Chapter 6

The Fourier Transform

In this chapter we will turn to the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can represent analog signals. Analog signals are continuous signals which may be sums over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent.

We will see how to rewrite our trigonometric Fourier series as complex exponential series. Then we will extend our series to signals with infinite periods. In later chapters we will see the connection between analog and digital signals.

6.1 Complex Exponential Fourier Series

We first recall the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{6.1}$$

where the Fourier coefficients were found as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots \tag{6.2}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like terms. This gives

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{inx} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-inx}. \end{aligned} \tag{6.3}$$

The coefficients can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \tag{6.4}$$

Then we also have that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \tag{6.5}$$

This gives our representation as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Reindexing the first sum, by letting $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty}\bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Now, we can define

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Finally, we note that we can take $c_0 = \frac{a_0}{2}$. So, we can write the complex exponential Fourier series representation as

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{-inx}, \tag{6.6}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\ c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\ c_0 &= \frac{a_0}{2}. \end{aligned} \tag{6.7}$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots$.

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n) \\ &= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}+e^{-inx}}{2}\right)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}-e^{-inx}}{2i}\right)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx \end{aligned} \tag{6.8}$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, for all n we have show that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \tag{6.9}$$

We have converted our trigonometric series for functions defined on $[-\pi, \pi]$ to a complex exponential series in Equation (6.6) with Fourier coefficients given by (6.9). We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

This can be rewritten in an exponential Fourier series of the form

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{-in\pi x/L}$$

with

$$c_n = \frac{1}{2\pi}\int_{-L}^{L} e^{in\pi x/L}\,dx.$$

6.2 Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions in term of sums over a discrete set of frequencies for functions of finite period. On intervals $[-L, L]$ the period is $2L$. Writing the arguments in terms of frequencies, we have $2\pi f = \frac{n\pi}{L}$, or the sums are over frequencies $f = \frac{n}{2L}$. This is a discrete, or countable, set of frequencies.

We would now like to extend our interval to $x \in (-\infty, \infty)$ and to extend the discrete set of frequencies to a continuous set of frequencies. One can do this rigorously, but it amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed. We define $\omega = 2\pi f$ and the sum over a continuous set of frequencies becomes an integral. Formally, we arrive at the *Fourier transform*

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx. \tag{6.10}$$

This is a generalization of the Fourier coefficients (6.9). Once we know the Fourier transform, then we can *reconstruct* our function using the *inverse Fourier transform*, which is given by

$$F^{-1}[f] = f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}\,d\omega. \tag{6.11}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Such functions are said to be L_1.

The Fourier transfrom and inverse Fourier transform are inverse operations. This means that

$$F^{-1}[F[f]] = f(x)$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega).$$

We will now prove the first of these equations. The second follows in a similar way. This is done by inserting the definition of the Fourier transform into the inverse transform definition and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f]e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}\, d\xi\right] e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\, d\xi d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi. \qquad (6.12) \end{aligned}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we will define

$$D_L(x) = \int_{-L}^{L} e^{i\omega x}\, d\omega$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega = \lim_{L\to\infty} D_L(\xi - x).$$

We can compute $D_L(x)$. A simple evaluation yields

$$D_L(x) \quad = \quad \int_{-L}^{L} e^{i\omega x}\, d\omega$$

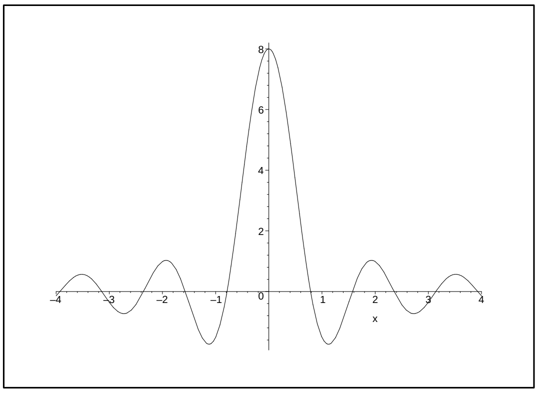

Figure 6.1: A plot of the function $D_L(x)$ for $L = 4$.

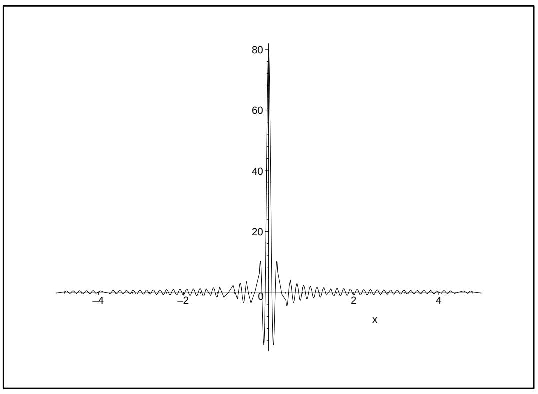

Figure 6.2: A plot of the function $D_L(x)$ for $L = 40$.

$$
\begin{aligned}
&= \frac{e^{i\omega x}}{ix}\Big|_{-L}^{L} \\
&= \frac{e^{ixL} - e^{-ixL}}{2ix} \\
&= \frac{2\sin xL}{x}.
\end{aligned} \tag{6.13}
$$

We can graph this function. As $x \to 0$, $D_L(x) \to 2L$. For large x, The function tends to zero. A plot of this function is in Figure 6.1. For large L the peak grows and the values of $D_L(x)$ for $x \neq 0$ tend to zero as show in Figure 6.2.

We note that in the limit $L \to$, $D_L(x) = 0$ for $x \neq 0$ and it is infinite at

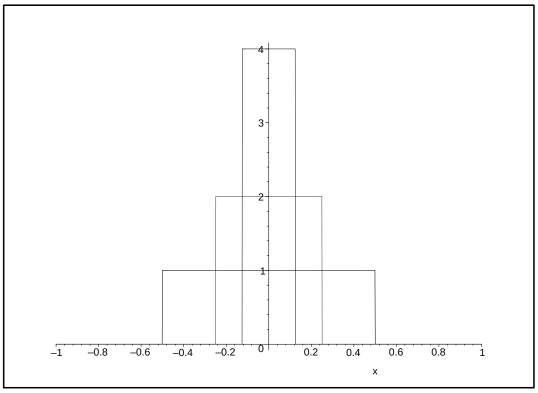

Figure 6.3: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

$x = 0$. However, the area is constant for each L. In fact,

$$\int_{-\infty}^{\infty} D_L(x)\, dx = 2\pi.$$

This behavior can be represented by the limit of other sequences of functions. Define the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n} \\ \frac{n}{2}, & |x| < \frac{1}{n} \end{cases}$$

This is a sequence of functions as shown in Figure 6.4. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

The limit is not really a function. It is a *generalized function*. It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

Before returning to the proof, we state one more property of the Dirac delta function, which we will prove in the next section. We have that

$$\int_{-\infty}^{\infty} \delta(x - a) f(x)\, dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega = \lim_{L\to\infty} D_L(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (6.12), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\, d\xi. \\ &= f(x). \end{aligned} \tag{6.14}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f.

6.3 The Dirac Delta Function

In the last section we introduced the Dirac delta function, $\delta(x)$. This is one example of what is known as a *generalized function* or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

More generally, the integration over a more general interval gives

$$\int_a^b \delta(x)\, dx = 1, \quad 0 \in [a, b]$$

and

$$\int_a^b \delta(x)\, dx = 0, \quad 0 \text{ not in } [a, b]$$

.

The other property that was used was that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a).$$

Other occurrences of the delta function are integrals of the form $\int_{-\infty}^{\infty} \delta(f(x))\, dx$. Such integrals can be converted into a useful form depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x)|} \delta(x - x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|} \delta(x - x_1).$$

More generally, one can show that when $f(x_j) = 0$ for x_j, $j = 1, 2, \ldots, n$, then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|} \delta(x - x_j).$$

Finally, one can show that there is a relationship between the Heaviside, or step, function and the Dirac delta function. We define the Heaviside function as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Then, it is easy to see that $H'(x) = \delta(x)$. In some texts the notation $\theta(x)$ is used for the step function, or $H(x)$.

Example Evaluate $\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx$.

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x) = 3x-2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(x-\frac{2}{3})x^2\,dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

6.4 Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications our functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega x}\,dt, \tag{6.15}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx. \tag{6.16}$$

ω is called the angular frequency and is related to the frequency ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the wavenumber. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

1. For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

   $$F[f+g] = F[f] + F[g]$$

   and

   $$F[af] = aF[f].$$

   These simply folow from the properties of integration and establish the linearity of the Fourier transform.

2. $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

   This property can be shown using integration by parts.

$$\begin{aligned} F[\frac{df}{dx}] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx \\ &= \lim_{L\to\infty} \left(f(x)e^{ikx}\right)\big|_{-L}^{L} - ik\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx. \end{aligned} \quad (6.17)$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The integral is recognized as the Fourier transform of f, proving the given property.

3. $F\left[\frac{d^n f}{dx^n}\right] = (-ik)^n\hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F[\frac{d^2 f}{dx^2}] &= F[\frac{d}{dx} f'] \\ &= -ikF[\frac{df}{dx}] = (-ik)^2\hat{f}(k). \end{aligned} \quad (6.18)$$

   This result will be true if both $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to\pm\infty} f'(x) = 0$. Generalization to the transform of the nth derivative easily follows.

4. $F\left[xf(x)\right] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and being able to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)\, dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \quad (6.19)$$

5. **Shifting Properties** For constant a, we have the following shifting properties:

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{6.20}$$
$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{6.21}$$

Here we have denoted the Fourier transform pairs as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform, or inverse Fourier transform. The first shift property is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

Now perform the substitution $y = x - a$. Then,

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy = e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy = e^{ika}\hat{f}(k).$$

The second shift property follows in a similar way.

6. **Convolution** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dx. \tag{6.22}$$

Then

$$F[f*g] = \hat{f}(k)\hat{g}(k). \tag{6.23}$$

We will return to the proof of this property in a later section.

6.4.1 Fourier Transform Examples

In this section we will compute some Fourier transforms of several functions.

Example 1 $f(x) = e^{-ax^2/2}$.

This function is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a

Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{6.24}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\,dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned} -\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\ &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\ &= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a} \end{aligned} \tag{6.25}$$

We now substitute this expression in the integral and then make the substitution $y = x - \frac{ik}{a}$.

$$\begin{aligned} \hat{f}(k) &= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2}\,dx \\ &= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2}\,dy. \end{aligned} \tag{6.26}$$

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane. We can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we can safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Let I be given by

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy.$$

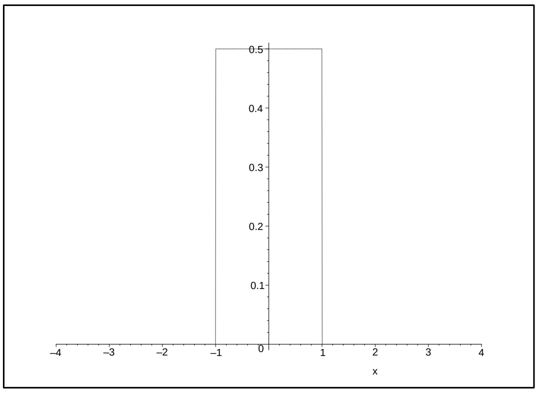

Figure 6.4: A plot of the box function in Example 2.

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Note that we needed to change the integration variable. We can now write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2}\, r^2 drd\theta.$$

This integral is doable, giving $I^2 = \frac{\pi}{\beta}$. So, the final result is gotten by taking the square root of both sides:

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{6.27}$$

Example 2 $f(x) = \begin{cases} b, & |x| \le a \\ 0, & |x| > a \end{cases}.$

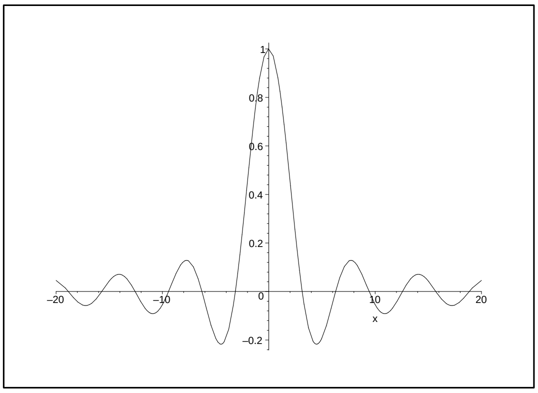

Figure 6.5: A plot of the Fourier transform of the box function in Example 2.

This function is called the box function, or gate function. It is shown in Figure 6.4. The Fourier transform of the box function is relatively easy to compute. It is given by

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= \int_{-a}^{a} be^{ikx}\,dx \\ &= \frac{b}{ik}e^{ikx}|_{-a}^{a} \\ &= \frac{2b}{k}\sin ka. \end{aligned} \tag{6.28}$$

We can rewrite this as

$$\hat{f}(k) = 2ab\frac{\sin ka}{ka} \equiv 2ab\,\text{sinc}\;ka.$$

A plot of this function is shown in Figure 6.5. We will consider special limiting values for the box function and its transform.

(a) $a \to \infty$ and b fixed.
In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 6.5 with Figure DL. In fact,

$\hat{f}(k) = bD_a(k)$. So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit gives us the fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes wider, the Fourier transform becomes more localized. In fact, we have arrived at the result that

$$\int_{-\infty}^{\infty} e^{ikx} = 2\pi\delta(k). \tag{6.29}$$

(b) $b \to \infty$, $a \to 0$, and $2ab = 1$.
In this case our box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

$$\int_{-\infty}^{\infty} \delta(x)e^{ikx} = 1. \tag{6.30}$$

In this case we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.

(c) **The Uncertainty Principle**
The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx of the box function as

$$\Delta x = 2a.$$

The width of the Fourier transform is a little trickier. This function actually extends the entire k-axis. However, as $\hat{f}(k)$ became more localized, the central peak became narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe. Thus,

$$\Delta k = \frac{2\pi}{a}.$$

Combining these two relations, we find that

$$\Delta x \Delta k = 4\pi.$$

Thus, the more localized a signal, the less localized its transform. This notion is referred to as the Uncertainty Principle. For more general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq 1.$$

Example 3 $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \ a > 0.$

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\ &= \int_{o}^{\infty} e^{ikx-ax}\, dx \\ &= \frac{1}{a-ik}. \end{aligned} \tag{6.31}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 4 $\hat{f}(k) = \frac{1}{a-ik}$.

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\, dk.$$

This integral can be evaluated using contour integral methods. We recall Jordan's Lemma from the last chapter:

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{ikz}\, dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$. A similar result applies for $k < 0$, but one closes the contour in the lower half plane.

In this example, we have to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz}\, dz.$$

According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$. The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk \\ &= \pm\frac{1}{2\pi}\oint_C \frac{e^{-ixz}}{a-iz}\,dz \\ &= \begin{cases} 0, & x<0 \\ -\frac{1}{2\pi}2\pi i \text{ Res } [z=-ia], & x>0 \end{cases} \\ &= \begin{cases} 0, & x<0 \\ e^{-ax}, & x>0 \end{cases}. \end{aligned} \tag{6.32}$$

Example 5 $\hat{f}(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$.

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega-\omega_0)$. This is a shifted function. From the Shift Theorems we have

$$e^{i\omega_0 t}f(t) \leftrightarrow \hat{f}(\omega-\omega_0).$$

Recalling from a previous example that

$$\int_{-\infty}^{\infty} 1e^{i\omega t}\,dt = 2\pi\delta(\omega),$$

we have

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}e^{-i\omega_0 t}.$$

The other term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0] = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 6 The Finite Wave Train $f(x) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}.$

For our last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. A straight forward computation gives

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt \\
&= \int_{-a}^{a} \cos\omega_0 t e^{i\omega t}\,dt \\
&= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\,dt \\
&= \frac{1}{2}\int_{-a}^{a} [\cos(\omega_0+\omega)t + \cos(\omega_0-\omega)t]\,dt \\
&= \frac{\sin(\omega_0+\omega)a}{\omega+\omega_0} + \frac{\sin(\omega_0-\omega)a}{\omega-\omega_0}. \qquad (6.33)
\end{aligned}$$

6.5 The Convolution Theorem

In our list of properties, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt. \qquad (6.34)$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is *faltung*, which means 'folding'.

First, we not that the convolution is commutative: $f*g = g*f$. This is easily shown by replacing $x-t$ with a new variable, y.

$$\begin{aligned}
(g*f)(x) &= \int_{-\infty}^{\infty} g(t)f(x-t)\,dt \\
&= -\int_{\infty}^{-\infty} g(x-y)f(y)\,dy \\
&= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \\
&= (f*g)(x). \qquad (6.35)
\end{aligned}$$

Example Graphical Convolution.

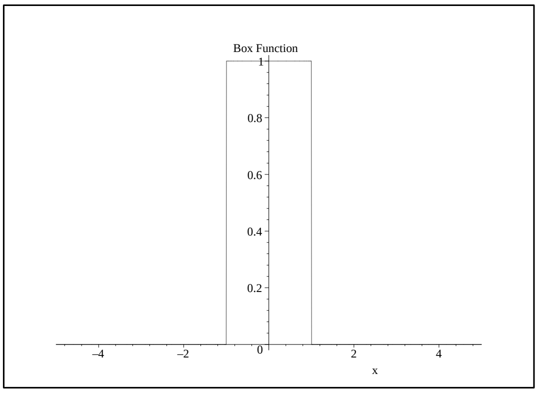

Figure 6.6: A plot of the box function $f(x)$.

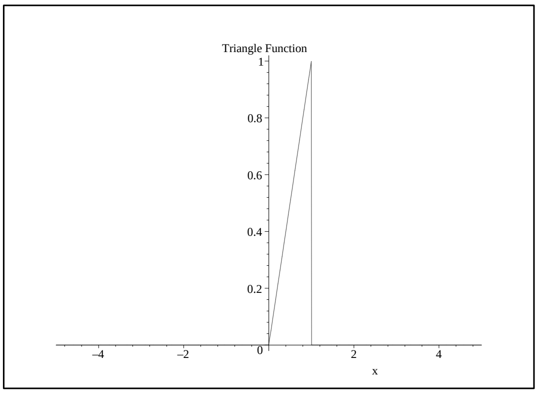

Figure 6.7: A plot of the triangle function.

In order to understand the convolution operation, we need to apply it to several functions. We will do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

as shown in Figures 6.6 and 6.7.

In order to determine the contributions to the integrand, we look at the shifted and reflected function $g(t-x)$ for various values of t. For $t = 0$, we

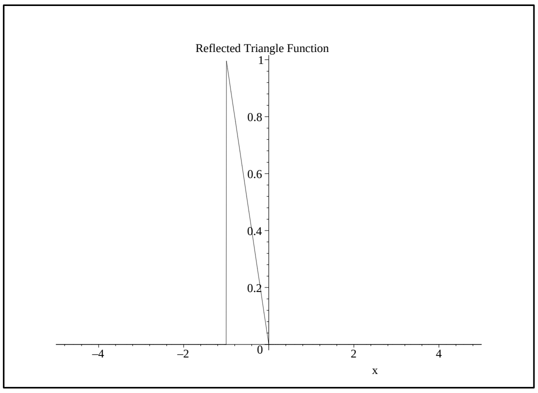

Figure 6.8: A plot of the reflected triangle function.

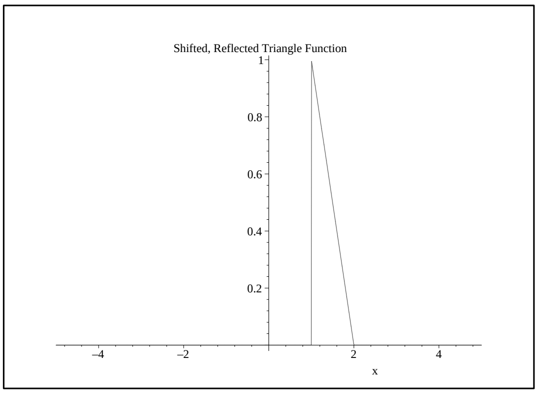

Figure 6.9: A plot of the reflected triangle function shifted by 2 units.

have $g(-x)$. This is a reflection of the triangle function as shown in Figure 6.8.

We then translate this function through horizontal shifts by t. In Figure 6.9 we show such a shifted and reflected $g(x)$ for $t = 2$. The following figures show other shifts superimposed on $f(x)$. The integrand is the product of $f(x)$ and $g(t-x)$ and the convolution evaluated at t is given by the shaded areas. In Figures 6.10 and 6.14 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in Figures 6.11-6.13 and the convolution is shown by the area under the product of the two functions.

We see that the value of the convolution integral builds up and then

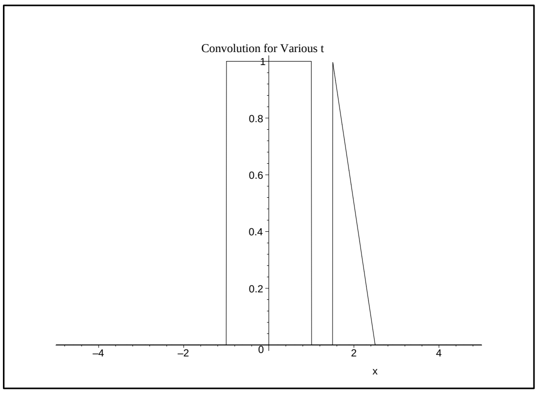

Figure 6.10: A plot of the box and triangle functions with the convolution indicated by the shaded area.

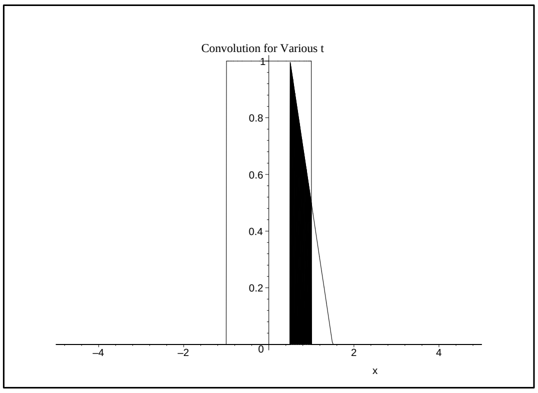

Figure 6.11: A plot of the box and triangle functions with the convolution indicated by the shaded area.

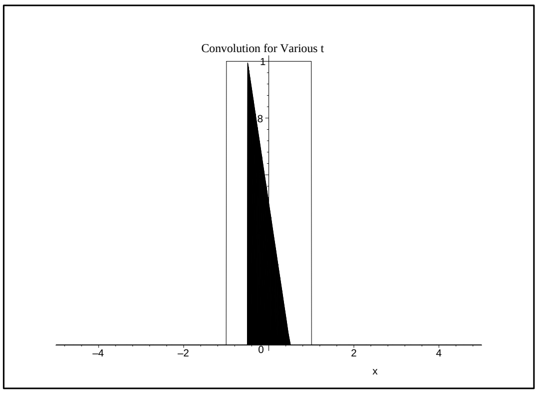

Figure 6.12: A plot of the box and triangle functions with the convolution indicated by the shaded area.

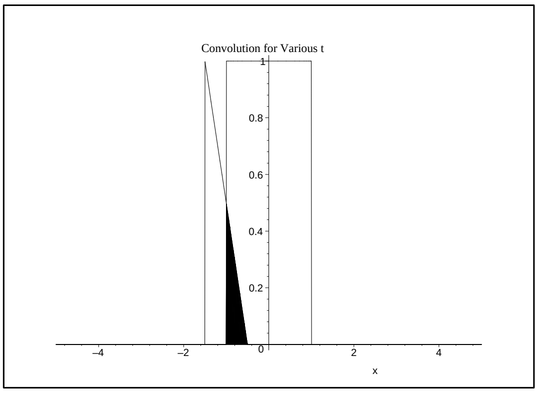

Figure 6.13: A plot of the box and triangle functions with the convolution indicated by the shaded area.

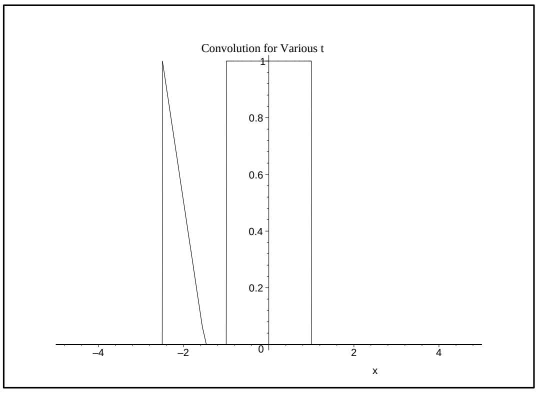

Figure 6.14: A plot of the box and triangle functions with the convolution indicated by the shaded area.

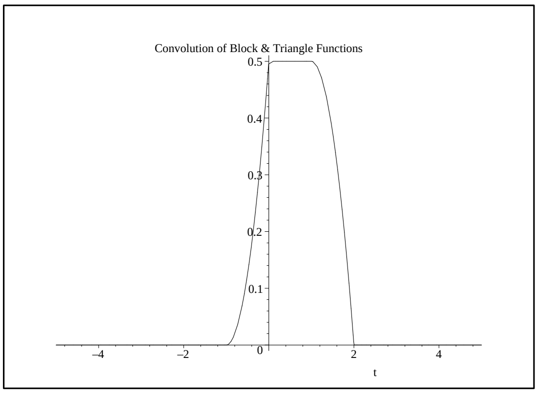

Figure 6.15: A plot of the convolution of the box and triangle functions.

quickly drops to zero. The plot of the convolution of the box and triangle functions is given in Figure 6.15.

Next we would like to computer the Fourier transform of the convolution integral. First, use the definitions of Fourier transform and convolution to write the transform as

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty} (f*g)(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)g(x-t)\,dt \right) e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t)e^{ikx}\,dx \right) f(t)\,dt. \end{aligned} \tag{6.36}$$

Next, we substitute $y = x - t$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f * g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{ik(y+t)}\, dy \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right) f(t) e^{ikt}\, dt. \end{aligned} \tag{6.37}$$

We see the the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{6.38}$$

Example Convolution of two Gaussian functions.

We would like to compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt = \int_{-\infty}^{\infty} e^{-at^2 - b(x-t)^2}\, dt.$$

This integral can be rewritten as

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2bxt}\, dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution. Recalling the Fourier transform of a Gaussian, we have

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a} \tag{6.39}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{2\pi}{b}} e^{-k^2/2b}.$$

Denoting the convolution function by $h(x) = (f * g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{2\pi}{\sqrt{ab}} e^{-k^2/2a} e^{-k^2/2b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{2\pi}{\sqrt{ab}} e^{-\frac{1}{2}(\frac{1}{a}+\frac{1}{b})k^2} = \frac{2\pi}{\sqrt{ab}} e^{-\frac{a+b}{2ab}k^2}. \tag{6.40}$$

To complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (6.40). We can do this by looking at Equation (6.39). We have first that

$$F^{-1}[\sqrt{\frac{2\pi}{a}} e^{-k^2/2a}] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/2a}] = \sqrt{\frac{a}{2\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{2a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/2\alpha}.$$

This is in the form needed to invert (6.40). Thus, for $\alpha = \frac{a+b}{2ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{2\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

6.6 Applications of the Convolution Theorem

There are many applications of the convolution operation. In this section we will describe a few of the applications, which are useful in studying signal analysis.

The first application is filtering. For a given signal there might be some noise in the signal, or some undesirable high frequencies. Or, the device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal. An example is provided in Figure 6.16. Recall that the Fourier transform gives the frequency

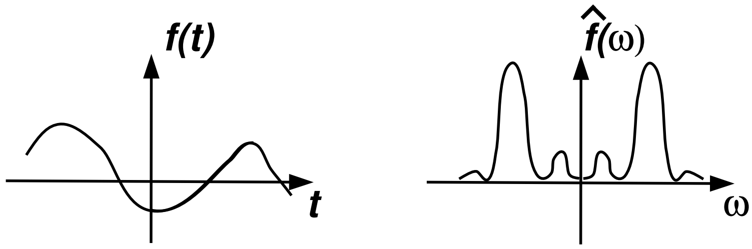

Figure 6.16: Plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

content of the signal and that $\omega = 2\pi\nu$, where ν is the frequency in Hertz, or cycles per second (cps).

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high frequencies, $|\omega| > \omega_0$ for some cutoff frequency ω_0. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could consider the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}. \tag{6.41}$$

Figure 6.17 shows how the gate function is used to filter the signal.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{6.42}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the *impulse response.* This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\,d\tau = h(t).$$

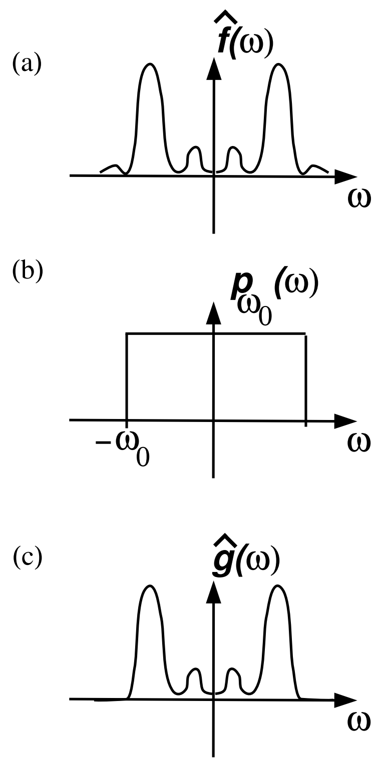

Figure 6.17: (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b).

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

We will later see that the effect of windowing would be to change the spectral content of the signal we are trying to analyze. We will study these natural windowing and filtering effects from recording data in the last chapter.

We can use the convolution theorem to derive **Parseval's Equality**:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega. \tag{6.43}$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, sometimes referred as Plancherel's formula, is simply a statement that the energy is invariant under the transform.

Let's rewrite the Convolution Theorem in the form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f * g)(t). \tag{6.44}$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty} f(t-u)g(u)\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t} \, d\omega.$$

Setting $t = 0$,

$$\int_{-\infty}^{\infty} f(-u)g(u)\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) \, d\omega. \tag{6.45}$$

Now, let $g(t) = \overline{f(-t)}$, or $f(-t) = \overline{g(t)}$. Then, the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \overline{f(-t)} e^{i\omega t} \, dt$$

$$\begin{aligned} &= -\int_{\infty}^{-\infty} \overline{f(\tau)} e^{-i\omega\tau}\, d\tau \\ &= \overline{\int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau}\, d\tau} = \overline{\hat{f}(\omega)}. \end{aligned} \tag{6.46}$$

So, inserting this result into Equation (6.45), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

which implies Parseval's Equality.