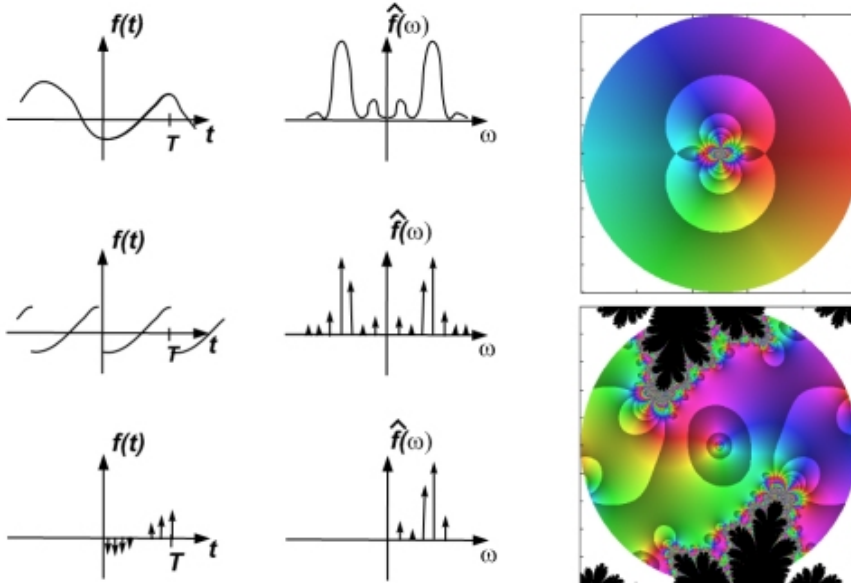
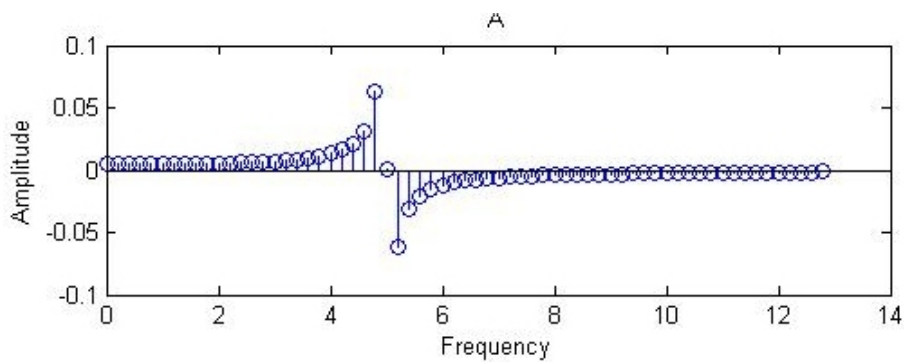


RUSSELL L. HERMAN

# AN INTRODUCTION TO FOURIER AND COMPLEX ANALYSIS WITH APPLICATIONS TO THE SPECTRAL ANALYSIS OF SIGNALS



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*Dedicated to those students who have endured  
the various editions of AN INTRODUCTION  
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