Chapter 4

Complex Representations of Functions

4.1 Complex Representations of Signals

We have seen that we can seek the frequency content of a signal $f(t)$ defined on an interval $[0, T]$ by looking for the the Fourier coefficients in the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}.$$

The coefficients take forms like

$$a_n = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi nt}{T} \, dt.$$

However, trigonometric functions can be written in a complex exponential form. Recall Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Replacing i with $-i$, we also have

$$e^{-i\theta} = \cos\theta - i \sin\theta.$$

Adding these expressions, we have

$$2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

Subtracting the exponentials leads to an expression for the sine function. Thus, we have that sines and cosines can be written as complex exponentials:

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned} \tag{4.1}$$

So, we can write

$$\cos\frac{2\pi nt}{T} = \frac{1}{2}(e^{\frac{2\pi int}{T}} + e^{-\frac{2\pi int}{T}}).$$

Later we will see that we can use this information to rewrite our series as a sum over complex exponentials in the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi int}{T}}$$

where the Fourier coefficients now take the form

$$c_n = \int_0^T f(t) e^{-\frac{2\pi int}{T}}.$$

In fact, in order to connect our analysis to ideal signals over an infinite interval and containing a continuum of frequencies, we will see the above sum become an integral and we will naturally find ourselves needing to work with functions of complex variables and perform complex integrals.

We can also extend these ideas to develop a complex representation for waves. The solution of the wave equation for waves on finite length strings can be written in the form

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n(x+ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x-ct)\right]. \tag{4.2}$$

We can replace the sines with their complex forms as

$$\begin{aligned} u(x,t) = \frac{1}{4i}\Bigg[&\sum_{n=1}^{\infty} A_n \left(e^{ik_n(x+ct)} - e^{-ik_n(x+ct)}\right) \\ &+ \sum_{n=1}^{\infty} A_n \left(e^{ik_n(x-ct)} - e^{-ik_n(x-ct)}\right)\Bigg]. \end{aligned} \tag{4.3}$$

Now, defining $k_{-n} = -k_n$, we can rewrite this solution in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{ik_n(x+ct)} + d_n e^{ik_n(x-ct)} \right]. \tag{4.4}$$

Such representations are also possible for waves propagating over the entire real line. In such cases we are not restricted to discrete frequencies and wave numbers. The sum of the harmonics will then be a sum over a continuous range, which means that our sums become integrals. So, we are then lead to the complex representation

$$u(x,t) = \int_{-\infty}^{\infty} \left[c(k) e^{ik(x+ct)} + d(k) e^{ik(x-ct)} \right] dk. \tag{4.5}$$

The forms $e^{ik(x+ct)}$ and $e^{ik(x-ct)}$ are complex representations of what are called plane waves in one dimension. The integral represents a general wave form consisting of a sum over plane waves, typically representing wave packets. The Fourier coefficients in the representation can be complex valued functions and the evaluation of the integral may be done using methods from complex analysis. We would like to be able to compute such integrals.

With the above ideas in mind, we will now take a tour of complex analysis. We will first review some facts about complex numbers and then introduce complex functions. This will lead us to the calculus of functions of a complex variable, including differentiation and complex integration.

4.2 Complex Numbers

Complex numbers were first introduced in order to solve some simple problems. The history of complex numbers only extends about two thousand years. In essence, it was found that we need to find the roots of equations such as $x^2 + 1 = 0$. The solution is $x = \pm\sqrt{-1}$. Due to the usefulness of this concept, which was not realized at first, a special symbol was introduced - the imaginary unit, $i = \sqrt{-1}$.

A complex number is a number of the form $z = x + iy$, where x and y are real numbers. x is called the real part of z and y is the imaginary part of

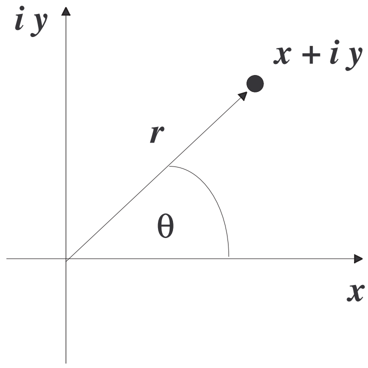

Figure 4.1: The Argand diagram for plotting complex numbers in the complex z-plane.

z. Examples of such numbers are $3 + 3i$, $-1i$, $4i$ and 5. Note that $5 = 5 + 0i$ and $4i = 0 + 4i$.

There is a geometric representation of complex numbers in a two dimensional plane, known as the complex plane C. This is given by the Argand diagram as shown in Figure 4.1. Here we can think of the complex number $z = x + iy$ as a point (x, y) in the complex plane or as a vector. The magnitude, or length, of this vector is called the complex modulus of z, denoted by $|z| = \sqrt{x^2 + y^2}$.

We can also use the geometric picture to develop a polar representation of complex numbers. From Figure 4.1 we can see that in terms of r and θ we have that

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{4.6}$$

Thus,

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{4.7}$$

So, given r and θ we have $z = re^{i\theta}$. However, given the Cartesian form, $z = x + iy$, we can also determine the polar form, since

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned} \tag{4.8}$$

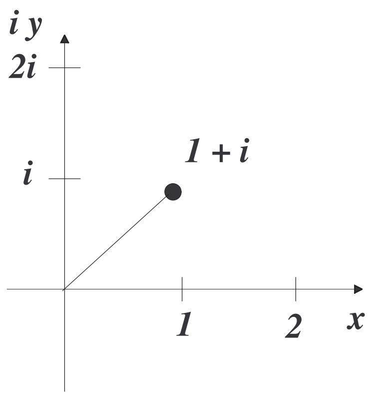

Figure 4.2: Locating $1+i$ in the complex z-plane.

Note that $r = |z|$.

Example Write $1+i$ in polar form.

If one locates $1+i$ in the complex plane, then it might be possible to immediately determine the polar form from the angle and length of the "complex vector". This is shown in Figure 4.2.

If one did not see the polar form from the plot in the z-plane, then one can systematically determine the results. We want to write $1+i$ in polar form: $1+i = re^{i\theta}$ for some r and θ. Using the above relations, we have $r = \sqrt{x^2+y^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = 1$. This gives $\theta = \frac{\pi}{4}$. So, we have found that

$$1+i = \sqrt{2}e^{i\pi/4}.$$

We also have the usual operations. We can add two complex numbers and obtain a complex number. This is simply done by adding the real parts and the imaginary parts. So,

$$(3+2i)+(1-i) = 4+i.$$

We can also multiply two complex numbers just like we multiply any binomials, though we now can use the fact that $i^2 = -1$. For example, we have

$$(3+2i)(1-i) = 3+2i-3i+2i(-i) = 5-i.$$

We can even divide one complex number into another one and get a complex number as the quotient. Before we do this, we need to introduce the complex conjugate, $\bar{z}$, of a complex number. The complex conjugate of $z = x + iy$, where x and y are real numbers, is given as

$$\overline{z} = x - iy.$$

Complex conjugates satisfy the following relations for complex numbers z and w and real number x.

$$\begin{aligned}\overline{z+w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{z}\,\overline{w} \\ \overline{\overline{z}} &= z \\ \overline{x} &= x.\end{aligned} \tag{4.9}$$

One consequence is that the complex conjugate of $re^{i\theta}$ is

$$\overline{re^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = re^{-i\theta}.$$

Another consequence is that

$$z\overline{z} = re^{i\theta}re^{-i\theta} = r^2.$$

Thus, the product of a complex number with its complex conjugate is a real number. We can also write this result in the form

$$z\overline{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2.$$

Now we are in a position to write the quotient of two complex numbers in the standard form of a real plus an imaginary number. As an example, we want to rewrite $\frac{3+2i}{1-i}$. This is accomplished by multiplying the numerator and denominator of this expression by the complex conjugate of the denominator:

$$\frac{3+2i}{1-i} = \frac{3+2i}{1-i}\frac{1+i}{1+i} = \frac{1+5i}{2}.$$

Therefore, we have the quotient is $\frac{1}{2} + \frac{5}{2}i$.

We can also look at powers of complex numbers. For example,

$$(1+i)^2 = 2i,$$

$$(1+i)^3 = (1+i)(2i) = 2i - 2.$$

But, what is $(1+i)^{1/2} = \sqrt{1+i}$?

In general, we want to find the nth root of a complex number. Let $t = z^{1/n}$. To find t in this case is the same as asking for the solution of

$$z = t^n$$

given z. But, this is the root of an nth degree equation, for which we expect n roots. We can answer our question if we write z in polar form, $z = re^{i\theta}$. Then,

$$\begin{aligned} z^{1/n} &= \left(re^{i\theta}\right)^{1/n} \\ &= r^{1/n} e^{i\theta/n} \\ &= r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right]. \end{aligned} \tag{4.10}$$

Now, let's attempt an answer to our problem: $(1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}$. But this is only one solution. We expected two solutions.

The problem is that the polar representation for z is not unique. We note that

$$e^{2k\pi i} = 1, \quad k = 0, \pm 1, \pm 2, \ldots.$$

So, we can rewrite z as $z = re^{i\theta}e^{2k\pi i} = re^{i(\theta+2k\pi)}$. Now, we have that

$$z^{1/n} = r^{1/n} e^{i(\theta+2k\pi)/n}.$$

We note that we only get different values for $k = 0, 1, \ldots, n-1$.

Now, we can finish our example.

$$\begin{aligned} (1+i)^{1/2} &= \left(\sqrt{2}e^{i\pi/4}e^{2k\pi i}\right)^{1/2} \\ &= 2^{1/4} e^{i(\pi/8 + k\pi)} \\ &= 2^{1/4}e^{i\pi/8}, 2^{1/4}e^{9\pi i/8}. \end{aligned} \tag{4.11}$$

Finally, what is $\sqrt[n]{1}$? Our first guess would be $\sqrt[n]{1} = 1$. But, we know that there should be n roots. These roots are called the nth roots of unity. Using the above result with $r = 1$ and $\theta = 0$, we have that

$$\sqrt[n]{1} = \left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right], \quad k = 0, \ldots, n-1.$$

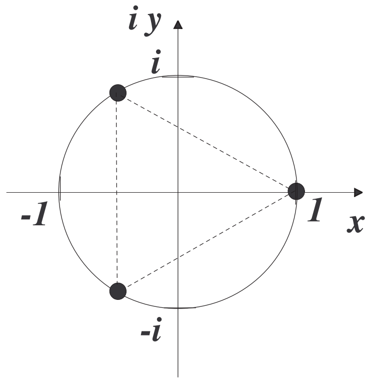

Figure 4.3: Locating the cube roots of unity in the complex z-plane.

For example, we have

$$\sqrt[3]{1} = \left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right], \quad k = 0, 1, 2.$$

These three roots can be written out as

$$\sqrt[3]{1} = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

We can locate these cube roots of unity in the complex plane. In Figure 4.3 we see that these points lie on the unit circle and are at the vertices of an equilateral triangle. In fact, all nth roots of unity lie on the unit circle and are the vertices of a regular n-gon.

4.3 Complex Valued Functions

We would like to next explore complex functions and the calculus of complex functions. We begin by defining a function that takes complex numbers into complex numbers, $f : C \to C$. It is difficult to visualize such functions. One typically uses two copies of the complex plane to indicate how such functions behave. We will call the domain the z-plane and the image will lie in the w-plane. We show this is Figure 4.4.

We let $z = x + iy$ and $w = u + iv$. Then we can define our function as

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

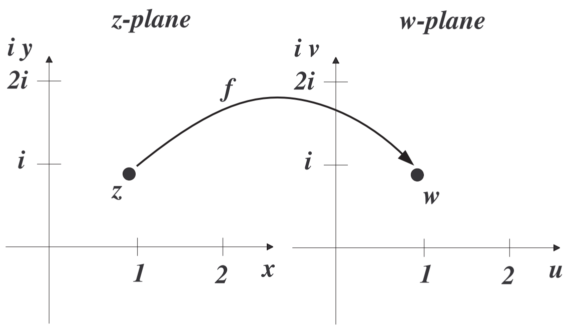

Figure 4.4: Defining a complex valued function on C.

We see that one can view this function as a function of z or a function of x and y. Often, we have an interest in writing out the real and imaginary parts of the function, which can be viewed as functions of two variables.

Example 1: $f(z) = z^2$.

For example, we can look at the simple function $f(z) = z^2$. It is a simple matter to determine the real and imaginary parts of this function. Namely, we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Therefore, we have that

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Example 2: $f(z) = e^z$.

For this case, we make use of Euler's Formula.

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned} \tag{4.12}$$

Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

Example 3: $f(z) = \ln z$.

In this case we make use of the polar form, $z = re^{i\theta}$. Our first thought would be to simply compute

$$\ln z = \ln r + i\theta.$$

However, the natural logarithm is multivalued, just like the nth root. Recalling that $e^{2\pi ik} = 1$ for k an integer, we have $z = re^{i(\theta+2\pi k)}$. Therefore,

$$\ln z = \ln r + i(\theta + 2\pi k), \qquad k = \text{ integer}.$$

The natural logarithm is a multivalued function. In fact there are an infinite number of values for a given z. Of course, this contradicts the definition of a function that you were first taught. Thus, one typically will only report the *principal value*, $\ln z = \ln r + i\theta$, for θ restricted to some interval of length 2π, such as $[0, 2\pi)$.

There are ways to handle multivalued functions. This involves introducing branch cuts and (at a more sophisticated level) Riemann surfaces. We will not go into these types of functions here, but refer the interested reader to other texts.

4.4 Complex Differentiation

Next we want to differentiate complex functions. We generalize our definition from single variable calculus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \qquad (4.13)$$

provided this limit exists.

The computation of this limit is similar to what we faced in multivariable calculus. Letting $\Delta z \to 0$ means that we get closer to z. There are many paths that one can take that will approach z. [See Figure 4.5.]

It is sufficient to look at two paths in particular. We first consider the path $y =$ constant. Such a path is shown in Figure 4.6. For this path,

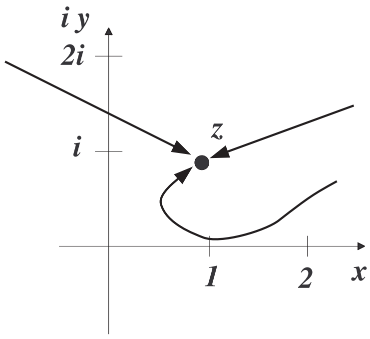

Figure 4.5: There are many paths that approach z as $\Delta z \to 0$.

$\Delta z = \Delta x + i\Delta y = \Delta x$, since y does not change along the path. The derivative, if it exists, is then computed as

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x,y) + iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}. \end{aligned} \tag{4.14}$$

The last two limits are easily identified as partial derivatives of real valued functions of two variables. Thus, we have shown that when $f'(z)$ exists,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{4.15}$$

A similar computation can be made if instead we take a path corresponding to $x =$ constant. In this case $\Delta z = i\Delta y$ and

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x,y) + iv(x,y))}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}. \end{aligned} \tag{4.16}$$

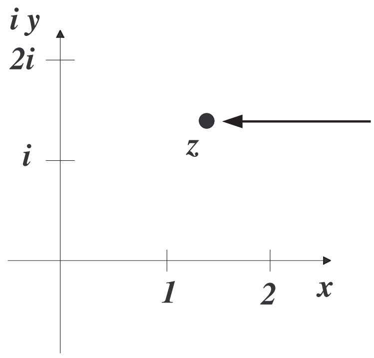

Figure 4.6: A path that approaches z with $y =$ constant.

Therefore,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{4.17}$$

We have found two different expressions for $f'(z)$ by following two different paths to z. If the derivative exists, then these two expressions must be the same. Equating the real and imaginary parts of these expressions, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{4.18}$$

These are known as the Cauchy-Riemann equations.

Theorem $f(z)$ is holomorphic (differentiable) if and only if the Cauchy-Riemann equations are satisfied.

Example 1: $f(z) = z^2$.

In this case we have already seen that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We first check the Cauchy-Riemann equations.

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= 2y = -\frac{\partial u}{\partial y}. \end{aligned} \tag{4.19}$$

Therefore, $f(z) = z^2$ is differentiable.

We can further compute the derivative using either Equation (4.15) or Equation (4.17). Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2z.$$

This result is not surprising.

Example 2: $f(z) = \bar{z}$.

In this case we have $f(z) = x - iy$. Therefore, $u(x, y) = x$ and $v(x, y) = -y$. But, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$. Thus, the Cauchy-Riemann equations are not satisfied and we conclude the $f(z) = \bar{z}$ is not differentiable.

4.5 Harmonic Functions and Laplace's Equation

Another consequence of the Cauchy-Riemann equations is that both $u(x, y)$ and $v(x, y)$ are *harmonic functions.* A real-valued function $u(x, y)$ is harmonic if it satisfies Laplace's equation in 2D, $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem f is differentiable if and only if u and v are harmonic functions.

This is easily proven using the Cauchy-Riemann equations.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\
&= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= -\frac{\partial^2 u}{\partial y^2}.
\end{aligned} \tag{4.20}$$

1. Is $u(x, y) = x^2 + y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 \neq 0.$$

   No, it is not.

2. Is $u(x, y) = x^2 - y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

   Yes, it is.

Given a harmonic function $u(x, y)$, can one find a function, $v(x, y)$, such $f(z) = u(x, y) + iv(x, y)$ is differentiable?

Example $u(x, y) = x^2 - y^2$ is harmonic, find $v(x, y)$ so that $u + iv$ is differentiable.

The Cauchy-Riemann equations tell us the following about the unknown function, $v(x, y)$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y,$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

We can integrate the first of these equations to obtain

$$v(x, y) = \int 2y\, dx = 2xy + c(y).$$

Here $c(y)$ is an arbitrary function of y. One can check to see that this works by simply differentiating the result with respect to x. However, the second equation must also hold. So, we differentiate our result with respect to y to find that

$$\frac{\partial v}{\partial y} = 2x + c'(y).$$

Since we were supposed to get $2x$, we have that $c'(y) = 0$. Thus, $c(y) = k$ is a constant.

We have just shown that we get an infinite number of functions,

$$v(x, y) = 2xy + k,$$

such that

$$f(z) = x^2 - y^2 + i(2xy + k)$$

is differentiable. In fact, for $k = 0$ this is nothing other than $f(z) = z^2$.

4.6 Complex Integration

In the last chapter we were introduced to functions of a complex variable. We have also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some real integrals. We will first turn to some definitions.

4.6.1 Complex Path Integrals

In this section we will investigate the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ, we would like to define the integral of $f(z)$ along Γ,

$$\int_\Gamma f(z)\, dz.$$

A natural procedure would be to work in real variables, by writing

$$\int_\Gamma f(z)\, dz = \int_\Gamma [u(x, y) + iv(x, y)]\,(dx + idy).$$

In order to carry out the integration, we then have to find a parametrization of the path and use methods from our third semester calculus class.

Before carrying this out with some examples, we first provide some definitions.

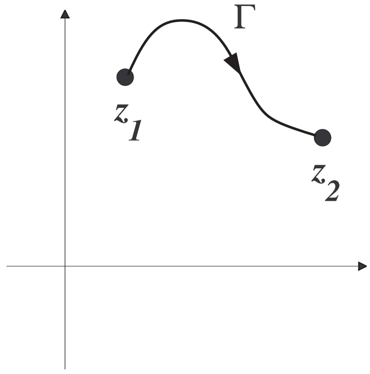

Figure 4.7: We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

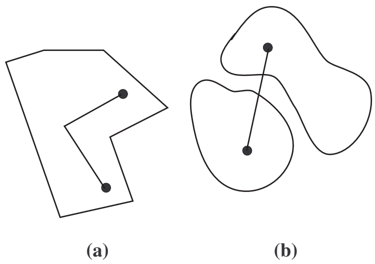

Figure 4.8: Examples of (a) a connected set and (b) a disconnected set.

Definition A set D is *connected* if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called *disconnected.* Examples are shown in Figure 4.8

Definition A set D is *open* if and only if for all z_0 in D there exists an open disk $|z - z_0| < \rho$ in D.

In Figure 4.9 we show a region with two disks. For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

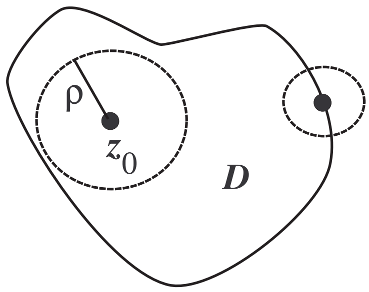

Figure 4.9: Locations of open disks inside and on the boundary of a region.

Definition D is called a *domain* if it is both open and connected.

Definition Let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \leq t \leq t_1$. Then

$$\int_\Gamma f(z)\,dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))]\,(\frac{dx}{dt} + i\frac{dy}{dt})dt. \tag{4.21}$$

It is easy to see how this definition arises. We see that we can write

$$f(z) = u(x, y) + iv(x, y)$$

and $z = x(t) + iy(t)$. Then,

$$dz = dx + idy = \frac{dx}{dt}dt + i\frac{dy}{dt}dt.$$

Inserting these expressions in the integral leads to the above definition.

This definition gives us a prescription for computing path integrals. Let's see how this works with a couple of examples.

Example 1 $\int_C z^2\,dz$, $C =$ the arc of the unit circle in the first quadrant as shown in Figure 4.10.

We first specify the parametrization. There are two ways we could do this. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

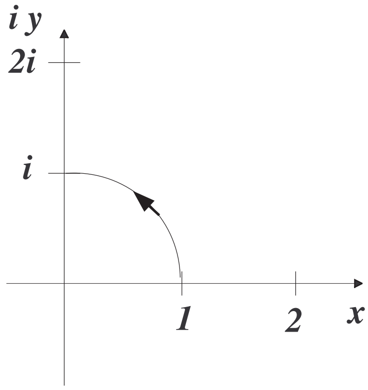

Figure 4.10: Contour for Example 1.

This is simply the result of using the polar forms

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{4.22}$$

for $r = 1$ and restricting θ to trace out a quarter of a circle. Then, we have

$$z = \cos\theta + i\sin\theta$$

and

$$dz = (-\sin\theta + i\cos\theta)d\theta.$$

Using this parametrization, the path integral becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)\,d\theta.$$

We can multiply this out and integrate, having to perform some trigonometric integrations:

$$\int_0^{\frac{\pi}{2}} [\sin^3\theta - 3\cos^2\theta\sin\theta + i(\cos^3\theta - 3\cos\theta\sin^2\theta)]\,d\theta.$$

While this is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ on C. So, $dz = ie^{i\theta}d\theta$. The integration then becomes

$$\int_C z^2\,d \;=\; \int_0^{\frac{\pi}{2}} (e^{i\theta})^2 ie^{i\theta}\,d\theta$$

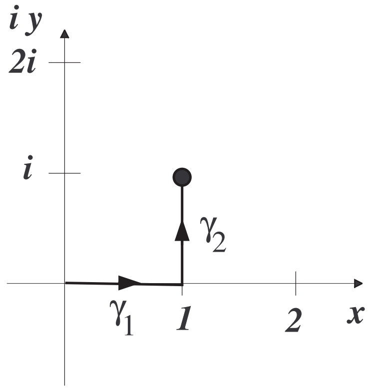

Figure 4.11: Contour for Example 2.

$$
\begin{aligned}
&= i \int_0^{\frac{\pi}{2}} e^{3i\theta}\, d\theta \\
&= \left[\frac{ie^{3i\theta}}{3i}\right]_0^{\frac{\pi}{2}} \\
&= -\frac{1+i}{3}.
\end{aligned}
\tag{4.23}
$$

Example 2 $\int_\Gamma z\, dz$, Γ is the path shown in Figure 4.11.

In this problem we have a path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding up the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 4.11.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0, 1]$. It is natural to take x as the parameter. So, $dz = dx$ and we have

$$\int_{\gamma_1} z\, dz = \int_0^1 x\, dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1 + iy$ for $y \in [0, 1]$. Thus, $dz = idy$. The integral becomes

$$\int_{\gamma_2} z\, dz = \int_0^1 (1 + iy)\, idy = i - \frac{1}{2}.$$

Combining these results, we have $\int_\Gamma z\, dz = \frac{1}{2} + (i - \frac{1}{2}) = i$.

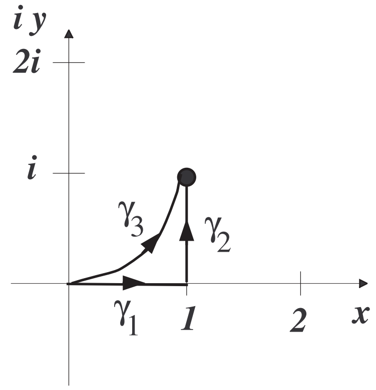

Figure 4.12: Contour for Example 3.

Example 3 $\int_{\gamma_3} z\,dz$, γ_3 is the path shown in Figure 4.12.

In this case we take a path from $z = 0$ to $z = 1 + i$ along a different path. Let $\gamma_3 = \{(x, y)|y = x^2, x \in [0, 1]\} = \{z|z = x + ix^2, x \in [0, 1]\}$. Then, $dz = (1 + 2ix)\ dx$.

The integral becomes

$$\begin{aligned}\int_{\gamma_1} z\,dz &= \int_0^1 (x + ix^2)(1 + 2ix)\,dx \\ &= \int_0^1 (x + 2ix^2 - 2x^3)\,dx = i.\end{aligned} \tag{4.24}$$

In the last case we found the same answer as in Example 2. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. We will now look into this notion of path independence.

Definition The integral $\int f(z)\,dz$ is *path independent* if

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all paths from z_1 to z_2.

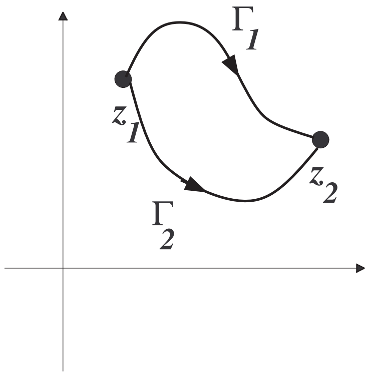

Figure 4.13: $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.

If $\int f(z)\,dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)\,dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)\,dz$. But first we have to define what we mean by a closed loop.

Definition A *simple closed contour* is a path satisfying

a The end point is the same as the beginning point. (This makes the loop closed.)

b The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over a closed loop C as shown in Figure 4.14. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite direction. Then,

$$\begin{aligned} \oint_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2^-} f(z)\,dz. \end{aligned} \tag{4.25}$$

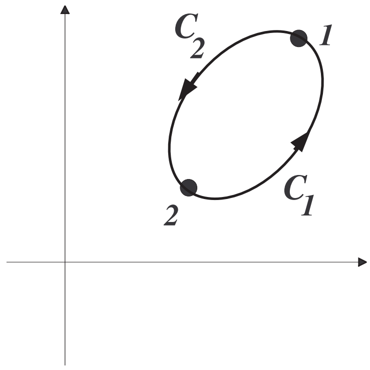

Figure 4.14: The integral $\oint_C f(z)\,dz$ around C is zero if the integral $\int_\Gamma f(z)\,dz$ is path independent.

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)\,dz = 0$.

4.6.2 Cauchy's Theorem

Next we want to investigate if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour.

Definition A curve with parametrization $(x(t), y(t))$ has a *normal* $(n_x, n_y) = (-\frac{dx}{dt}, \frac{dy}{dt})$.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points toward the interior of a closed curve. We will define a positively oriented contour as one that is traversed with the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

We now consider $\oint_C (u+iv)\,dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u+iv)\,dz &= \int_C (u+iv)(dx+i\,dy) \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \end{aligned} \tag{4.26}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Theorem : Green's Theorem in the Plane Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable functions on and inside the simple closed curve C. Denoting the enclosed region S, we have

$$\int_C P\,dx + Q\,dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy. \tag{4.27}$$

Using Green's Theorem to rewrite the first integral in (4.26), we have

$$\int_C u\,dx - v\,dy = \int\int_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dxdy.$$

If u and v satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

then the integrand in the double integral vanishes. Therefore,

$$\int_C u\,dx - v\,dy = 0.$$

In a similar fashion, one can show that

$$\int_C v\,dx + u\,dy = 0.$$

We have thus proven the following theorem:

Theorem If u and v satisfy the Cauchy-Riemann equations inside and on the simple closed contour C, then

$$\oint (u+iv)\,dz = 0. \tag{4.28}$$

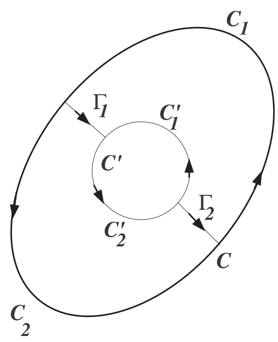

Figure 4.15: The contours needed to prove that $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$. when $f(z)$ is holomorphic between the contours C and C'.

Corollary $\oint_C f(z)\,dz = 0$ when f is differentiable in domain D with $C \subset D$.

Either one of these is referred to as **Cauchy's Theorem**.

Example Consider $\oint_{|z-1|=3} z^4\,dz$. Since $f(z) = z^4$ is differentiable inside the circle $|z-1| = 3$, this integral vanishes.

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

Theorem If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$.

We consider the two curves as shown in Figure 4.15. Now connect the two contours with contours Γ_1 and Γ_2 as shown. This splits C into contours C_1 and C_2 and C' into contours C_1' and C_2'.

$f(z)$ is differentiable inside the newly formed regions between the curves and the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that integrals of $f(z)$ over these regions are zero. Noting that integrations over contours opposite to the positive orientation are the negative of integrals in the opposite directions, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)\,dz + \int_{\Gamma_1} f(z)\,dz - \int_{C_1'} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

and

$$\int_{C_2} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{C_2'} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0.$$

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards and Γ_2. The second integral denotes the integration over the lower region, but going backwards over all contours except for C_2.

Adding these two equations, we have

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_1'} f(z)\,dz - \int_{C_2'} f(z)\,dz = 0.$$

Rewriting and noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$, as was to be proven.

Example Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2,2]\times[-2i,2i]$.

We can do this integral by computing four separate integrals over the sides of this square in the complex plane. One simply parametrizes each line segment, performs the integration and sums the four results. The last theorem tells us that we could instead integrate over a simpler contour by deforming the square into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the square and the circle. Thus, we can use the unit circle, as shown in Figure 4.16

The theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

The latter integral can be computed using the parametrization $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned} \oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned} \tag{4.29}$$

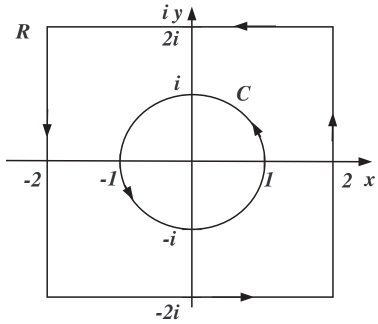

Figure 4.16: The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

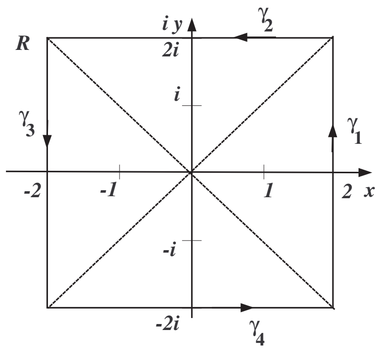

Figure 4.17: The contours used to compute $\oint_R \frac{dz}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ by deforming the original simple closed contour.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 4.17. The lower segment, γ_1 of the square can be simply parametrized by noting that along this segment $z = x - 2i$ for $x \in [-2, 2]$. Then, we have

$$\begin{aligned}
\oint_{\gamma_1} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\
&= \ln|x-2i|_{-2}^{2} \\
&= (\ln(2\sqrt{2}) + \frac{7\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{5\pi i}{4}) \\
&= \frac{\pi i}{2}. \qquad (4.30)
\end{aligned}$$

We note that the arguments of the logarithms are determined from the angles made by the diagonals provided in Figure 4.17. [The reader should verify this!]

Similarly, the integral along the top segment is computed as

$$\begin{aligned}
\oint_{\gamma_3} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\
&= \ln|x+2i|_{2}^{-2} \\
&= (\ln(2\sqrt{2}) + \frac{3\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{\pi i}{4}) \\
&= \frac{\pi i}{2}. \qquad (4.31)
\end{aligned}$$

The integral over the right side is

$$\begin{aligned}
\oint_{\gamma_2} \frac{dz}{z} &= \int_{-2}^{2} \frac{idy}{2+iy} \\
&= \ln|2+iy|_{-2}^{2} \\
&= (\ln(2\sqrt{2}) + \frac{\pi i}{4}) - (\ln(2\sqrt{2}) - \frac{\pi i}{4}) \\
&= \frac{\pi i}{2}. \qquad (4.32)
\end{aligned}$$

Finally, the integral over the left side is

$$\oint_{\gamma_4} \frac{dz}{z} = \int_{2}^{-2} \frac{idy}{-2+iy}$$

$$\begin{aligned} &= \ln|-2+iy|_{-2}^{2} \\ &= (\ln(2\sqrt{2})+\frac{5\pi i}{4})-(\ln(2\sqrt{2})+\frac{3\pi i}{4}) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{4.33}$$

Therefore, we have that

$$\oint_R \frac{dz}{z} = 4(\frac{\pi i}{2}) = 2\pi i$$

Note that we had obtained the same result using Cauchy's Theorem. However, it took quite a bit of computation!

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)\,dz = 0$ does not imply that $f(z)$ is differentiable. What we do have though, is **Morera's Theorem**.

Theorem Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. Then f is differentiable in D.

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

4.6.3 Analytic Functions and Cauchy's Integral Formula

In the previous section we saw that Cauchy's Theorem was useful for computing certain integrals without having to parametrize the contours, or deforming certain contours to simpler ones. The integrand needs to possess certain differentiability properties. In this section, we will generalize our integrand slightly so that we can integrate a larger family of complex functions. This will take the form of what is called Cauchy's Integral Formula, which is different from Cauchy's Theorem. We first need to explore the concept of analytic functions.

Definition $f(z)$ is *analytic* in D if for every open disk $|z-z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n.$$

This series converges uniformly and absolutely inside the circle of convergence, $|z - z_0| < R$, with radius of convergence R.

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C (z - z_0)^n \, dz$. As we will see in the homework exercises, these integrals evaluate to zero. Thus, we can show that for $f(z)$ analytic in D and any C lying in D, $\oint_C f(z) \, dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

Let's recall some manipulations from the study of series of real functions. The reader might need to recall how to sum geometric series. A review is given at the end of this section.

Example $f(z) = \frac{1}{1+z}$ for $z_0 = 0$.

This case is simple. $f(z)$ is the sum of a geometric series for $|z| < 1$. We have

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Thus, this series expansion converges inside the unit circle in the complex plane.

Example $f(z) = \frac{1}{1+z}$ for $z_0 = \frac{1}{2}$.

We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor' formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in powers of $z - \frac{1}{2}$. So, we rewrite the function in a form that has this term. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. We can get the

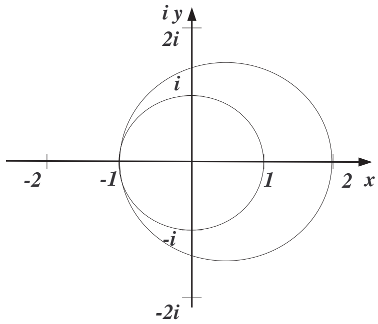

Figure 4.18: Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

denominator into such a form by factoring out the $\frac{3}{2}$. Then we would have

$$f(z) = \frac{2}{3}\frac{1}{1+\frac{2}{3}(z-\frac{1}{2})}.$$

The second factor now has the form $\frac{1}{1-r}$, which would be the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{2}{3}(z - \frac{1}{2})$ provided that $|r| < 1$. Therefore, we have found that

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty}\left[-\frac{2}{3}(z-\frac{1}{2})\right]^n$$

for

$$|-\frac{2}{3}(z-\frac{1}{2})| < 1.$$

This convergence interval can be rewritten as

$$|z-\frac{1}{2}| < \frac{3}{2}.$$

This is a circle centered at $z = \frac{1}{2}$ with radius $\frac{3}{2}$.

In Figure 5.6 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives us that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a region even further outside to the region $|z - \frac{1}{2}| < \frac{3}{2}$. We will see later that there are expansions outside of these regions, though some are expansions involving negative powers of $z - z_0$.

We now present the **Cauchy Integral Formula**.

Theorem Let $f(z)$ be analytic in $|z - z_0| < \rho$ and let C be the boundary (circle) of this disk. Then,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{4.34}$$

In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned} \frac{f(z)}{z - z_0} &= \frac{1}{z - z_0}\left[\sum_{n=0}^{\infty} c_n (z - z_0)^n\right] \\ &= \frac{1}{z - z_0}\left[c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots\right] \\ &= \frac{c_0}{z - z_0} + \underbrace{c_1 + c_2(z - z_0) + \ldots}_{\text{analytic function}}. \end{aligned} \tag{4.35}$$

As noted the integrand can be written as

$$\frac{f(z)}{z - z_0} = \frac{c_0}{z - z_0} + h(z),$$

where $h(z)$ is an analytic function, since it is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)\, dz = 0$. Noting also that $c_0 = f(z_0)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_C \left[\frac{c_0}{z - z_0} + h(z)\right] dz = f(z_0) \oint_C \frac{1}{z - z_0}\, dz.$$

We need only compute the integral $\oint_C \frac{1}{z - z_0}\, dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $|z - z_0| = \rho$ as shown in Figure 4.19. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

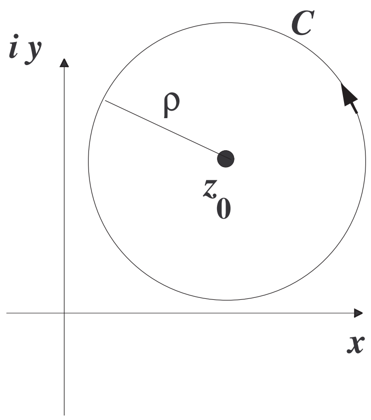

Figure 4.19: Circular contour used in proving the Cauchy Integral Formula.

(Note that this has the right complex modulus since $|e^{i\theta}| = 1$.) Then $dz = i\rho e^{i\theta} d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i.$$

Therefore,

$$\oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \oint_C \frac{1}{z - z_0}\, dz = 2\pi f(z_0),$$

as was to be shown.

Example - Using the Cauchy Integral Formula

We now compute $\oint_{|z|=4} \frac{\cos z}{z^2 - 6z + 5}\, dz$.

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2 - 6z + 5 = (z - 1)(z - 5)$. We next locate the locations of the zeroes of the denominator. In Figure 4.20 we see the contour and the points $z = 1$ and $z = 5$. The only point inside the region bounded by the contour is $z = 1$. Therefore, we can apply the Cauchy Integral Formula for $f(z) = \frac{\cos z}{z - 5}$ to the integral

$$\int_{|z|=4} \frac{\cos z}{(z - 1)(z - 5)}\, dz = \int_{|z|=4} \frac{f(z)}{(z - 1)}\, dz = 2\pi i f(1).$$

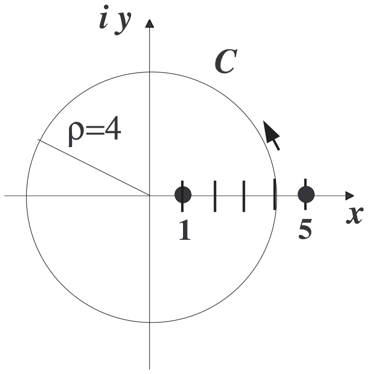

Figure 4.20: Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\, dz$.

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\, dz = -\frac{\pi i \cos(1)}{2}.$$

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z - z_0| < \rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz. \tag{4.36}$$

This can be proven following a derivation similar to that for the Cauchy Integral Formula. One needs to recall the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

We also need the following lemma

Lemma

$$\oint_C \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 0, & n \neq 0 \\ 2\pi i, & n = 0. \end{cases} \tag{4.37}$$

This will be a homework problem. The integrals are similar to the $n = 0$ case above.

4.6.4 Geometric Series

In this section we have made use of geometric series. A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^2 + \ldots + ar^n + \ldots . \tag{4.38}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

The sum of a geometric series, when it converges, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{4.39}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{4.40}$$

Subtracting these two equations, while noting the many cancellations, we have

$$(1 - r)s_n = a - ar^n. \tag{4.41}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1 - r^n)}{1 - r}. \tag{4.42}$$

Recalling that the sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting n get large in the partial sum (4.42) we need only evaluate $\lim_{n\to\infty} r^n$. From our special limits we know that this limit is zero for $|r| < 0$. Thus, we have the sum of the geometric series:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}. \tag{4.43}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

Example 2. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example 3. $\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n})$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed. Thus, we have

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

4.7 Complex Series Representations

Until this point we have only talked about series whose terms have nonnegative powers of $z - z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series were shown in Figure 5.6. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found.

Example 1. $f(z) = \frac{1}{1+z}$ for $|z| > 1$.

As before, we make use of the geometric series. Since $|z| > 1$, we instead rewrite our function as

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{1}{z}$. We note that $|z| > 1$ implies that $|r| < 1$. Thus, we have the geometric series

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}(-\frac{1}{z})^n.$$

This can be re-indexed as

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$, has negative powers of z.

Example 2. $f(z) = \frac{1}{1+z}$ for $|z - \frac{1}{2}| > \frac{3}{2}$.

In this case, we write as before

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

Instead of factoring out the $\frac{3}{2}$ we factor out the $(z - \frac{1}{2})$ term. Then, we obtain

$$f(z) = \frac{1}{1+z} = \frac{1}{(z - \frac{1}{2})}\frac{1}{(1 + \frac{3}{2}(z - \frac{1}{2})^{-1})}.$$

Again, we identify $a = 1$ and $r = -\frac{3}{2}(z - \frac{1}{2})^{-1}$. This leads to the series

$$f(z) = \frac{1}{z - \frac{1}{2}}\sum_{n=0}^{\infty}(-\frac{3}{2}(z - \frac{1}{2})^{-1})^n.$$

This converges for $|z - \frac{1}{2}| > \frac{3}{2}$ and can also be re-indexed to verify that this series involves negative powers of $z - \frac{1}{2}$.

This leads to the following theorem:

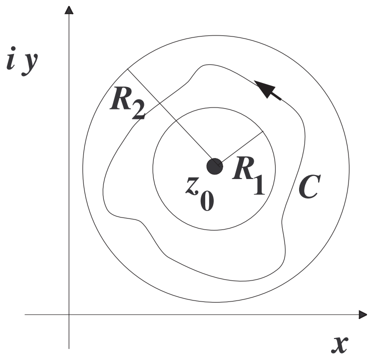

Figure 4.21: This figure shows an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

Theorem Let $f(z)$ be analytic in an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 4.21. Then,

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} b_j (z - z_0)^{-j},$$

with

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{j+1}}\, dz,$$

and

$$b_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-j+1}}\, dz.$$

The above series can be written in the more compact form

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j.$$

Such a series expansion is called a *Laurent* expansion.

Example Expand $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$.

Using partial fractions, we can write this as

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z| > 1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z| < 2$. This is done as follows. First, we write

$$\frac{1}{2+z} = \frac{1}{2[1-(-\frac{z}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{z}{2})^n.$$

Then we write

$$\frac{1}{1-z} = -\frac{1}{z[1-\frac{1}{z}]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\begin{aligned}\frac{1}{(1-z)(2+z)} &= \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}(-\frac{z}{2})^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6(2^n)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}. \qquad (4.44)\end{aligned}$$

4.8 Singularities and the Residue Theorem

In the last section we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$ where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a singularity of $g(z)$, as $g(z)$ is not defined there. As we saw from the proof of the Cauchy Integral Formula, $g(z)$ has a Laurent series expansion about $z = z_0$,

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2 + \ldots.$$

We will first classify singularities.

Definition A *singularity* of $f(z)$ is a point at which $f(z)$ fails to be analytic.

Typically these are isolated singularities. In order to classify the singularities of $f(z)$, we look at the principal part of the Laurent series: $\sum_{j=1}^{\infty} b_j(z-z_0)^{-j}$.

1. If $f(z)$ is bounded near z_0, then z_0 is a *removable singularity.*
2. If there are a finite number of terms in the principal part with the degree no less than $-n$, then one has a *pole of order n.*
3. If there are an infinite number of terms in the principal part, then one has an *essential singularity.*

Example: Removable $f(z) = \frac{\sin z}{z}$.

At first it looks like there is a possible singularity at $z = 0$. However, we know from the first semester of calculus that $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Furthermore, we can expand $\sin z$ about $z = 0$ and see that

$$\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{6!} + \ldots) = 1 - \frac{z^2}{6!} + \ldots.$$

Thus, there are only nonnegative powers in the series expansion. So, this is an example of a removable singularity.

Example: Poles $f(z) = \frac{e^z}{(z-1)^n}$.

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$ called a simple pole. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1}e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series expansion about $z = 1$ has only one term.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2}e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. This is a pole of order 2.

Example: Essential $f(z) = e^{\frac{1}{z}}$.

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0^\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two. In general, we can define poles of order k.

Definition $f(z)$ has a *pole of order* k at z_0 if and only if $(z - z_0)^k f(z)$ has a removable singularity at z_0, but $(z - z_0)^{k-1} f(z)$ for $k > 0$ does not.

Let $\phi(z) = (z - z_0)^k f(z)$ be analytic. Then it has a Taylor series expansion about z_0. As we had seen in Equation (4.36) in the last section, we can write the integral representation of the $(k-1)$st derivative of an analytic function as

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C \frac{\phi(z)}{(z - z_0)^k}\, dz.$$

Inserting the definition of $\phi(z)$ we then have

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C f(z)\, dz.$$

Dividing out the factorial factor and evaluating the $\phi(z)$ derivative, we have that

$$\begin{aligned} \frac{1}{2\pi i} \oint_C f(z)\, dz &= \frac{1}{(k-1)!} \phi^{(k-1)}(z_0) \\ &= \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]_{z=z_0}. \end{aligned} \tag{4.45}$$

We note that from the integral representation of the coefficients for a Laurent series, this gives c_{-1}, or b_1. This particular coefficient plays a role in helping to compute contour integrals surrounding poles. It is called the *residue* of $f(z)$ at $z = z_0$. Thus, for a pole of order k we define the residue as

$$\text{Res}[f(z); z_0] = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right] \tag{4.46}$$

Again, the residue is the coefficient of the $(z - z_0)^{-1}$ term.

Referring to the last derivation, we have shown that if $f(z)$ has one pole,z_0, of order k inside a simple closed contour C, then

$$\oint_C f(z)\, dz = 2\pi i \ \text{Res}[f(z); z_0] \tag{4.47}$$

Example $\oint_{|z|=1} \frac{dz}{\sin z}$.
We begin by looking for the singularities of the integrand, which is when $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \ldots$ are the singularities. However, only $z = 0$ lies inside the contour, as shown in Figure 4.22. We note further that $z = 0$ is a simple pole, since

$$\lim_{z \to 0} (z - 0) \frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2 - 1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 4.24. One could do a partial fraction decomposition and have two integrals with one pole each. However, in cases in which we have many poles, we can use the following theorem, known as the Residue Theorem.

Theorem Let $f(z)$ be a function which has poles z_j, $j = 1, \ldots, N$ inside a simple closed contour C and no other singularities in this region. Then,

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} \text{Res}[f(z); z_j], \tag{4.48}$$

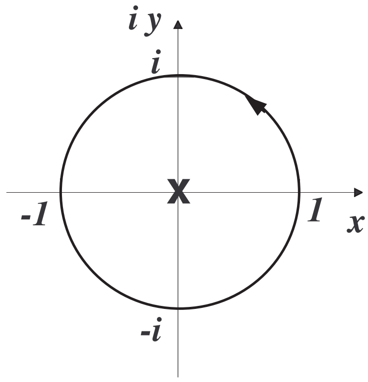

Figure 4.22: Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

where the residues are computed using Equation (4.46).

The proof of this theorem is based upon the contours shown in Figure 4.23. One constructs a new contour C' by encircling each pole, as shown in the figure. Then one connects a path from C to each circle. In the figure two paths are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)\,dz = \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have be cut out. Solving for $\oint_C f(z)\,dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)\,dz$ is $2\pi i$ times the sum of the residues.

Example 1 $\oint_{|z|=2} \frac{dz}{z^2-1}$.

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

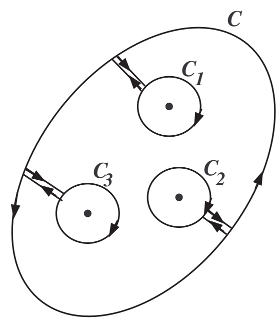

Figure 4.23: A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

In Figure 4.24 we plot the contour and the two poles, each denoted by an "x". Since both poles are inside the contour, we need to compute the residues for each one. They are both simple poles, so we have

$$\begin{aligned} \operatorname{Res}\left[\frac{1}{z^2-1}; z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1} \\ &= \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2}, \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} \operatorname{Res}\left[\frac{1}{z^2-1}; z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1} \\ &= \lim_{z\to -1}\frac{1}{z-1} = -\frac{1}{2}. \end{aligned} \tag{4.50}$$

Then,

$$\oint_{|z|=2}\frac{dz}{z^2-1} = 2\pi i\left(\frac{1}{2}-\frac{1}{2}\right) = 0.$$

Example 2 $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply methods from our calculus class to do this integral, attempting to write $1+\cos\theta = 2\cos^2\frac{\theta}{2}$. We do not, however, get very far.

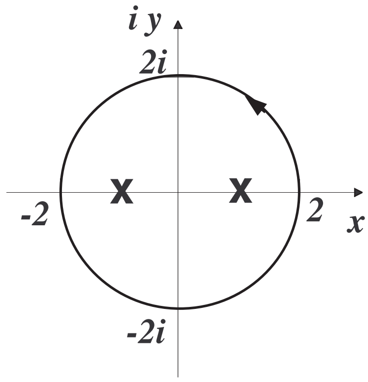

Figure 4.24: Contour for computing $\oint_{|z|=2} \frac{dz}{z^2-1}$.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta, \sin\theta)$, is to transform the integration to the complex plane through the transformation $z = e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i}{2}\left(z - \frac{1}{z}\right).$$

Under this transformation, $z = e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}\, d\theta = iz\, d\theta$, we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= -i \oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}(z^2+1)} \\ &= -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1}. \end{aligned} \tag{4.51}$$

We can apply the Residue Theorem to the resulting integral. The singularities occur for $z^2 + 4z + 1 = 0$. Using the quadratic formula, we have the roots $z = -2 \pm \sqrt{3}$. The locations of these poles are shown in Figure 4.25. Only $z = -2 + \sqrt{3}$ lies inside the integration

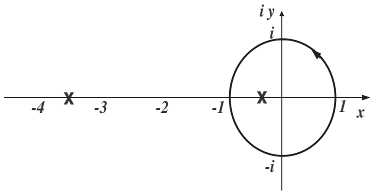

Figure 4.25: Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$\begin{aligned}
\operatorname{Res}[f(z); z=-2+\sqrt{3}] &= \lim_{z\to -2+\sqrt{3}} (z-(-2+\sqrt{3}))\frac{-2i}{z^2+4z+1} \\
&= -2i \lim_{z\to -2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))} \\
&= -2i \lim_{z\to -2+\sqrt{3}} \frac{1}{z-(-2-\sqrt{3})} \\
&= \frac{-2i}{-2+\sqrt{3}-(-2-\sqrt{3})} \\
&= \frac{-i}{\sqrt{3}} \\
&= \frac{-i\sqrt{3}}{3} \qquad (4.52)
\end{aligned}$$

Therefore, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i \left(\frac{-i\sqrt{3}}{3}\right) = \frac{2\pi\sqrt{3}}{3}. \qquad (4.53)$$

Example 3 $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz$.

In this example there are two poles $z = 1, -2$ inside the contour. $z = 1$ is a second order pole and $z = 2$ is a simple pole. See Figure 4.26. Therefore, we need the residues at each pole for $f(z) = \frac{z^2+1}{(z-1)^2(z+2)}$:

$$\operatorname{Res}[f(z); z=1] = \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2 \frac{z^2+1}{(z-1)^2(z+2)}\right]$$

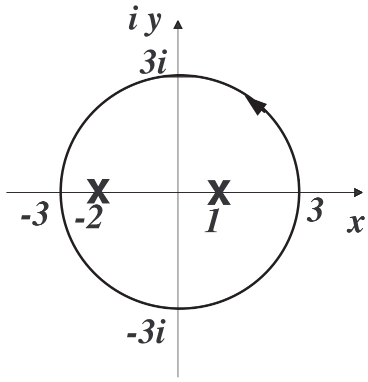

Figure 4.26: Contour for computing $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz$.

$$
\begin{aligned}
&= \lim_{z\to 1} \left(\frac{z^2+4z-1}{(z+2)^2} \right) \\
&= \frac{4}{9}. \qquad (4.54)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Res}[f(z); z=-2] &= \lim_{z\to -2} (z+2) \frac{z^2+1}{(z-1)^2(z+2)} \\
&= \lim_{z\to -2} \frac{z^2+1}{(z-1)^2} \\
&= \frac{5}{9}. \qquad (4.55)
\end{aligned}
$$

The evaluation of the integral is now $2\pi i$ times the sum of the residues:

$$
\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz = 2\pi i \left(\frac{4}{9} + \frac{5}{9} \right) = 2\pi i.
$$

4.9 Computing Real Integrals

As our final application of complex integration techniques, we will turn to the evaluation of infinite integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$. These types of integrals will appear later in the text and will help to tie in what seems to be a digression in our study of Fourier Analysis. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall that such integrals are improper integrals and you had seen them in your calculus classes. The way that one determines if such integrals exist, or converge, is to compute the integral using a limit:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

For example,

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \left(-\frac{2}{R}\right) = 0.$$

Similarly,

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{(-R)^2}{2}\right) = 0.$$

However, the integrals $\int_0^\infty x\,dx$ and $\int_{-\infty}^0 x\,dx$ do not exist. Note that

$$\int_0^{\infty} x\,dx = \lim_{R\to\infty} \int_0^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2}\right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

does not exist while $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the *Cauchy Principal Value Integral* and is denoted with either a P or PV prefix:

$$P\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{4.56}$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

We now proceed to the evaluation of such principal value integrals using complex integration methods. We want to evaluate the integral

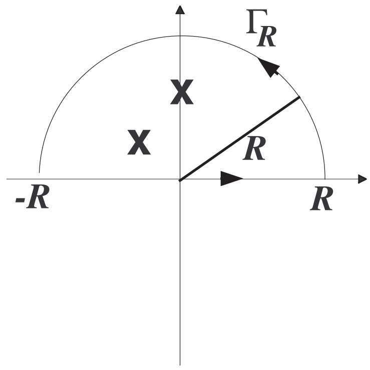

Figure 4.27: Contours for computing $P\int_{-\infty}^{\infty} f(x)\,dx$.

$\int_{-\infty}^{\infty} f(x)\,dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$). We then consider the integral $\int_{-R}^{R} f(x)\,dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 4.27. Note, a similar construction is sometimes needed to extend the integration into the lower half plane ($\text{Im}(z) < 0$) when $f(z)$ is analytic there.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz. \tag{4.57}$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)\,dx = \oint_{C} f(z)\,dz - \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz, \tag{4.58}$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We can parametrize the contour Γ_R using $z = Re^{i\theta}$. We assume that the function is bounded by some function

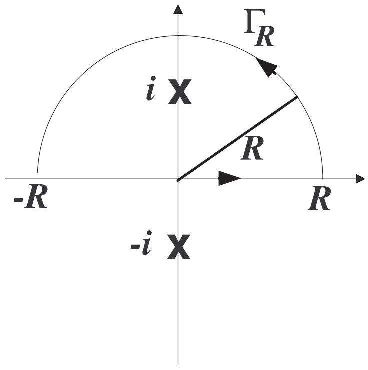

Figure 4.28: Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

of R. Denote this functon by $M(R)$. Thus, for $|f(z)| < M(R)$, we have

$$\begin{aligned} \int_{\Gamma_R} f(z)\,dz &\leq \left|\int_{\Gamma_R} f(z)\,dz\right| \\ &= \left|\int_0^{2\pi} f(Re^{i\theta})Re^{i\theta}\,d\theta\right| \\ &\leq R\int_0^{2\pi} \left|f(Re^{i\theta})\right|\,d\theta \\ &< RM(R)\int_0^{2\pi}\,d\theta \\ &= 2\pi RM(R). \end{aligned} \tag{4.59}$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$.

We show how this applies some examples.

Example 1 $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We already know how to do this integral from our calculus classes. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left(2\tan^{-1} R\right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 4.28 and the poles of the integrand are at $z = \pm i$.

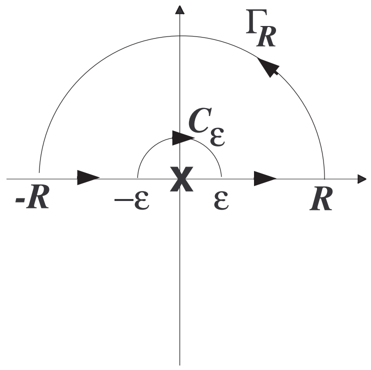

Figure 4.29: Contour for computing $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

We first note that $f(z) = \frac{1}{1+z^2}$ goes to zero fast enough on Γ_R as R gets large.

$$R|f(z)| = \frac{R}{|1+R^2e^{2i\theta}}| = \frac{R}{\sqrt{1+2R^2\cos\theta + R^4}}.$$

Thus, as $R \to \infty$, $R|f(z) \to 0$. So,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$\text{Res}[f(z); z = i] = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(\frac{1}{2i}\right) = \pi.$$

Example 2 $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

There are several new techniques that have to be introduced in order to carry out this integration. We need to handle the pole at $z = 0$ in a special way and we need something called Jordan's Lemma to guarantee that the contour on the contour Γ_R vanishes, since the integrand does not satisfy our previous bound condition.

For this example the integral is unbounded at $z = 0$. Constructing the contours as before, we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with our computation by carefully going around this pole with a small semicircle of radius ϵ, as indicated in Figure 4.29. Then our principal value integral computation becomes

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{\sin x}{x}\, dx + \int_{\epsilon}^{\infty} \frac{\sin x}{x}\, dx\right). \tag{4.60}$$

We will also need to rewrite the sine function in terms of exponentials in this integral.

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx\right). \tag{4.61}$$

We now employ **Jordan's Lemma**. If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}\, dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$. A similar result applies for $k < 0$, but one closes the contour in the lower half plane.

We now put these ideas together to compute the given integral. According to Jordan's lemma, we will need to compute the above exponential integrals using two different contours. We first consider $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$. We use the contour in Figure 4.29. Then we have

$$\oint_{C_R} \frac{e^{iz}}{z}\, dz = \int_{\Gamma_R} \frac{e^{iz}}{z}\, dz + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\, dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}\, dz.$$

The integral $\oint_{C_R} \frac{e^{iz}}{z}\, dz$ vanishes, since there are no poles enclosed in the contour! The integral over Γ_R will vanish as R gets large, according to Jordan's Lemma. The sum of the second and fourth integrals is the integral we seek as $\epsilon \to 0$ and $R \to \infty$.

The remaining integral around the small circle has to be done separately. We have

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \int_{\pi}^{0} \frac{\exp(i\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta = -\int_{0}^{\pi} i\exp(i\epsilon e^{i\theta})\, d\theta.$$

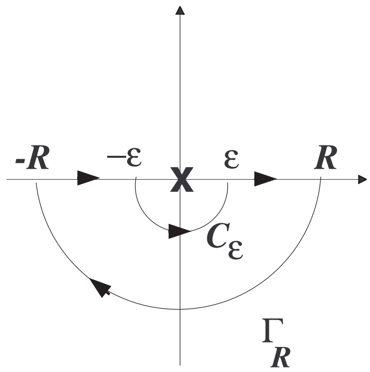

Figure 4.30: Contour in the lower half plane for computing $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$.

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz = -\pi i.$$

So far, we have that

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = -\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz = \pi i.$$

We can compute $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$ in a similar manner, being careful with the sign changes due to the orientations of the contours. In this case, we find the same value

$$P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx = \pi i.$$

Finally, we can compute the original integral as

$$\begin{aligned} P\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx - P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx\right) \\ &= \frac{1}{2i}(\pi i + \pi i) \\ &= \pi. \end{aligned} \tag{4.62}$$