence.

These Theorems can be relaxed to include piecewise C^1 functions. **Lemma 4** needs to be changed for such functions to the result that $\lim\limits_{n\to\infty} \frac{1}{L} \int\limits_{-L}^{L} D_n(x)h(x)\,dx = \frac{1}{2}[h(0^+) + h(0^-)]$ by splitting the integral into integrals over$[-L, 0]$, $[0, L]$ and applying a one-sided L'Hospital's Rule. Proving uniform convergence under the conditions in **Theorem 4** takes a little more effort, but it can be done.

3.7 General Orthogonal Function Expansions