2

Review of Sequences and Infinite Series

"Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth." Sherlock Holmes (by Sir Arthur Conan Doyle, 1859-1930)

In this chapter we will review and extend some of the concepts and definitions related to infinite series that you might have seen previously in your calculus class ([1], [2], [3]). Working with infinite series can be a little tricky and we need to understand some of the basics before moving on to the study of series of trigonometric functions in the next chapter.

The material in this chapter is a review of material covered in a standard course in calculus with some additional notions from advanced calculus. It is provided as a review before encountering the notion of Fourier series and their convergence as seen in the next chapter.

[1] George B. Thomas and Ross L. Finney. *Calculus and Analytic Geometry*. Addison Wesley Publishing Company, ninth edition, 1995

[2] James Stewart. *Calculus: Early Transcendentals*. Brooks Cole, sixth edition, 2007

[3] Wilfred Kaplan. *Advanced Calculus*. Addison Wesley Publishing Company, fourth edition, 1991

For example, one can show that the infinite series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$. However, the terms can be rearranged to give

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots = \frac{3}{2}\ln 2.$$

In fact, other rearrangements can be made to give any desired sum!

As we will see, $\ln(1+x) = x - \frac{x}{2} + \frac{x}{3} - \ldots$. So, inserting $x = 1$ yields the first result - at least formally! It was shown in Cowen, Davidson and Kaufman (in *The American Mathematical Monthly*, Vol. 87, No. 10. (Dec., 1980), pp. 817-819) that expressions like

$$\begin{aligned} f(x) &= \frac{1}{2}\left[\ln\frac{1+x}{1-x} + \ln(1-x^4)\right] \\ &= \frac{1}{2}\ln\left[(1+x)^2(1+x^2)\right] \end{aligned}$$

lead to alternate sums of the rearrangement of the alternating harmonic series.

Other problems with infinite series can occur. Try to sum the following infinite series to find that

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2} \sim 0.937548\ldots.$$

A sum of even as many as 10^7 terms only gives convergence to four or five decimal places.

The series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots, \quad x > 0$$

diverges for all x. So, you might think this divergent series is useless. However, truncation of this divergent series leads to an approximation of the integral

$$\int_0^\infty \frac{e^{-t}}{x+t}\, dt, \quad x > 0.$$

So, can we make sense out of any of these, or other manipulations, of infinite series? We will not answer these questions now, but we will go back and review what you have seen in your calculus classes.

2.1 Sequences of Real Numbers

WE FIRST BEGIN with the definitions for sequences and series of numbers.

Definition 2.1. A *sequence* is a function whose domain is the set of positive integers, $a(n)$, $n \in N$ $[N = \{1, 2, \ldots.\}]$.

Examples are

1. $a(n) = n$ yields the sequence $\{1, 2, 3, 4, 5, \ldots\}$
2. $a(n) = 3n$ yields the sequence $\{3, 6, 9, 12, \ldots\}$

However, one typically uses subscript notation and not functional notation: $a_n = a(n)$. We then call a_n the nth term of the sequence. Furthermore, we will denote sequences by $\{a_n\}_{n=1}^{\infty}$. Sometimes we will only give the nth term of the sequence and will assume that $n \in N$ unless otherwise noted.

Another way to define a particular sequence is recursively.

Definition 2.2. A *recursive sequence* is defined in two steps:

1. The value of first term (or first few terms) is given.
2. A rule, or recursion formula, to determine later terms from earlier ones is given.

Example 2.1. *A typical example is given by the Fibonacci*[4] *sequence. It can be defined by the recursion formula* $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ *and the starting values of* $a_1 = 0$ *and* $a_1 = 1$. *The resulting sequence is* $\{a_n\}_{n=1}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$. *Writing the general expression for the nth term is possible, but it is not as simply stated. Recursive definitions are often useful in doing computations for large values of* n.

2.2 Convergence of Sequences

NEXT WE ARE INTERESTED in the behavior of the sequence as n gets large. For the sequence defined by $a_n = n - 1$, we find the behavior as shown in Figure 2.1. Notice that as n gets large, a_n also gets large. This sequence is said to be divergent.

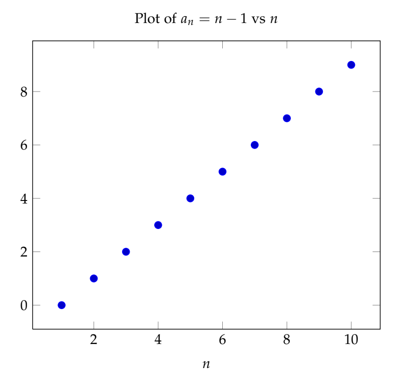

Figure 2.1: Plot of $a_n = n - 1$ for $n = 1 \ldots 10$.

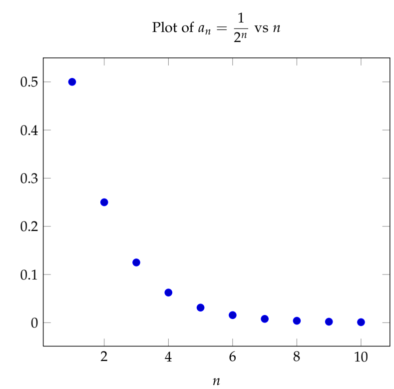

Figure 2.2: Plot of $a_n = \dfrac{1}{2^n}$ for $n = 1 \ldots 10$.

[4] Leonardo Pisano Fibonacci (c.1170-c.1250) is best known for this sequence of numbers. This sequence is the solution of a problem in one of his books: *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive* http://www-history.mcs.st-and.ac.uk

On the other hand, the sequence defined by $a_n = \frac{1}{2^n}$ approaches a limit as n gets large. This is depicted in Figure 2.2. Another related series, $a_n = \frac{(-1)^n}{2^n}$, is shown in Figure 2.3. This sequence is the alternating sequence $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\}$.

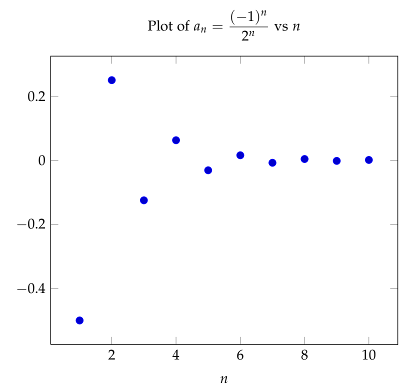

Figure 2.3: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$.

Definition 2.3. The sequence a_n *converges* to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a - L| < \epsilon.$$

If no such number exists, then the sequence is said to *diverge*.

In Figures 2.4-2.5 we see what this means. For the sequence given by $a_n = \frac{(-1)^n}{2^n}$, we see that the terms approach $L = 0$. Given an $\epsilon > 0$, we ask for what value of N the nth terms $(n > N)$ lie in the interval $[L - \epsilon, L + \epsilon]$. In these figures this interval is depicted by a horizontal band. We see that for convergence, sooner, or later, the tail of the sequence ends up entirely within this band.

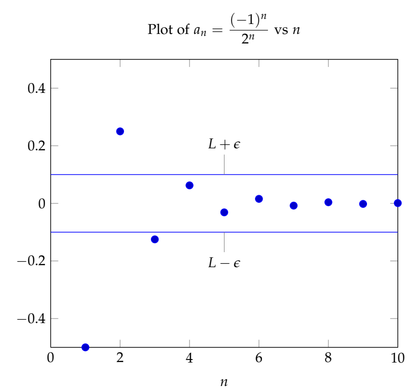

Figure 2.4: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.1$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 3$.

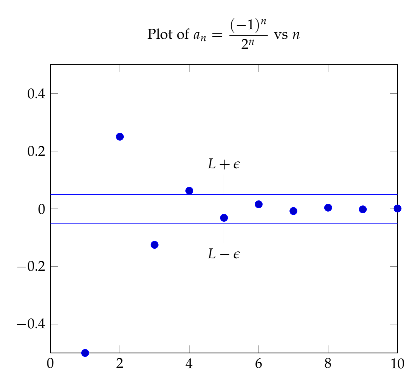

Figure 2.5: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.05$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 4$.

If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit L, then we write either $a_n \to L$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = L$. For example, we have already seen in Figure 2.3 that $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$.

2.3 *Limit Theorems*

ONCE WE HAVE DEFINED the notion of convergence of a sequence to some limit, then we can investigate the properties of the limits of sequences. Here we list a few general limit theorems and some special limits, which arise often.

Limit Theorem

Theorem 2.1. *Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ and a number k. Assume that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then we have*

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B.$
2. $\lim_{n\to\infty}(kb_n) = kB.$
3. $\lim_{n\to\infty}(a_n b_n) = AB.$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0.$

Some special limits are given next. These are generally first encountered in a second course in calculus.

Special Limits

Theorem 2.2. *The following are special cases:*

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$
2. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$
3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0.$
4. $\lim_{n\to\infty} x^n = 0, \quad |x| < 1.$
5. $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x.$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

The proofs generally are straight forward and found in beginning calculus texts. For example, one can prove the first limit by first realizing that $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$. This limit is indeterminate as $x \to \infty$ in its current form since the numerator and the denominator get large for large x. In such cases one employs L'Hopital's Rule: One computes

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

L'Hopital's Rule is used often in computing limits. We recall this powerful rule here as a reference for the reader.

Theorem 2.3. *Let c be a finite number or $c = \infty$. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

The second limit in Theorem 2.2 can be proven by first looking at

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Now, if $\lim_{n\to\infty} \ln f(n) = 0$, then $\lim_{n\to\infty} f(n) = e^0 = 1$. Thus proving the second limit.[5]

The third limit can be done similarly. The reader is left to confirm the other limits. We finish this section with a few selected examples.

[5] We should note that we are assuming something about limits of composite functions. Let a and b be real numbers. Suppose f and g are continuous functions, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to b} g(x) = b$, and $g(b) = a$. Then, $\lim_{x\to b} f(g(x)) = f(\lim_{x\to b} g(x)) = f(g(b)) = f(a)$.

Example 2.2. $\lim_{n\to\infty} \frac{n^2+2n+3}{n^3+n}$

Divide the numerator and denominator by n^2. Then

$$\lim_{n\to\infty} \frac{n^2 + 2n + 3}{n^3 + n} = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{n + \frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Another approach to this example is to consider the behavior of the numerator and denominator as $n \to \infty$. As n gets large, the numerator behaves like n^2, since $2n + 3$ becomes negligible for large enough n. Similarly, the denominator behaves like n^3 for large n. Thus,

$$\lim_{n\to\infty} \frac{n^2 + 2n + 3}{n^3 + n} = \lim_{n\to\infty} \frac{n^2}{n^3} = 0.$$

Example 2.3. $\lim_{n\to\infty} \frac{\ln n^2}{n}$

Rewriting $\frac{\ln n^2}{n} = \frac{2\ln n}{n}$, we find from identity 1 of the Theorem 2.2 that

$$\lim_{n\to\infty} \frac{\ln n^2}{n} = 2 \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Example 2.4. $\lim_{n\to\infty}(n^2)^{\frac{1}{n}}$

To compute this limit, we rewrite

$$\lim_{n\to\infty}(n^2)^{\frac{1}{n}} = \lim_{n\to\infty}(n)^{\frac{1}{n}}(n)^{\frac{1}{n}} = 1,$$

using identity 2 of the Theorem 2.2.

Example 2.5. $\lim_{n\to\infty}(\frac{n-2}{n})^n$

This limit can be written as

$$\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-2)}{n}\right)^n = e^{-2}.$$

Here we used identity 5 of the Theorem 2.2.

2.4 *Infinite Series*

IN THIS SECTION we investigate the meaning of infinite series, which are infinite sums of the form

$$a_1 + a_2 + a_2 + \ldots. \tag{2.1}$$

A typical example is the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots. \tag{2.2}$$

How would one evaluate this sum? We begin by just adding the terms. For example,

$$\begin{aligned} 1 + \frac{1}{2} &= \frac{3}{2}, \\ 1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{31}{16}, \end{aligned} \tag{2.3}$$

etc. The values tend to a limit. We can see this graphically in Figure 2.6.

In general, we want to make sense out of Equation (2.1). As with the example, we look at a sequence of partial sums . Thus, we consider the sums

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned} \tag{2.4}$$

There is story that's described in E.T. Bell's "Men of Mathematics" about Carl Friedrich Gauß (1777-1855). Gauß' third grade teacher needed to occupy the students, so she asked the class to sum the first 100 integers thinking that this would occupy the students for a while. However, Gauß was able to do so in practically no time. He recognized the sum could be written as $(1+100) + (2+99) + \ldots (50+51) = 50(101)$. $\sum_{k=1}^n k = \frac{n(n+1)}{2}$. This is an example of an arithmetic progression which is a finite sum of terms.

E. T. Bell. *Men of Mathematics*. Fireside Books, 1965

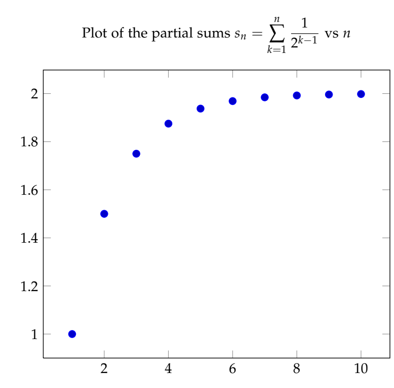

Figure 2.6: Plot of $s_n = \sum_{k=1}^n \frac{1}{2^{k-1}}$ for $n = 1\ldots10$.

etc. In general, we define the nth partial sum as

$$s_n = a_1 + a_2 + \ldots + a_n.$$

If the infinite series (2.1) is to make any sense, then the sequence of partial sums should converge to some limit. We define this limit to be the sum of the infinite series, $S = \lim_{n\to\infty} s_n$.

Definition 2.4. If the sequence of partial sums converges to the limit L as n gets large, then the infinite series is said to have the sum L.

We will use the compact summation notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots.$$

Here n will be referred to as the index and it may start at values other than $n = 1$.

2.5 Geometric Series

INFINITE SERIES OCCUR often in mathematics and physics. In this section we look at the special case of a geometric series. A geometric series is of the form

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots. \tag{2.5}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

Example 2.6. *For example,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

is an example of a geometric series. We can write this using summation notation,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} 1 \left(\frac{1}{2}\right)^n.$$

Thus, $a = 1$ is the first term and $r = \frac{1}{2}$ is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.

The sum of a geometric series, when it converges, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{2.6}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{2.7}$$

Subtracting these two equations, while noting the many cancelations, we have

$$\begin{aligned} (1-r)s_n &= (a + ar + \ldots + ar^{n-2} + ar^{n-1}) \\ &\quad -(ar + ar^2 + \ldots + ar^{n-1} + ar^n) \\ &= a - ar^n \\ &= a(1-r^n). \end{aligned} \tag{2.8}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1-r^n)}{1-r}. \tag{2.9}$$

Recall that the sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting n get large in the partial sum (2.9), we need only evaluate $\lim_{n\to\infty} r^n$. From our special limits we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series

The sum of the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1. \tag{2.10}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 2.7. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ *and* $r = \frac{1}{2}$. *Therefore, this infinite series converges and the sum is*

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure 2.6.

Example 2.8. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. *So,* $a = \frac{4}{9}$. *Also,* $r = \frac{1}{3}$. *So,*

$$S = \frac{\frac{4}{9}}{1-\frac{1}{3}} = \frac{2}{3}.$$

Example 2.9. $\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right)$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed [6]. *Thus, we have*

[6] A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent.

$$\sum_{n=1}^{\infty}\left(\frac{3}{2^n}-\frac{2}{5^n}\right)=\sum_{n=1}^{\infty}\frac{3}{2^n}-\sum_{n=1}^{\infty}\frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty}\left(\frac{3}{2^n}-\frac{2}{5^n}\right)=\frac{\frac{3}{2}}{1-\frac{1}{2}}-\frac{\frac{2}{5}}{1-\frac{1}{5}}=3-\frac{1}{2}=\frac{5}{2}.$$

Geometric series are important because they are easily recognized and summed. Other series, which can be summed, are special cases of Taylor series, as we will see later. Another type of series that can be summed is a *telescoping series* as seen in the next example.

Example 2.10. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ *The first few terms of this series are*

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots.$$

It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1},$$

then we find the partial sums can be written as

$$\begin{aligned} s_k &= \sum_{n=1}^{k}\frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right). \end{aligned} \tag{2.11}$$

We see that there are many cancelations of neighboring terms, leading to the series collapsing (like a telescope) to something manageable:

$$s_k=1-\frac{1}{k+1}.$$

Taking the limit as $k\to\infty$, we find $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$

2.6 *Convergence Tests*

Given a general infinite series, it would be nice to know if it converges, or not. Often, we are only interested in the convergence and not the actual sum as it is often difficult to determine the sum even if the series converges. In this section we will review some of the standard tests for convergence, which you should have seen in Calculus II.

First, we have the nth term divergence test. This is motivated by two examples:

1. $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \ldots$.
2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots$.

In the first example it is easy to see that each term is getting larger and larger, and thus the partial sums will grow without bound. In the second case, each term is bigger than one. Thus, the series will be bigger than adding the same number of ones as there are terms in the sum. Obviously, this series will also diverge.

The nth Term Divergence Test.

This leads to the nth Term Divergence Test:

Theorem 2.4. *If* $\lim a_n \neq 0$ *or if this limit does not exist, then* $\sum_n a_n$ *diverges.*

This theorem does not imply that just because the terms are getting smaller, the series will converge. Otherwise, we would not need any other convergence theorems.

For the next theorems, we will assume that the series has nonnegative terms.

1. **Comparison Test**

The Comparison Test.

The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ such that $a_n \leq c_n$ for all $n > N$ for some N. The series $\sum a_n$ diverges if there is a divergent series $\sum d_n$ such that $d_n \leq a_n$ for all $n > N$ for some N.

This is easily seen. In the first case, we have

$$a_n \leq c_n, \forall n > N.$$

Summing both sides of the inequality, we have

$$\sum_n a_n \leq \sum_n c_n.$$

If $\sum c_n$ converges, $\sum c_n < \infty$, the $\sum a_n$ converges as well. A similar argument applies for the divergent series case.

For this test one has to dream up a second series for comparison. Typically, this requires some experience with convergent series. Often it is better to use other tests first if possible.

Example 2.11. $\sum_{n=0}^{\infty} \frac{1}{3^n}$

We already know that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges. So, we compare these two series. In the above notation, we have* $a_n = \frac{1}{3^n}$ *and* $c_n = \frac{1}{2^n}$. *Since* $\frac{1}{2^n} \leq \frac{1}{3^n}$ *for* $n \geq 0$ *and* $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges, then* $\sum_{n=0}^{\infty} \frac{1}{3^n}$ *converges by the Comparison Test.*

2. **Limit Comparison Test**

The Limit comparison Test.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, then $\sum a_n$ and $\sum b_n$ converge together or diverge together.

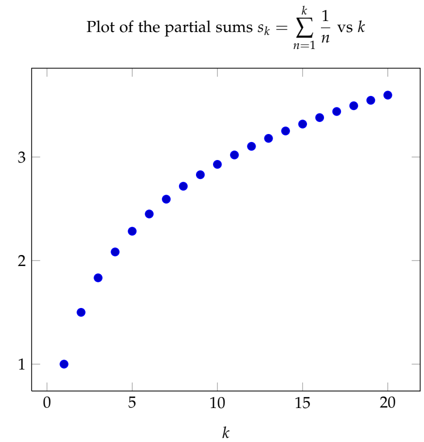

Figure 2.7: Plot of the partial sums, $s_k = \sum_{n=1}^{k} \frac{1}{n}$, for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Example 2.12. $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

In order to establish the convergence, or divergence, of this series, we look to see how the terms, $a_n = \frac{2n+1}{(n+1)^2}$, behave for large n. As n gets large, the numerator behaves like $2n$ and the denominator behaves like n^2. So, a_n behaves like $\frac{2n}{n^2} = \frac{2}{n}$. The factor of 2 does not really matter. So, will compare the infinite series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$. Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{(n+1)^2} = 2$. Thus, these two series both converge, or both diverge. If we knew the behavior of the second series, then we could draw a conclusion. Using the next test, we will prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ diverges by the Limit Comparison Test. Another example of this test is given in Example 2.14.

The Integral Test.

3. **Integral Test**

Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$. Then, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge. Here we mean that the integral converges or diverges as an improper integral.

Example 2.13. *The harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$*

We are interested in the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we saw in the Limit Comparison Test example. This infinite series is famous and is called the harmonic series. The plot of the partial sums is given in Figure 2.7. It appears that the series could possibly converge or diverge. It is hard to tell graphically.

In this case we can use the Integral Test. In Figure 2.8 we plot $f(x) = \frac{1}{x}$ and at each integer n we plot a box from n to $n+1$ of height $\frac{1}{n}$. We can see from the figure that the total area of the boxes is greater than the area under the curve. Since the area of each box is $\frac{1}{n}$, then we have that

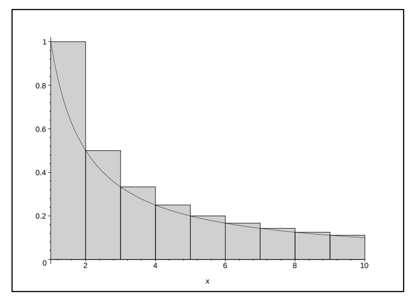

Figure 2.8: Plot of $f(x) = x$ and boxes of height $\frac{1}{n}$ and width 1.

$$\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

But, we can compute the integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{x\to\infty} (\ln x) = \infty.$$

Thus, the integral diverges. However, the infinite series is larger than this! So, the harmonic series diverges by the Integral Test.

p-series and p-test.

The Integral Test provides us with the convergence behavior for a class of infinite series called a p-series . These series are of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Recalling that the improper integrals $\int_1^{\infty} \frac{dx}{x^p}$ converge for $p > 1$ and diverge otherwise, we have the p-test :

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1$$

and diverges otherwise.

Example 2.14. $\sum_{n=1}^{\infty} \frac{n+1}{n^3-2}$.

We first note that as n gets large, the general term behaves like $\frac{1}{n^2}$ *since the numerator behaves like n and the denominator behaves like* n^3. *So, we expect that this series behaves like the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Thus, by the Limit Comparison Test,*

$$\lim_{n\to\infty} \frac{n+1}{n^3-2}(n^2) = 1.$$

These series both converge, or both diverge. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test since* $p = 2$. *Therefore, the original series converges.*

4. **Ratio Test**

The Ratio Test.

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the behavior of the infinite series can be determined from the conditions

$$\begin{aligned} \rho < 1, &\quad \text{converges} \\ \rho > 1, &\quad \text{diverges} \end{aligned}$$

Example 2.15. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$.

We compute

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \frac{10^n}{10^{n+1}} \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \frac{1}{10} \\ &= \frac{1}{10} < 1. \end{aligned} \tag{2.12}$$

Therefore, the series is said to converge by the Ratio Test.

Example 2.16. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

In this case we make use of the fact that[7] $(n+1)! = (n+1)n!$. *We compute*

[7] Note that the Ratio Test works when factorials are involved because using $(n+1)! = (n+1)n!$ helps to reduce the needed ratios into something manageable.

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1 \end{aligned} \tag{2.13}$$

This series also converges by the Ratio Test.

5. ***n*th Root Test**

The nth Root Test.

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} {a_n}^{1/n}$. Then the behavior of the infinite series can be determined using

$$\begin{aligned} \rho < 1, &\quad \text{converges} \\ \rho > 1, &\quad \text{diverges} \end{aligned}$$

Example 2.17. $\sum_{n=0}^{\infty} e^{-n}$.

We use the nth Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$. *Thus, this series converges by the nth Root Test.*[8]

[8] Note that the Root Test works when there are no factorials and simple powers are involved. In such cases special limit rules help in the evaluation.

Example 2.18. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$.

This series also converges by the nth Root Test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2^n} = 0 < 1.$$

We next turn to series which have both positive and negative terms. We can toss out the signs by taking absolute values of each of the terms. We note that since $a_n \le |a_n|$, we have

$$-\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n|.$$

If the sum $\sum_{n=1}^{\infty} |a_n|$converges, then the original series converges. This type of convergence is useful, because we can use the previous tests to establish convergence of such series. Thus, we say that a series *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series converges, but does not converge absolutely, then it is said to *converge conditionally.*

Conditional and absolute convergence.

Example 2.19. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$. *This series converges absolutely because* $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a p-series with* $p = 2$.

Finally, there is one last test that we recall from your introductory calculus class. We consider the alternating series, given by $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. The convergence of an alternating series is determined from **Leibniz's Theorem**[9] .

Convergence of alternating series.

[9] Gottfried Wilhelm Leibniz (1646-1716) developed calculus independently of Sir Isaac Newton (1643-1727).

Theorem 2.5. *The series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges if*

1. a_n*'s are positive.*
2. $a_n \ge a_{n+1}$ *for all* n.
3. $a_n \to 0$.

The first condition guarantees that we have alternating signs in the series. The next condition says that the magnitude if the terms gets smaller and the last condition imposes further that the terms approach zero.

Example 2.20. *The alternating harmonic series:* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

First of all, this series is an alternating series. The a_n's in Leibniz's Theorem are given by $a_n = \frac{1}{n}$. Condition 2 for this case is

$$\frac{1}{n} \geq \frac{1}{n+1}.$$

This is certainly true, as condition 2 just means that the terms are not getting bigger as n increases. Finally, condition 3 says that the terms are in fact going to zero as n increases. This is true in this example. Therefore, the the alternating harmonic series converges by Leibniz's Theorem. Note: The alternating harmonic series converges conditionally, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ gives the (divergent) harmonic series. So, the alternating harmonic series does not converge absolutely.

Example 2.21. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ *also passes the conditions of Leibniz's Theorem. It should be clear that the terms of this alternating series are getting smaller and approach zero. Furthermore, this series converges absolutely!*

2.7 *Sequences of Functions*

OUR IMMEDIATE GOAL is to prepare for studying Fourier series, which are series whose terms are functions. So, in this section we begin to discuss series of functions and the convergence of such series. Once more we will need to resort to the convergence of the sequence of partial sums. This means we really need to start with sequences of functions.

Definition 2.5. A *sequence of functions* is simply a set of functions $f_n(x)$, $n = 1, 2, \ldots$ defined on a common domain D. A frequently used example will be the sequence of functions $\{1, x, x^2, \ldots\}, x \in [-1, 1]$.

Evaluating each sequences of functions at a given value of x, we obtain a sequence of real numbers. As before, we can ask if this sequence converges. Doing this for each point in the domain D, we then ask if the resulting collection of limits defines a function on D. More formally, this leads us to the idea of pointwise convergence.

Definition 2.6. A sequence of functions f_n *converges pointwise* on D to a limit g if

Pointwise convergence.

$$\lim_{n \to \infty} f_n(x) = g(x)$$

for each $x \in D$. More formally, we write that

$$\lim_{n \to \infty} f_n = g \text{ (pointwise on } D)$$

if given $x \in D$ and $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N.$$

Example 2.22. *Consider the sequence of functions*

$$f_n(x) = \frac{1}{1+nx}, \quad |x| < \infty, \quad n = 1, 2, 3, \ldots.$$

The limits depends on the value of x. We consider two cases, $x = 0$ and $x \neq 0$.

1. *$x = 0$. Here $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1$.*
2. *$x \neq 0$. Here $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0$.*

Therefore, we can say that $f_n \to g$ pointwise for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2.14}$$

We also note that in Definition 2.6 N generally depends on both x and ϵ.

Example 2.23. *We consider the functions $f_n(x) = x^n, x \in [0,1], n = 1, 2, \ldots.$ We recall that the definition for pointwise convergence suggests that for each x we seek an N such that $|f_n(x) - g(x)| < \epsilon, \forall n \geq N$. This is not at first easy to see. So, we will provide some simple examples showing how N can depend on both x and ϵ.*

1. *$x = 0$. Here we have $f_n(0) = 0$ for all n. So, given $\epsilon > 0$ we seek an N such that $|f_n(0) - 0| < \epsilon, \forall n \geq N$. Inserting $f_n(0) = 0$, we have $0 < \epsilon$. Since this is true for all n, we can pick $N = 1$.*
2. *$x = \frac{1}{2}$. In this case we have $f_n(\frac{1}{2}) = \frac{1}{2^n}$, for $n = 1, 2, \ldots.$ As n gets large, $f_n \to 0$. So, given $\epsilon > 0$, we seek N such that $|\frac{1}{2^n} - 0| < \epsilon, \quad \forall n \geq N$. This means that $\frac{1}{2^n} < \epsilon$. Solving the inequality for n, we have $n > -\frac{\ln \epsilon}{\ln 2}$ We choose $N \geq -\frac{\ln \epsilon}{\ln 2}$. Thus, our choice of N depends on ϵ. For, $\epsilon = 0.1$, this gives*

   $$N \geq -\frac{\ln 0.1}{\ln 2} = \frac{\ln 10}{\ln 2} \approx 3.32.$$

   So, we pick $N = 4$ and we have $n > N = 4$.
3. *$x = \frac{1}{10}$. This can be examined like the last example. We have $f_n(\frac{1}{10}) = \frac{1}{10^n}$, for $n = 1, 2, \ldots.$ This leads to $N \geq -\frac{\ln \epsilon}{\ln 10}$ For $\epsilon = 0.1$, this gives $N \geq 1$, or $n > 1$.*
4. *$x = \frac{9}{10}$. This can be examined like the last two examples. We have $f_n(\frac{9}{10}) = \left(\frac{9}{10}\right)^n$, for $n = 1, 2, \ldots.$ So given an $\epsilon > 0$, we seek an N such that $\left(\frac{9}{10}\right)^n < \epsilon$ for all $n > N$. Therefore,*

   $$n > N \geq \frac{\ln \epsilon}{\ln\left(\frac{9}{10}\right)}.$$

   For $\epsilon = 0.1$, we have $N \geq 21.85$, or $n > N = 22$.

So, for these cases, we have shown that N can depend on both x and ϵ. These cases are shown in Figure 2.9.

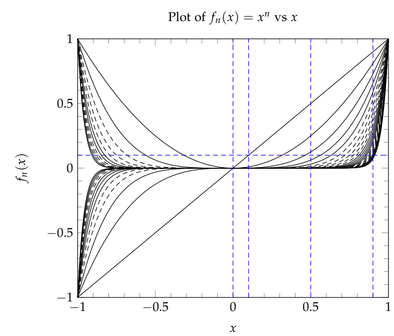

Figure 2.9: Plot of $f_n(x) = x^n$ showing how N depends on $x = 0, 0.1, 0.5, 0.9$ (the vertical lines) and $\epsilon = 0.1$ (the horizontal line). Look at the intersection of a given vertical line with the horizontal line and determine N from the number of curves not under the intersection point.

There are other questions that can be asked about sequences of functions. Let the sequence of functions f_n be continuous on D. If the sequence of functions converges pointwise to g on D then we can ask the following.

1. Is g continuous on D?
2. If each f_n is integrable on $[a, b]$, then does
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx?$$
3. If each f_n is differentiable at c, then does
$$\lim_{n\to\infty} f_n'(c) = g'(c)?$$

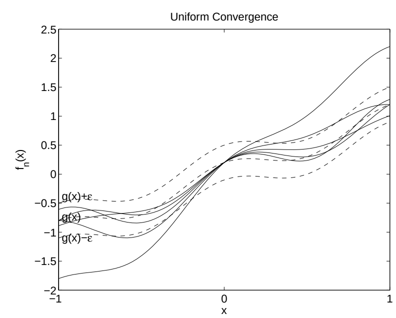

Figure 2.10: For uniform convergence, as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.

It turns out that pointwise convergence is not enough to provide an affirmative answer to any of these questions. Though we will not prove it here, what we will need is uniform convergence.

Definition 2.7. Consider a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on D. Let $g(x)$ be defined for $x \in D$. Then the sequence *converges uniformly* on D, or
$$\lim_{n\to\infty} f_n = g \text{ uniformly on } D,$$
if given $\epsilon > 0$, there exists an N such that
$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N \text{ and } \forall x \in D.$$

This definition almost looks like the definition for pointwise convergence. However, the seemingly subtle difference lies in the fact that N does not depend upon x. The sought N works for all x in the domain. As seen in Figure 2.10 as n gets large, $f_n(x)$ lies in the band $(g(x) - \epsilon, g(x) + \epsilon)$.

Example 2.24. $f_n(x) = x^n$*, for* $x \in [0, 1]$.
Note that in this case as n gets large, $f_n(x)$ *does not lie in the band* $(g(x) - \epsilon, g(x) + \epsilon)$*. This is displayed in Figure 2.11.*

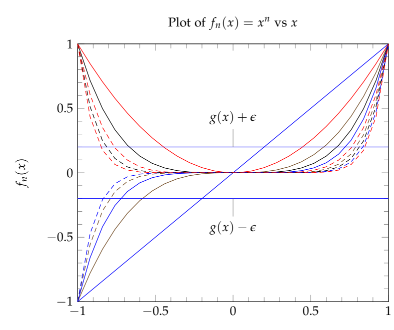

Figure 2.11: Plot of $f_n(x) = x^n$ on [-1,1] for $n = 1 \ldots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

Example 2.25. $f_n(x) = \cos(nx)/n^2$ *on* $[-1, 1]$.
For this example we plot the first several members of the sequence in Figure 2.12. We can see that eventually $(n \geq N)$ *members of this sequence do lie inside a band of width* ϵ *about the limit* $g(x) \equiv 0$ *for all values of* x*. Thus, this sequence of functions will converge uniformly to the limit.*

Finally, we should note that if a sequence of functions is uniformly convergent then it converges pointwise. However, the examples should bear out that the converse is not true.

2.8 *Infinite Series of Functions*

WE NOW TURN our attention to infinite series of functions, which will form the basis of our study of Fourier series. An infinite *series of functions* is given by $\sum_{n=1}^{\infty} f_n(x), \quad x \in D$. Using powers of x again, an example would be $\sum_{n=1}^{\infty} x^n, \quad x \in [-1,1]$. In order to investigate the convergence of this series; i.e., we would substitute values for x and determine if the resulting series of numbers converges. This means that we would need to consider the Nth partial sums

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Does this sequence of functions converge? We begin to answer this question by defining pointwise and uniform convergence.

Definition 2.8. $\sum f_j(x)$ *converges pointwise* to $f(x)$ on D if given $x \in D$, and $\epsilon > 0$, there exists and N such that

Pointwise convergence.

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$.

Definition 2.9. $\sum f_j(x)$ *converges uniformly* to $f(x)$ on D given $\epsilon > 0$, there exists and N such that

Uniform convergence.

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Again, we state without proof the following:

Uniform convergence give nice properties under some additional conditions, such as being able to integrate, or differentiate, term by term.

1. Uniform convergence implies pointwise convergence.
2. If f_n is continuous on D, and $\sum_n^{\infty} f_n$ converges uniformly to f on D, then f is continuous on D.
3. If f_n is continuous on $[a,b] \subset D$, $\sum_n^{\infty} f_n$ converges uniformly on D, and $\int_a^b f_n(x)\,dx$ exists, then
$$\sum_n^{\infty} \int_a^b f_n(x)\,dx = \int_a^b \sum_n^{\infty} f_n(x)\,dx = \int_a^b g(x)\,dx.$$
4. If f_n' is continuous on $[a,b] \subset D$, $\sum_n^{\infty} f_n$ converges pointwise to g on D, and $\sum_n^{\infty} f_n'$ converges uniformly on D, then $\sum_n^{\infty} f_n'(x) = \frac{d}{dx}(\sum_n^{\infty} f_n(x)) = g'(x)$ for $x \in (a,b)$.

Since uniform convergence of series gives so much, like term by term integration and differentiation, we would like to be able to recognize when we have a uniformly convergent series. One test for such convergence is the **Weierstraß M-Test**[10].

[10] Karl Theodor Wilhelm Weierstraß (1815-1897) was a German mathematician who may be thought of as the father of analysis.

Theorem 2.6. Weierstraß M-Test *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on D. If $|f_n(x)| \leq M_n$, for $x \in D$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly of D.*

Proof. First, we note that for $x \in D$,

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} M_n.$$

Thus, since by the assumption that $\sum_{n=1}^{\infty} M_n$ converges, we have that $\sum_{n=1}^{\infty} f_n$ converges absolutely on D. Therefore, $\sum_{n=1}^{\infty} f_n$ converges pointwise on D. So, let $\sum_{n=1}^{\infty} f_n = g$.

We now want to prove that this convergence is in fact uniform. Given $\epsilon > 0$, we need to find an N such that

$$|g(x) - \sum_{j=1}^{n} f_j(x)| < \epsilon$$

if $n \geq N$ for all $x \in D$.

So, for any $x \in D$,

$$\begin{aligned} |g(x) - \sum_{j=1}^{n} f_j(x)| &= |\sum_{j=1}^{\infty} f_j(x) - \sum_{j=1}^{n} f_j(x)| \\ &= |\sum_{j=n+1}^{\infty} f_j(x)| \\ &\leq \sum_{j=n+1}^{\infty} |f_j(x)|, \quad \text{by the triangle inequality} \\ &\leq \sum_{j=n+1}^{\infty} M_j. \end{aligned} \tag{2.15}$$

Now, the sum over the M_j's is convergent, so we can choose N such that

$$\sum_{j=n+1}^{\infty} M_j < \epsilon, \quad n \geq N.$$

Then, we have from above that

$$|g(x) - \sum_{j=1}^{n} f_j(x)| \leq \sum_{j=n+1}^{\infty} M_j < \epsilon$$

for all $n \geq N$ and $x \in D$. Thus, $\sum f_j \to g$ uniformly on D. □

We now given an example of how to use the Weierstraß M-Test.

Example 2.26. *We consider the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ defined on $[-\pi, \pi]$. Each term is bounded by $\left|\frac{\cos nx}{n^2}\right| = \frac{1}{n^2} \equiv M_n$. We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Thus, we can conclude that the original series converges uniformly, as it satisfies the conditions of the Weierstraß M-Test.*

2.9 Power Series

A TYPICAL EXAMPLE OF A SERIES of functions that the student has encountered in previous courses is the power series. Examples of such series were provided by Taylor and Maclaurin series.[11]

[11] Actually, what are now known as Taylor and Maclaurin series were known long before they were named. James Gregory (1638-1675) has been recognized for discovering Taylor series, which were later named after Brook Taylor (1685-1731) . Similarly, Colin Maclaurin (1698-1746) did not actually discover Maclaurin series, but because of his particular use of them.

Definition 2.10. A *power series* expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n (x-a)^n$.

For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

An example of such a power series is the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots . \tag{2.16}$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to which infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (2.10), we have the sum of the series is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide approximations to the function $(1-x)^{-1}$. If x is small enough, we can write

$$(1-x)^{-1} \approx 1 + x.$$

In Figure 2.13 we see that for small values of x these functions do agree.

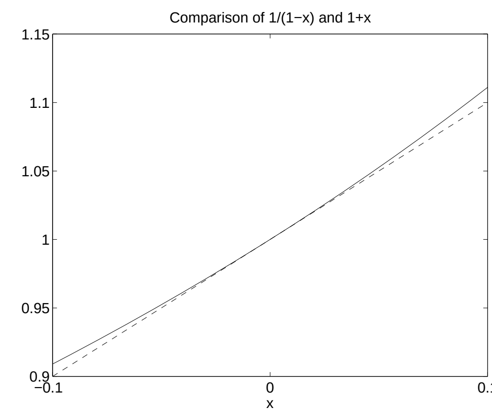

Figure 2.13: Comparison of $\frac{1}{1-x}$ (solid) to $1+x$ (dashed) for $x \in [-0.1, 0.1]$.

Of course, if we want better agreement, we select more terms. In Figure 2.14 we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure 2.15 shows that keeping only quadratic terms may not be good enough. Keeping the cubic terms gives better agreement over the interval.

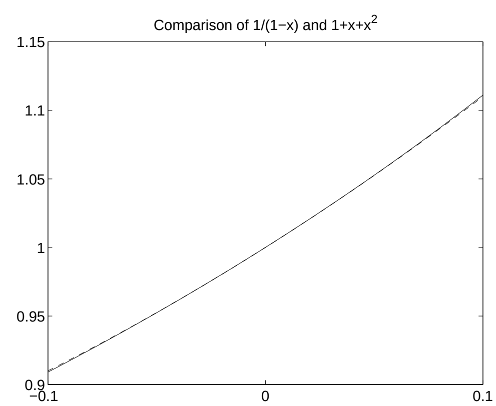

Figure 2.14: Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) for $x \in [-0.1, 0.1]$.

Finally, in Figure 2.16 we show the sum of the first 21 terms over the entire interval $[-1, 1]$. Note that there are problems with approximations near the endpoints of the interval, $x = \pm 1$.

Such polynomial approximations are called *Taylor polynomials.* Thus, $T_3(x) = 1 + x + x^2 + x^3$ is the third order Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$.

With this example we have seen how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with a function and then find its series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and Maclaurin series.

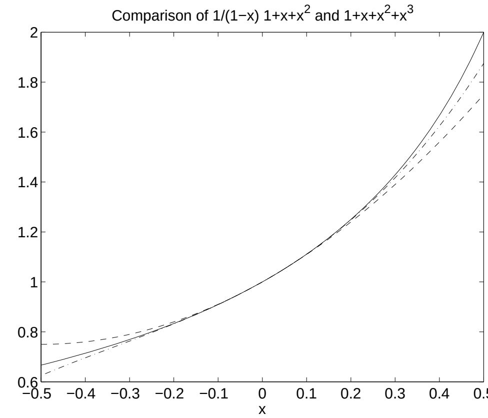

Figure 2.15: Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) and $1+x+x^2+x^3$ (dash-dot) for $x \in [-0.5, 0.5]$.

Taylor series expansion.

Definition 2.11. A *Taylor series* expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{2.17}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{2.18}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those Taylor series for which the expansion is about $x = 0$.

Maclaurin series expansion.

Definition 2.12. A *Maclaurin series* expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{2.19}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{2.20}$$

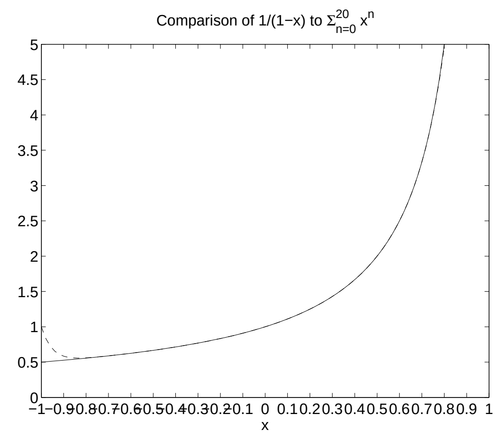

Figure 2.16: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{\infty} x^n$ for $x \in [-1, 1]$.

Example 2.27. *Expand $f(x) = e^x$ about $x = 0$.*

We begin by creating a table. In order to compute the expansion coefficients, c_n, we will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then we only need to evaluate the second column at $x = 0$ and divide by $n!$.

n	$f^{(n)}(x)$	c_n
0	e^x	$\frac{e^0}{0!} = 1$
1	e^x	$\frac{e^0}{1!} = 1$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$

Next, one looks at the last column and tries to determine some pattern so as to write down the general term of the series. If there is only a need to get a polynomial approximation, then the first few terms may be sufficient.

In this case, we have that the pattern is obvious: $c_n = \frac{1}{n!}$. *So,*

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 2.28. *Expand* $f(x) = e^x$ *about* $x = 1$.

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n(x-1)^n$. *We could create a table like the last example. In fact, the last column would have values of the form* $\frac{e}{n!}$. *(You should confirm this.) However, we could make use of the Maclaurin series expansion for* e^x *and get the result quicker. Note that* $e^x = e^{x-1+1} = ee^{x-1}$. *Now, apply the known expansion for* e^x *:*

$$e^x \sim e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \ldots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

Example 2.29. *Expand* $f(x) = \frac{1}{1-x}$ *about* $x = 0$.

This is the example with which we started our discussion. We set up a table again. We see from the last column that we get back our geometric series (2.16).

n	$f^{(n)}(x)$	c_n
0	$\frac{1}{1-x}$	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$\frac{3!}{3!} = 1$

So, we have found

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n.$$

We can replace $\sim$ by equality if we can determine the range of x-values for which the resulting infinite series converges. We will investigate such convergence shortly.

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided below.

Series Expansions You Should Know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (2.21)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (2.22)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (2.23)$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad (2.24)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad (2.25)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad (2.26)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n \quad (2.27)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (2.28)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad (2.29)$$

What is still left to be determined is for what values do such power series converge. The first five of the above expansions converge for all reals, but the others only converge for $|x| < 1$.

We consider the convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

Theorem 2.7. *If $\sum_{n=0}^{\infty} c_n (b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely for all x satisfying $|x-a| < |b-a|$.*

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may only converge at $x = a$.
2. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Interval and radius of convergence.

The number R is called the *radius of convergence* of the power series and $(a-R, a+R)$ is called the *interval of convergence.* Convergence at the endpoints of this interval has to be tested for each power series.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty} |c_n (x-a)^n| = \sum_{n=0}^{\infty} |c_n||x-$

$a|^n$. This is easily done using either the ratio test or the nth root test. We first identify the nonnegative terms $a_n = |c_n||x-a|^n$, using the notation from Section 2.4. Then we apply one of our convergence tests.

For example, the nth Root Test gives the convergence condition

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{|c_n|}|x-a| < 1.$$

Thus,

$$|x-a| < \left(\lim_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1} \equiv R.$$

This, R is the radius of convergence.

Similarly, we can apply the Ratio Test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}|x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \equiv R.$$

Example 2.30. $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$.

Since there is a factorial, we will use the Ratio Test with $a = 0$..

$$\rho = \lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}|x| = \lim_{n\to\infty} \frac{1}{n+1}|x| = 0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x. We also can say that the radius of convergence is infinite.

Example 2.31. $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$.

In this example we will use the nth Root Test with $a = 0$.

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.

In this example we could have also used the Ratio Test. Thus,

$$\rho = \lim_{n\to\infty} \frac{1}{1}|x| = |x| < 1.$$

We have obtained the same result as when we used the nth Root Test.

Example 2.32. $\sum_{n=1}^\infty \frac{3^n(x-2)^n}{n}$.

In this example, we have an expansion about $x = 2$. Using the nth Root Test we find that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $\left(2 - \frac{1}{3}, 2 + \frac{1}{3}\right) = \left(\frac{5}{3}, \frac{7}{3}\right)$.

As for the endpoints, we need to first test at $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n (\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = \frac{5}{3}$ we get the alternating harmonic series, which does converge. So, we have convergence on $[\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the left endpoint, $x = \frac{5}{3}$.

Example 2.33. *Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.*

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x-1$, since we are expanding about $x = 1$. This is easily done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. Factoring out a 3, we can rewrite this as a sum of a geometric series. Namely, we use the expansion for

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \ldots . \end{aligned} \tag{2.30}$$

and then we rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3[1+\frac{1}{3}(x-1)]} \\ &= \frac{1}{3}\frac{1}{1+\frac{1}{3}(x-1)}. \end{aligned} \tag{2.31}$$

Note that $f(x) = \frac{1}{3} g(\frac{1}{3}(x-1))$ for $g(z) = \frac{1}{1+z}$ So, the expansion becomes

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \ldots\right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \ldots .$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $|-\frac{1}{3}(x-1)| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for both $x = -2$ and $x = 4$. You should verify this!

As a final application, we can derive Euler's Formula ,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots.$$

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is an important formula and will be used throughout the text.

Next we note that each term has a power of i. The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots\}$. See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also have

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

Equating these two expressions, we are led to de Moivre's Formula, named after Abraham de Moivre (1667-1754),

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.32}$$

This formula is useful for deriving identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$ in Equation (2.32), we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Here we see elegant proofs of well known trigonometric identities. We will later make extensive use of these identities. Namely, you should know:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \tag{2.33}$$
$$\sin 2\theta = 2\sin\theta\cos\theta, \tag{2.34}$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \tag{2.35}$$
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta). \tag{2.36}$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Combining this with Euler's Formula, we have that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Trigonometric functions can be written in terms of complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^{x} + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.

Hyperbolic functions and trigonometric functions are intimately related.

$$\cos(ix) = \cosh x,$$
$$\sin(ix) = -i\sinh x.$$

2.10 Binomial Series

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$ in powers of a and b. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p. While the binomial expansion can be obtained using Taylor series, we will provide a more interesting derivation here to show that

$$(a+b)^p = \sum^{\infty} C_p^r a^{n-r} b^r, \tag{2.37}$$

where the C_p^r are called the *binomial coefficients*.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= a+b \\ (a+b)^2 &= a^2 + 2ab + b^2 \end{aligned}$$

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ &\cdots\end{aligned} \tag{2.38}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficient in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{array}{rccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{2.39}$$

This pattern is the famous Pascal's triangle.[12] There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3$:

$$\begin{array}{rccccccccc} n=2: & & 1 & & & 2 & & & 1 & \\ & & & \searrow & \swarrow & & \searrow & \swarrow & & \\ n=3: & 1 & & 3 & & & 3 & & & 1 \end{array} \tag{2.40}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{rccccccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{2.41}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5$: $1+4=5$, $4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired n. Fortunately, there is a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$.

[12] Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of number were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 (1, 11, 121, 1331, etc.).
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28... results in a perfect square.
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number.
- Sums along certain diagonals leads to the Fibonacci sequence.

Let $r = k - 1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \left(\begin{array}{c} n \\ r \end{array} \right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange r products of b with $n - r$ products of a. There are n slots to place the b's. For example, the $r = 2$ case for $n = 4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) a^{n-r} b^r. \qquad (2.42)$$

Now consider the geometric series $1 + x + x^2 + \ldots$. We have seen that such a series converges for $|x| < 1$, giving

$$1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$.

This is again a binomial to a power, but the power is not an integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (2.42).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \left(\begin{array}{c} p \\ r \end{array} \right) x^r. \qquad (2.43)$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\left(\begin{array}{c} -1 \\ 1 \end{array} \right) = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned}\begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \qquad (2.44)\end{aligned}$$

With this in mind we now state the theorem:

General Binomial Expansion

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (2.42). For p real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \qquad (2.45)$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \qquad (2.46)$$

Example 2.34. *Approximate* $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ *for* $v \ll c$. *This can be rewritten as*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

Using the binomial expansion for $p = -1/2$, *we have*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For celestial speeds, this is an appropriate approximation.

Example 2.35. *Small differences in large numbers.*

As an example, we could compute $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. Inserting these values into a scientific calculator, one finds that*

$$f(6378164, 1) = \sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7} \ m.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg h$. *Therefore, the computation of* $f(R,h)$ *depends on how many digits the computing device can handle.*

The best way to get an answer is to use the binomial approximation. Writing $x = \frac{h}{R}$*, we have*

$$\begin{aligned} f(R,h) &= \sqrt{R^2+h^2} - R \\ &= R\sqrt{1+x^2} - R \\ &\simeq R\left[1+\frac{1}{2}x^2\right] - R \\ &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8}\ m. \end{aligned} \tag{2.47}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we show how computations taking a more general form can be handled. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example 2.36. *Obtain an approximation to* $(a+b)^p$ *when* a *is much larger than* b*, denoted by* $a \gg b$*.*

If we neglect b *then* $(a+b)^p \simeq a^p$*. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of* b/a *of the first neglected term in this expansion?*

In order to do this we first divide out a *as*

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$*. According to what we have seen earlier, we can use the binomial expansion to write*

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\left(\begin{array}{c} p \\ r \end{array}\right)\left(\frac{b}{a}\right)^r. \tag{2.48}$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$*. Since* $a \gg b$*, most of these terms can be neglected. So, we can write*

$$\left(1+\frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$*, big-Oh notation, to indicate the size of the first neglected term. (This notation is formally defined in another section.)*

Summarizing, this then gives

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p$$

$$
\begin{aligned}
&= a^p \left(1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right)\right) \\
&= a^p + pa^p\frac{b}{a} + a^pO\left(\left(\frac{b}{a}\right)^2\right). \qquad (2.49)
\end{aligned}
$$

Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$*, with an error on the order of* b^2a^{p-2}*. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that* $(a+b)^p \simeq a^p$*, but it is not typically good enough in applications because the error in this case is of the order* ba^{p-1}*.*

Example 2.37. *Approximate* $f(x) = (a+x)^p - a^p$ *for* $x \ll a$*.*

In an earlier example we computed $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. We can make use of the binomial expansion to determine the behavior of similar functions in the form* $f(x) = (a+x)^p - a^p$*. Inserting the binomial expression into* $f(x)$*, we have as* $\frac{x}{a} \to 0$ *that*

$$
\begin{aligned}
f(x) &= (a+x)^p - a^p \\
&= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\
&= a^p\left[\frac{px}{a} + O\left(\left(\frac{x}{a}\right)^2\right)\right] \\
&= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a} \to 0. \qquad (2.50)
\end{aligned}
$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a} \to 0.$$

We could use this approximation to answer the original question by letting $a = R^2$*,* $x = 1$ *and* $p = \frac{1}{2}$*. Then, our approximation would be of order*

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2+1} - 6378164 \approx a^{p-1}px$$

where

$$a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

This is the same result we had obtained before.

2.11 *The Order of Sequences and Functions*

Often we are interested in comparing the rates of convergence of sequences or asymptotic behavior of functions. This is useful in approximation theory as we had seen in the last section. We begin with the comparison of sequences and introduce *big-Oh* notation. We will then extend this to functions of continuous variables.

Definition 2.13. Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Then if there are numbers N and K (independent of N) such that

$$\left|\frac{a_n}{b_n}\right| < K \quad \text{whenever} \quad n > N,$$

then we say that a_n is *of the order of* b_n. We write this as

$$a_n = O(b_n) \quad \text{as} \quad n \to \infty$$

and say a_n is "big O" of b_n.

Example 2.38. *Consider the sequences given by* $a_n = \frac{2n+1}{3n^2+2}$ *and* $b_n = \frac{1}{n}$.

In this case we consider the ratio,

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{2n+1}{3n^2+2}}{\frac{1}{n}}\right| = \left|\frac{2n^2+n}{3n^2+2}\right|.$$

We want to find a bound on the last expression as n gets large. We divide the numerator and denominator by n^2 *and find that*

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{2+1/n}{3+2/n^2}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right|.$$

The last expresssion is largest for $n = 1$. *This gives*

$$\left|\frac{a_n}{b_n}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right| \le \frac{2}{3}\left|\frac{1+1/2}{1+2/3}\right| = \frac{9}{10}.$$

Thus, for $n > 1$, *we have that*

$$\left|\frac{a_n}{b_n}\right| \le \frac{9}{10} < 1 \equiv K.$$

We then conclude from Definition 2.13 that

$$a_n = O(b_n) = O\left(\frac{1}{n}\right).$$

In practice one is often given a sequence like a_n, but the second simpler sequence needs to be found by looking at the large n behavior of a_n.

Referring to the last example, we are given $a_n = \frac{2n+1}{3n^2+2}$. We look at the large n behavior. The numerator behaves like $2n$ and the denominator behaves like $3n^2$. Thus, $a_n = \frac{2n+1}{3n^2+2} \sim \frac{2n}{3n^2} = \frac{2}{3n}$ for large n. Therefore, we say that $a_n = O(\frac{1}{n})$ for large n. Note that we are only interested in the n-dependence and not the multiplicative constant since $\frac{1}{n}$ and $\frac{2}{3n}$ have the same growth rate.

In a similar way, we can compare functions. We modify our definition of big-Oh for functions of a continuous variable.

Definition 2.14. $f(x)$ is *of the order of* $g(x)$, or $f(x) = O(g(x))$ as $x \to x_0$ if

$$\lim_{x\to x_0} \left| \frac{f(x)}{g(x)} \right| < K$$

for some K independent of x_0.

Example 2.39. *Show that*

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \quad \textit{as } x \to 0.$$

This should be apparent from the Taylor series expansion for $\cos x$,

$$\cos x = 1 - \frac{x^2}{2} + O(x^4) \textit{ as } x \to 0.$$

However, we will show that $\cos x - 1 + \frac{x^2}{2}$ *is of the order of* $O(x^4)$ *using the above definition.*

We need to compute

$$\lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right|.$$

The numerator and denominator separately go to zero, so we have an indeterminate form. This suggests that we need to apply L'Hopital's Rule. In fact, we apply it several times to find that

$$\begin{aligned} \lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| &= \lim_{x\to 0} \left| \frac{-\sin x + x}{4x^3} \right| \\ &= \lim_{x\to 0} \left| \frac{-\cos x + 1}{12x^2} \right| \\ &= \lim_{x\to 0} \left| \frac{\sin x}{24x} \right| = \frac{1}{24}. \end{aligned}$$

Thus, for any number $K > \frac{1}{24}$, *we have that*

$$\lim_{x\to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| < K.$$

We conclude that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \quad \textit{as } x \to 0.$$

Example 2.40. *Determine the order of* $f(x) = (x^3 - x)^{1/3} - x$ *as* $x \to \infty$. *We can use a binomial expansion to write the first term in powers of* x. *However, since* $x \to \infty$, *we want to write* $f(x)$ *in powers of* $\frac{1}{x}$, *so that we can neglect higher order powers. We can do this by first factoring out the* x^3 :

$$\begin{aligned} (x^3 - x)^{1/3} - x &= x\left(1 - \frac{1}{x^2}\right)^{1/3} - x \\ &= x\left(1 - \frac{1}{3x^2} + O\left(\frac{1}{x^4}\right)\right) - x \\ &= -\frac{1}{3x} + O\left(\frac{1}{x^3}\right). \end{aligned} \tag{2.51}$$

Now we can see from the first term on the right that $(x^3 - x)^{1/3} - x = O\left(\frac{1}{x}\right)$ *as* $x \to \infty$.

Problems

1. For those sequences that converge, find the limit $\lim_{n\to\infty} a_n$.

a. $a_n = \frac{n^2+1}{n^3+1}$.

b. $a_n = \frac{3n+1}{n+2}$.

c. $a_n = \left(\frac{3}{n}\right)^{1/n}$.

d. $a_n = \frac{2n^2+4n^3}{n^3+5\sqrt{2+n^6}}$.

e. $a_n = n \ln\left(1 + \frac{1}{n}\right)$.

f. $a_n = n \sin\left(\frac{1}{n}\right)$.

g. $a_n = \frac{(2n+3)!}{(n+1)!}$.

2. Find the sum for each of the series:

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 3}{4^n}$.

b. $\sum_{n=2}^{\infty} \frac{2}{5^n}$.

c. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$.

d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

3. Determine if the following converge, or diverge, using one of the convergence tests. If the series converges, is it absolute or conditional?

a. $\sum_{n=1}^{\infty} \frac{n+4}{2n^3+1}$.

b. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

c. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

d. $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2n^2-3}$.

e. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

f. $\sum_{n=1}^{\infty} \frac{100^n}{n^{200}}$.

g. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+3}$.

h. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{5n}}{n+1}$.

4. Do the following:

a. Compute: $\lim_{n\to\infty} n \ln\left(1 - \frac{3}{n}\right)$.

b. Use L'Hopital's Rule to evaluate $L = \lim_{x\to\infty} \left(1 - \frac{4}{x}\right)^x$. Hint: Consider $\ln L$.

c. Determine the convergence of $\sum_{n=1}^{\infty}\left(\frac{n}{3n+2}\right)^{n^2}$.

d. Sum the series $\sum_{n=1}^{\infty}\left[\tan^{-1} n - \tan^{-1}(n+1)\right]$ by first writing the Nth partial sum and then computing $\lim_{N\to\infty} s_N$.

5. Consider the sum $\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}$.

a. Use an appropriate convergence test to show that this series converges.

b. Verify that

$$\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2} - \frac{n}{n+1}\right).$$

c. Find the nth partial sum of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2} - \frac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

6. Recall that the alternating harmonic series converges conditionally.

a. From the Taylor series expansion for $f(x) = \ln(1+x)$, inserting $x = 1$ gives the alternating harmonic series. What is the sum of the alternating harmonic series?

Since the alternating harmonic series does not converge absolutely, then a rearrangement of the terms in the series will result in series whose sums vary. One such rearrangement in alternating p positive terms and n negative terms leads to the following sum[13]:

$$\begin{aligned}\frac{1}{2}\ln\frac{4p}{n} &= \underbrace{\left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)}_{p \text{ terms}} - \underbrace{\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)}_{n \text{ terms}} \\ &\quad + \underbrace{\left(\frac{1}{2p+1}+\cdots+\frac{1}{4p-1}\right)}_{p \text{ terms}} - \underbrace{\left(\frac{1}{2n+2}+\cdots+\frac{1}{4n}\right)}_{n \text{ terms}} + \cdots.\end{aligned}$$

[13] This is discussed by Lawrence H. Riddle in the *Kenyon Math. Quarterly*, 1(2), 6-21.

Find rearrangements of the alternating harmonic series to give the following sums; i.e., determine p and n for the given expression and write down the above series explicitly; i.e, determine p and n leading to the following sums.

b. $\frac{5}{2}\ln 2$.

c. $\ln 8$.

d. 0.

e. A sum that is close to π.

7. Determine the radius and interval of convergence of the following infinite series:

a. $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{n}$.

b. $\sum_{n=1}^{\infty} \frac{x^n}{2^n n!}$.

c. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{5}\right)^n$

d. $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt{n}}$.

8. Find the Taylor series centered at $x = a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

a. $f(x) = \sinh x$, $a = 0$.

b. $f(x) = \sqrt{1+x}$, $a = 0$.

c. $f(x) = xe^x$, $a = 1$.

d. $f(x) = \frac{x-1}{2+x}$, $a = 1$.

9. Test for pointwise and uniform convergence on the given set. [The Weierstraß M-Test might be helpful.]

a. $f(x) = \sum_{n=1}^{\infty} \frac{\ln nx}{n^2}$, $x \in [1,2]$.

b. $f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \cos \frac{x}{2^n}$ on R.

10. Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

b. From this result, derive Gregory's series for π by inserting an appropriate value for x in the series expansion for $\tan^{-1} x$.

11. Use deMoivre's Theorem to write $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$. Hint: Focus on the imaginary part of $e^{3i\theta}$.

12. Evaluate the following expressions at the given point. Use your calculator and your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $\frac{1}{\sqrt{1+x^3}} - \cos x^2$ at $x = 0.015$.

b. $\ln\sqrt{\frac{1+x}{1-x}} - \tan x$ at $x = 0.0015$.

c. $f(x) = \frac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.

d. $f(R,h) = R - \sqrt{R^2 + h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.

e. $f(x) = 1 - \frac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.

13. Determine the order, $O(x^p)$, of the following functions. You may need to use series expansions in powers of x when $x \to 0$, or series expansions in powers of $1/x$ when $x \to \infty$.

a. $\sqrt{x(1-x)}$ as $x \to 0$.

b. $\frac{x^{5/4}}{1-\cos x}$ as $x \to 0$.

c. $\frac{x}{x^2-1}$ as $x \to \infty$.

d. $\sqrt{x^2+x} - x$ as $x \to \infty$.