
A Review of Sequences and Infinite Series

In this chapter we will review and extend some of the concepts and definitions related to infinite series that you might have seen previously in your calculus class. Working with infinite series can be a little tricky and we need to understand some of the basics before moving on to the study of series of trigonometric functions in the next chapter.

For example, one can show that the infinite series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$. However, the terms can be rearranged to give

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots = \frac{3}{2} \ln 2.$$

In fact, other rearrangements can be made to give any desired sum!

Other problems with infinite series can occur. Try to sum the following infinite series to find that

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2} \sim 0.937548 \dots$$

A sum of even as many as 10^7 terms only gives convergence to four or five decimal places.

The series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots, \quad x > 0$$

diverges for all x . So, you might think this divergent series is useless. However, truncation of this divergent series leads to an approximation of the integral

$$\int_0^{\infty} \frac{e^{-t}}{x+t} dt, \quad x > 0.$$

So, can we make sense out of any of these, or other manipulations of infinite series? We will not answer all of these questions now, but we will go back and review what you have seen in your calculus classes.

2.1 Sequences of Real Numbers

We first begin with the definitions for sequences and series of numbers.

Definition 2.1. *A sequence is a function whose domain is the set of positive integers.*

Examples are

1. $a(n) = n$ yields the sequence $\{1, 2, 3, 4, 5, \dots\}$
2. $a(n) = 3n$ yields the sequence $\{3, 6, 9, 12, \dots\}$

However, one typically uses subscript notation and not functional notation: $a_n = a(n)$. We then call a_n the n th term of the sequence.

Another way to define a particular sequence is recursively.

Definition 2.2. *A recursive sequence is defined in two steps:*

1. *The value of first term (or first few terms) is given.*
2. *A rule, or recursion formula, to determine later terms from earlier ones is given.*

A typical example is given by the Fibonacci sequence. It can be defined by the recursion formula $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ and the starting values of $a_1 = 0$ and $a_2 = 1$. The resulting sequence is $\{0, 1, 1, 2, 3, 5, 8, \dots\}$. Writing the general expression for the n th term is possible, but it is not as simply stated. Recursive definitions are often useful in doing computations for large values of n .

2.2 Convergence of Sequences

Next we are interested in the behavior of the sequence as n gets large. For the sequence defined by $a_n = n - 1$, we find the behavior as shown in Figure 2.1. Notice that as n gets large, a_n also gets large. This sequence is said to be divergent.

On the other hand, the sequence defined by $a_n = \frac{1}{2^n}$ approaches a limit as n gets large. This is depicted in Figure 2.2. Another related series, $a_n = \frac{(-1)^n}{2^n}$, is shown in Figure 2.3. This sequence is the alternating sequence $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\}$.

Definition 2.3. *The sequence a_n converges to the number L if to every positive number ϵ there corresponds an integer N such that for all n ,*

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

If no such number exists, then the sequence is said to diverge.

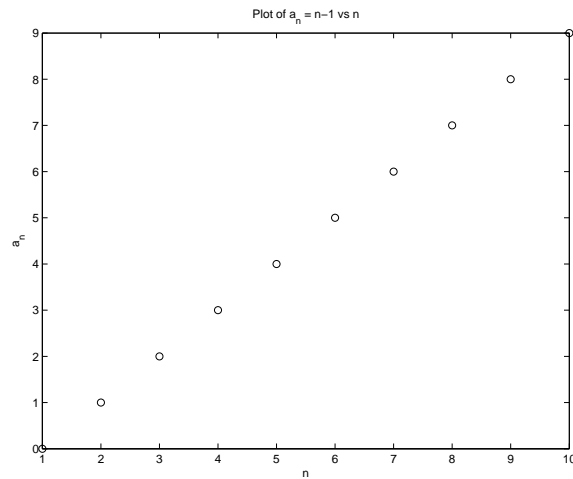


Fig. 2.1. Plot of $a_n = n - 1$ for $n = 1 \dots 10$.

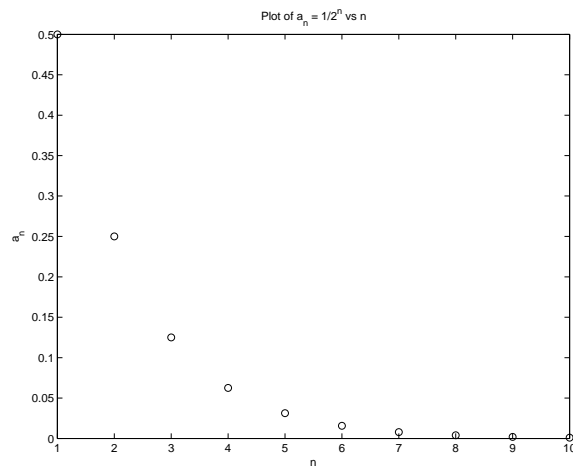


Fig. 2.2. Plot of $a_n = \frac{1}{2^n}$ for $n = 1 \dots 10$.

In Figures 2.4-2.5 we see what this means. For the given sequence, we see that $L = 0$. Given an $\epsilon > 0$, we ask for what value of N the n th terms ($n > N$) lie in the interval $[L - \epsilon, L + \epsilon]$. In these figures this interval is depicted by a horizontal band. We see that for convergence, sooner, or later, the tail of the sequence ends up entirely within this band.

If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit L , then we write either $a_n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} a_n = L$. For example, we have already seen that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n} = 0$.

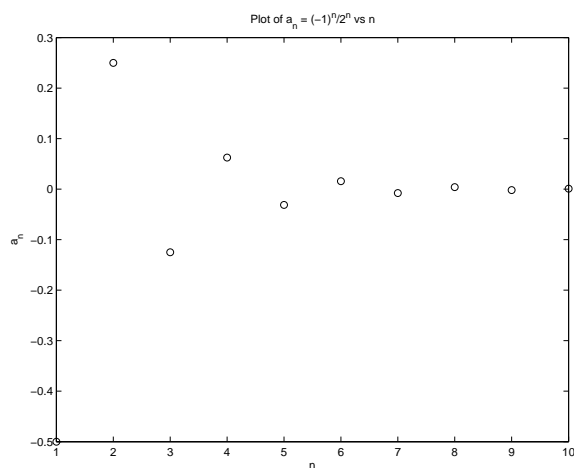


Fig. 2.3. Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$.

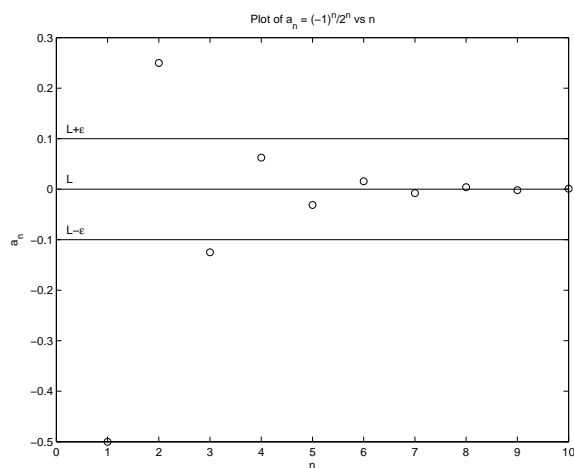


Fig. 2.4. Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$. Picking $\epsilon = 0.1$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 3$.

2.3 Limit Theorems

Once we have defined the notion of convergence of a sequence to some limit, then we can investigate certain properties of limits of sequences. Here we list a few results on limits theorems and some special limits, which arise often.

Theorem 2.4. Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ and a number k . Assume that $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$. Then we have

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B$.

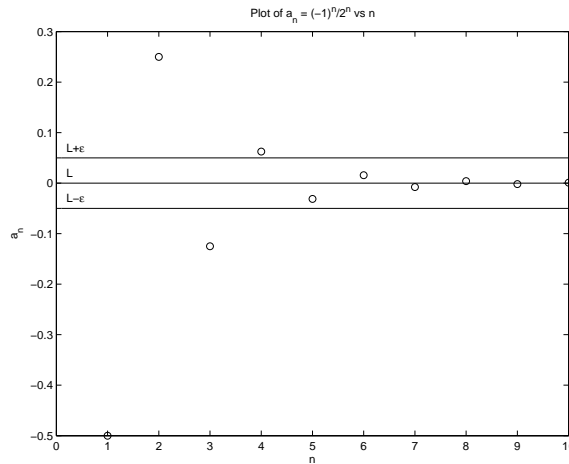


Fig. 2.5. Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$. Picking $\epsilon = 0.05$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 4$.

2. $\lim_{n \rightarrow \infty} (kb_n) = kB$.
3. $\lim_{n \rightarrow \infty} (a_n b_n) = AB$.
4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0$.

Some special limits are given next. These are generally first encountered in a second course in calculus.

Theorem 2.5. *The following are special cases:*

1. $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.
2. $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$.
3. $\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1, \quad x > 0$.
4. $\lim_{n \rightarrow \infty} x^n = 0, \quad |x| < 1$.
5. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.
6. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$.

The proofs generally are straight forward, depending upon L'Hopital's Rule and other manipulations. For example, one can prove the first limit by first realizing that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x}$. This limit is indeterminate as $x \rightarrow \infty$ in its current form since the numerator and the denominator get large for large x . In such cases one employs L'Hopital's Rule.

Theorem 2.6. *Let c be a finite number or $c = \infty$. If $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$, then*

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = \infty$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Thus, $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$.

The second limit in the list can be proven by first looking at

$$\lim_{n \rightarrow \infty} \ln n^{1/n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Now, if $\lim_{n \rightarrow \infty} \ln f(n) = 0$, then $\lim_{n \rightarrow \infty} f(n) = e^0 = 1$. Thus proving the second limit.

The third limit can be done similarly. The reader is left to confirm the other limits.

Example 2.7. $\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 3}{n^3 + n}$

Divide the numerator and denominator by n^2 . Then

$$\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 3}{n^3 + n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{n + \frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Example 2.8. $\lim_{n \rightarrow \infty} \frac{\ln(n^2)}{n}$

Rewriting $\frac{\ln(n^2)}{n} = \frac{2 \ln(n)}{n}$, we find from identity 1 of the Theorem 2.5 that

$$\lim_{n \rightarrow \infty} \frac{\ln(n^2)}{n} = 2 \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0.$$

Example 2.9. $\lim_{n \rightarrow \infty} (n^2)^{\frac{1}{n}}$

To compute this limit, we rewrite

$$\lim_{n \rightarrow \infty} (n^2)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (n)^{\frac{1}{n}} (n)^{\frac{1}{n}} = 1,$$

using identity 2 of the Theorem 2.5.

Example 2.10. $\lim_{n \rightarrow \infty} \left(\frac{n-2}{n}\right)^n$

This limit can be written as

$$\lim_{n \rightarrow \infty} \left(\frac{n-2}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2}.$$

Here we used identity 5 of the Theorem 2.5.

2.4 Infinite Series

In this section we investigate the meaning of series, which are infinite sums of the form

$$a_1 + a_2 + a_3 + \dots \tag{2.1}$$

A typical example is the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \quad (2.2)$$

How would one evaluate this sum? We begin by just adding the terms. For example,

$$\begin{aligned} 1 + \frac{1}{2} &= \frac{3}{2}, \\ 1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{31}{16}, \end{aligned} \quad (2.3)$$

etc. The values tend to a limit. We can see this graphically in Figure 2.6.

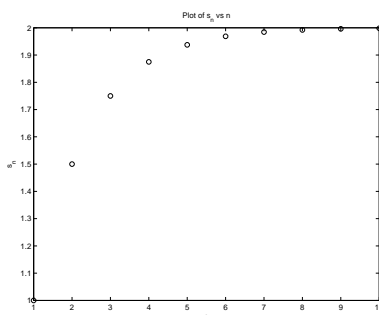


Fig. 2.6. Plot of $s_n = \sum_{k=1}^n \frac{1}{2^{k-1}}$ for $n = 1 \dots 10$.

In general, we want to make sense out of Equation (2.1). As with the example, we look at a sequence of partial sums. Thus, we consider the sums

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned} \quad (2.4)$$

etc. In general, we define the n th partial sum as

$$s_n = a_1 + a_2 + \dots + a_n.$$

If the infinite series (2.1) is to make any sense, then the sequence of partial sums should converge to some limit. We define this limit to be the sum of the infinite series, $S = \lim_{n \rightarrow \infty} s_n$.

Definition 2.11. *If the sequence of partial sums converges to the limit L as n gets large, then the infinite series is said to have the sum L .*

We will use the compact summation notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$$

Here n will be referred to as the index and it may start at values other than $n = 1$.

2.5 Geometric Series

Example (2.2) is an example of what is known as a geometric series. A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^2 + \dots + ar^n + \dots \quad (2.5)$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is $r \frac{ar^{n+1}}{ar^n}$.

The sum of a geometric series, when it converges, can easily be determined. We consider the n th partial sum:

$$s_n = a + ar + \dots + ar^{n-2} + ar^{n-1}. \quad (2.6)$$

Now, multiply this equation by r .

$$rs_n = ar + ar^2 + \dots + ar^{n-1} + ar^n. \quad (2.7)$$

Subtracting these two equations, while noting the many cancellations, we have

$$(1 - r)s_n = a - ar^n. \quad (2.8)$$

Thus, the n th partial sums can be written in the compact form

$$s_n = \frac{a(1 - r^n)}{1 - r}. \quad (2.9)$$

Recall that the sum, if it exists, is given by $S = \lim_{n \rightarrow \infty} s_n$. Letting n get large in the partial sum (2.9), we need only evaluate $\lim_{n \rightarrow \infty} r^n$. From our special limits we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series	
The sum of the geometric series is given by	
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad r < 1. \quad (2.10)$	

The reader should verify that the geometric series diverges for all other values of r . Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 2.12. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure 2.6.

Example 2.13. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example 2.14. $\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n})$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed. Thus, we have

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

2.6 Convergence Tests

Given a general infinite series, it would be nice to know if it converges, or not. Often, we are only interested in the convergence and not the actual sum. It is often difficult to determine the sum, when the series does converge. In this section we will review some of the standard tests for convergence.

First, we have the n th term divergence test. This can be motivated by two examples:

1. $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \dots$
2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots$

In the first example it is easy to see that each term is getting larger and larger, and thus the partial sums will grow without bound. In the second case, each term is bigger than one. Thus, the series will be bigger than adding the same number of ones as there are terms in the sum. Obviously, this series will also diverge.

This leads to the n th term divergence test:

Theorem 2.15. *If $\lim a_n \neq 0$ or if this limit does not exist, then $\sum_n a_n$ diverges.*

This theorem does not imply that just because the terms are getting smaller, the series will converge. Otherwise, we would not need any other convergence theorems.

For the next theorems, we will assume that the series has nonnegative terms.

1. Comparison Test

The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ such that $a_n \leq c_n$ for all $n > N$ for some N . The series $\sum a_n$ diverges if there is a divergent series $\sum d_n$ such that $d_n \leq a_n$ for all $n > N$ for some N .

For this test one has to dream up a second series for comparison. Typically, this requires some experience with convergent series. Often it is better to use other tests first if possible.

2. Limit Comparison Test

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ is finite then $\sum a_n$ and $\sum b_n$ converge together or diverge together.

For example, consider the infinite series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Then,

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{(n+1)^2} = 2$. Thus, these two series both converge, or both diverge. If we knew the behavior of the second series, then we could draw a conclusion. Using the next test, we will prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ diverges.

3. Integral Test

Consider the infinite series $\sum_{n=1}^{\infty} a_n$. Let $f(n) = a_n$. Then, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) dx$ both converge or both diverge. Here we mean that the integral converges or diverges as an improper integral.

We are interested in the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we saw in the Limit Comparison Test example. This infinite series is famous and is called the harmonic series. The plot of the partial sums is given in Figure 2.7. It appears that the series could possibly converge or diverge. It is hard to tell graphically.

In this case we can use the Integral Test. In Figure 2.8 we plot $f(x) = \frac{1}{x}$ and at each integer n we plot a box from n to $n+1$ of height $\frac{1}{n}$. We can see from the figure that the total area of the boxes is greater than the area under the curve. Since the area of each box is $\frac{1}{n}$, then we have that

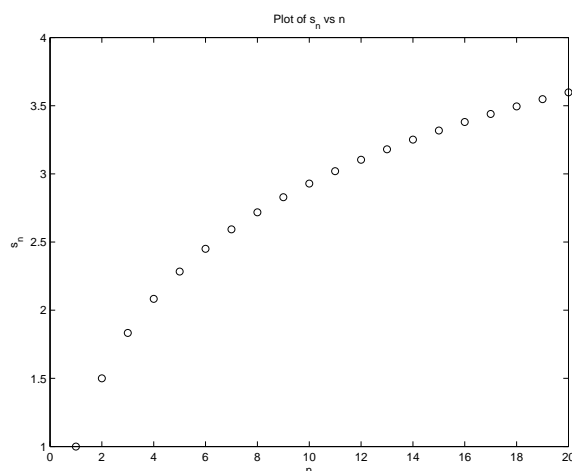


Fig. 2.7. Plot of the partial sums for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

But, we can compute the integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{x \rightarrow \infty} (\ln x) = \infty.$$

Thus, the integral diverges and the infinite series is larger than this! So, the harmonic series diverges.

The Integral Test provides us with the convergence behavior for a class of infinite series called p -series. These series are of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Recalling that the improper integrals $\int_1^{\infty} \frac{dx}{x^p}$ converge for $p > 1$ and diverge otherwise, we have the p -test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1$$

and diverges otherwise.

Example 2.16. $\sum_{n=1}^{\infty} \frac{n+1}{n^3-2}$.

We first note that as n gets large, the general term behaves like $\frac{1}{n^2}$ since the numerator behaves like n and the denominator behaves like n^3 . So, we expect that this series behaves like the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus, by the limit comparison test,

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^3-2} (n^2) = 1.$$

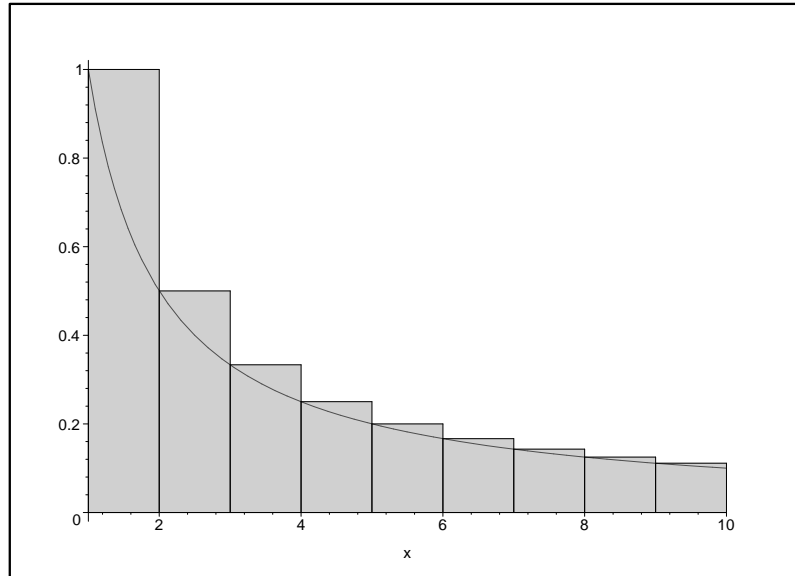


Fig. 2.8. Plot of $f(x) = \frac{1}{x}$ and boxes of height $\frac{1}{n}$ and width 1.

These series both converge, or both diverge. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -test since $p = 2$. Therefore, the original series converges.

4. Ratio Test

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$. Then the behavior of the infinite series can be determined from the conditions

$$\begin{aligned} \rho < 1, & \text{ converges} \\ \rho > 1, & \text{ diverges} \end{aligned}$$

Example 2.17. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$.

We compute

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^{10}}{n^{10}} \frac{10^n}{10^{n+1}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{10} \frac{1}{10} \\ &= \frac{1}{10} < 1. \end{aligned}$$

(2.11)

Therefore, the series is said to converge by the Ratio Test.

Example 2.18. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

In this case we make use of the fact that $(n+1)! = (n+1)n!$. We compute

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 < 1\end{aligned}\tag{2.12}$$

This series also converges by the Ratio Test.

5. Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n \rightarrow \infty} a_n^{1/n}$. Then the behavior of the infinite series can be determined using

$$\begin{aligned}\rho &< 1, \text{ converges} \\ \rho &> 1, \text{ diverges}\end{aligned}$$

Example 2.19. $\sum_{n=0}^{\infty} e^{-n}$.

We use the n th root test: $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} e^{-1} = e^{-1} < 1$. Thus, this series converges by the n th Root Test:

Example 2.20. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$.

This series also converges by the n th root test.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left(\frac{n^n}{2^{n^2}} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n}{2^n} = 0 < 1.$$

We next turn to series which have both positive and negative terms. We can toss out the signs by taking absolute values of each of the terms. We then note that since $a_n \leq |a_n|$ we have

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|.$$

If the sum $\sum_{n=1}^{\infty} |a_n|$ converges, then the original series converges. This type of convergence is useful, because we can use the previous tests to establish convergence of such series. Thus, we say that a series *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series converges, but does not converge absolutely, then it is said to *converge conditionally*.

Example 2.21. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$. This series converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p -series with $p = 2$.

Finally, there is one last test that we recall from your introductory calculus class. We consider the alternating series, given by $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. The convergence of alternating series is determined from **Leibniz's Theorem**.

Theorem 2.22. *The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if*

1. a_n 's are positive.
2. $a_n \geq a_{n+1}$ for all n .
3. $a_n \rightarrow 0$.

The first condition guarantees that we have alternating signs in the series. The next conditions say that the magnitude of the terms gets smaller and approaches zero.

Example 2.23. The alternating harmonic series converges. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Example 2.24. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ also passes the conditions of Leibniz's Theorem

Note that in the last example, we can show that the series is absolutely convergent. However, the series of absolute values for Example 1 is the harmonic series, so it is not absolutely convergent. Therefore, we have an example of a series that is conditionally convergent.

2.7 The Order of Sequences and Functions

Often we are interested in comparing the magnitudes of sequences or functions. This is useful in approximation theory. We begin with the comparison of sequences and introduce *big-Oh* notation. We will then extend this to functions of continuous variables.

Definition 2.25. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Then if there are numbers N and K (independent of N) such that*

$$\left| \frac{a_n}{b_n} \right| < K \quad \text{whenever } n > N,$$

then we say that a_n is of the order of b_n . We write this as

$$a_n = O(b_n) \quad \text{as } n \rightarrow \infty$$

and say a_n is "big O" of b_n .

Example 2.26. For example, consider the sequences given by $a_n = \frac{2n+1}{3n^2+2}$ and $b_n = \frac{1}{n}$. Then,

$$\left| \frac{a_n}{b_n} \right| = \left| \frac{\frac{2n+1}{3n^2+2}}{\frac{1}{n}} \right| = \left| \frac{2n^2 + n}{3n^2 + 2} \right|.$$

We want to find a bound on the last expression. We divide the numerator and denominator by n^2 and find that

$$\left| \frac{a_n}{b_n} \right| = \left| \frac{2 + 1/n}{3 + 2/n^2} \right| = \frac{2}{3} \left| \frac{1 + 1/2n}{1 + 2/3n^2} \right|.$$

The last term is largest for $n = 1$. This gives

$$\left| \frac{a_n}{b_n} \right| = \frac{2}{3} \left| \frac{1 + 1/2n}{1 + 2/3n^2} \right| \leq \frac{2}{3} \left| \frac{1 + 1/2}{1 + 2/3} \right| = \frac{9}{10}.$$

Thus, for $n > 1$, we have that

$$\left| \frac{a_n}{b_n} \right| \leq \frac{9}{10}.$$

We then conclude that

$$a_n = O(b_n) = O\left(\frac{1}{n}\right).$$

In practice one is given a sequence like a_n , but the second sequence needs to be found by looking at the large n behavior of a_n .

Referring to the last example, we are given $a_n = \frac{2n+1}{3n^2+2}$. We look at the large n behavior. The numerator behaves like $2n$ and the denominator behaves like $3n^2$. Thus, $a_n = \frac{2n+1}{3n^2+2} \sim \frac{2n}{3n^2} = \frac{2}{3n}$ for large n . Therefore, we say that $a_n = O\left(\frac{1}{n}\right)$. Note that we are only interested in the n -dependence and not the multiplicative constant since $\frac{1}{n}$ and $\frac{2}{3n}$ have the same growth rate. In a similar way, we can compare functions. We modify our definition of big-Oh for functions of a continuous variable.

Definition 2.27. $f(x)$ is of the order of $g(x)$, or $f(x) = O(g(x))$ as $x \rightarrow x_0$ if

$$\lim_{x \rightarrow x_0} \left| \frac{f(x)}{g(x)} \right| < K$$

for some K independent of x_0 .

Example 2.28. For example, we recall the Taylor series expansion for $\cos x$ gives us that

$$\cos x = 1 - \frac{x^2}{2} + O(x^4) \text{ as } x \rightarrow 0.$$

(We will review Taylor series in the last section of this chapter.)

We want to show that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \text{ as } x \rightarrow 0.$$

This should look obvious from the above series expansion. However, we want to verify this using our definition.

We need to compute

$$\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}.$$

The numerator and denominator separately go to zero, so we have an indeterminate form. This suggests that we need to apply L'Hopital's Rule. In fact, we apply it several times to find that

$$\begin{aligned} \lim_{x \rightarrow 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| &= \lim_{x \rightarrow 0} \left| \frac{-\sin x + x}{4x^3} \right| \\ &= \lim_{x \rightarrow 0} \left| \frac{-\cos x + 1}{12x^2} \right| \\ &= \lim_{x \rightarrow 0} \left| \frac{\sin x}{24x} \right| = \frac{1}{24}. \end{aligned}$$

Thus, for any number $K > \frac{1}{24}$, we have that

$$\lim_{x \rightarrow 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| > K.$$

We conclude that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \quad \text{as } x \rightarrow 0.$$

2.8 Sequences of Functions

Our immediate goal is to provide a preparation useful for studying Fourier series, which are series whose terms are functions. So, in this section we begin to discuss series of functions and the convergence of such series. Once more we will need to resort to the convergence of the sequence of partial sums. This means we really need to start with sequences of functions.

A *sequence of functions* is simply a set of functions $f_n(x)$, $n = 1, 2, \dots$ defined on a common domain D . A frequently used example is the sequence of functions $\{1, x, x^2, \dots\}$.

An infinite *series of functions* is given by $\sum_{n=1}^{\infty} f_n(x)$, $x \in D$. Using powers of x again, an example would be $\sum_{n=1}^{\infty} x^n$, $x \in [-1, 1]$. In order to investigate the convergence of this series, we really mean substitute values for x and determine if the resulting real series of number converges. This means that we would need to consider the N th partial sums

$$s_N(x) = \sum_{n=1}^N f_n(x).$$

Does this sequence of functions converge?

We say that a sequence of functions f_n converge pointwise on D to a limit g if

$$\lim_{n \rightarrow \infty} f_n(x) = g(x)$$

for each $x \in D$. More formally, we write that

$$\lim_{n \rightarrow \infty} f_n = g \text{ (pointwise on } D\text{)}$$

if given $x \in D$ and $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N.$$

Example 2.29. Consider the sequence of functions $f(x) = \frac{1}{1+nx}$, $|x| < \infty$, $n = 1, 2, 3, \dots$. The limits depends on the value of x .

1. $x = 0$. Here $\lim_{n \rightarrow \infty} f_n(0) = \lim_{n \rightarrow \infty} 1 = 1$.
2. $x \neq 0$. Here $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{1+nx} = 0$.

Therefore, we can say that $f_n \rightarrow g$ pointwise for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases} \quad (2.13)$$

We also note that in general N depends on both x and ϵ .

Example 2.30. We consider the functions $f_n(x) = x^n$, $x \in [0, 1]$, $n = 1, 2, \dots$. We recall that the above definition suggests that for each x we seek an N such that $|f_n(x) - g(x)| < \epsilon$, $\forall n \geq N$. Here are two examples:

1. $x = 0$. Here we have $f_n(0) = 0$ for all n . So, given $\epsilon > 0$ we seek an N such that $|f_n(0) - 0| < \epsilon$, $\forall n \geq N$, or $0 < \epsilon$. But all n work, so we can pick $N = 1$.
2. $x = \frac{1}{2}$. In this case we have $f_n(\frac{1}{2}) = \frac{1}{2^n}$, for $n = 1, 2, \dots$. As n gets large, $f_n \rightarrow 0$. So, given $\epsilon > 0$, we seek N such that $|\frac{1}{2^n} - 0| < \epsilon$, $\forall n \geq N$. This means that $\frac{1}{2^n} < \epsilon$, or $n > -\frac{\ln \epsilon}{\ln 2} \geq N$. Thus, our choice of N depends on ϵ .

There are other questions that can be asked about sequences of functions. Let the sequence of functions f_n be continuous on D . If the sequence of functions converges pointwise to g on D then we can ask the following.

1. Is g continuous on D ?
2. If each f_n is integrable on $[a, b]$, then does $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b g(x) dx$?
3. If each f_n is differentiable at c , then does $\lim_{n \rightarrow \infty} f'_n(c) = g'(c)$?

It turns out that pointwise convergence is not enough to provide an affirmative answer to any of these questions. Though we will not prove it here, what we will need is uniform convergence.

Definition 2.31. Consider a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on D . Let $g(x)$ be defined for $x \in D$. Then the sequence converges uniformly on D , or

$$\lim_{n \rightarrow \infty} f_n = g \text{ uniformly on } D,$$

if given $\epsilon > 0$, there exists an N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N \text{ and } \forall x \in D.$$

This definition almost looks like the definition for pointwise convergence. However, the seemingly subtle difference lies in the fact that N does not depend upon x . The sought N works for all x in the domain. As seen in Figure 2.9 as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) + \epsilon$.

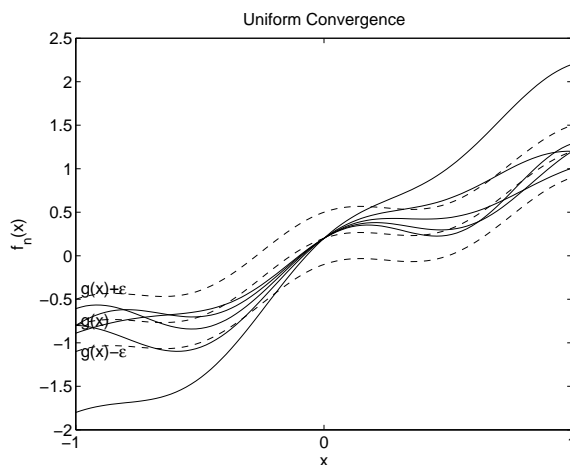


Fig. 2.9. For uniform convergence, as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) + \epsilon$.

Example 2.32. $f_n(x) = x^n$, for $x \in [0, 1]$. Note that in this case as n gets large, $f_n(x)$ does not lie in the band $(g(x) - \epsilon, g(x) + \epsilon)$. This is displayed in Figure 2.10.

Example 2.33. $f_n(x) = \cos(nx)/n^2$ on $[-1, 1]$. For this example we plot the first several members of the sequence in Figure 2.11. We can see that eventually ($n \geq N$) members of this sequence do lie inside a band of width ϵ about the limit $g(x) = 0$ for all values of x . Thus, this sequence of functions will converge uniformly to the limit.

Finally, we should note that if a sequence of functions is uniformly convergent then it converges pointwise. However, the examples should bear out that the converse is not true.

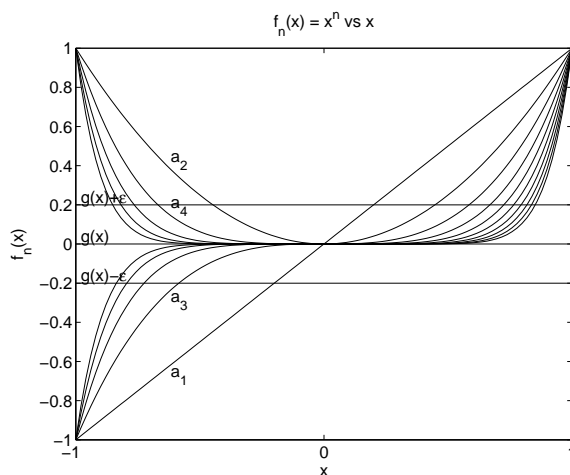


Fig. 2.10. Plot of $f_n(x) = x^n$ on $[-1, 1]$ for $n = 1 \dots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

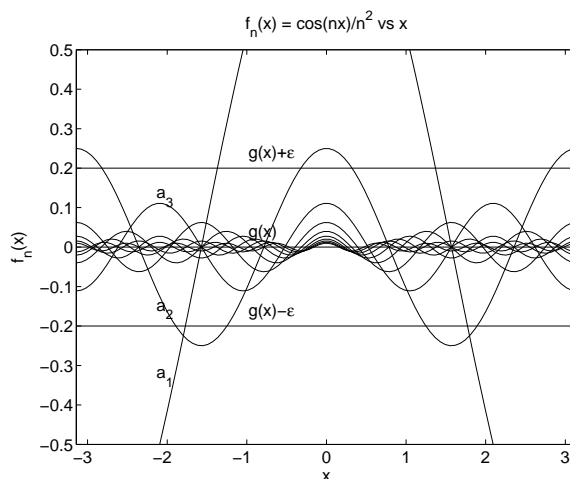


Fig. 2.11. Plot of $f_n(x) = \cos(nx)/n^2$ on $[-\pi, \pi]$ for $n = 1 \dots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

2.9 Infinite Series of Functions

We now turn our attention to infinite series of functions, which will form the basis of our study of Fourier series. Recall that we are interested in the convergence of the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n(x)$ for $x \in D$. But the sequence of partial sums is just a sequence of functions. So, it is natural to define the convergence of sequences of functions in terms of pointwise and uniform convergence.

We define the sequence of partial sums

$$s_n(x) = \sum_{j=1}^n f_j(x).$$

Then the definitions of pointwise and uniform convergence are as follows:

Definition 2.34. $\sum f_j(x)$ converges pointwise to $f(x)$ on D if given $x \in D$, and $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$.

Definition 2.35. $\sum f_j(x)$ converges uniformly to $f(x)$ on D given $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Again, we state without proof the following:

1. Uniform convergence implies pointwise convergence.
2. If f_n is continuous on D , and $\sum_n^\infty f_n$ converges uniformly to f on D , then f is continuous on D .
3. If f_n is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges uniformly on D , and $\int_a^b f_n(x) dx$ exists, then

$$\sum_n^\infty \int_a^b f_n(x) dx = \int_a^b \sum_n^\infty f_n(x) dx = \int_a^b g(x) dx.$$

4. If f'_n is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges pointwise to g on D , and $\sum_n^\infty f'_n$ converges uniformly on D , then $\sum_n^\infty f'_n(x) = \frac{d}{dx}(\sum_n^\infty f_n(x)) = g'(x)$ for $x \in (a, b)$.

Since uniform convergence of series gives so much, like term by term integration and differentiation, we would like to be able to recognize when we have a uniformly convergent series. One test for such convergence is the **Weierstrass M-Test**.

Theorem 2.36. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions on D . If $|f_n(x)| \leq M_n$, for $x \in D$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly of D .

Proof. First, we note that for $x \in D$

$$\sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty M_n.$$

Thus, since by the assumption that $\sum_{n=1}^{\infty} M_n$ converges, we have that $\sum_{n=1}^{\infty} f_n$ converges absolutely on D . Therefore, $\sum_{n=1}^{\infty} f_n$ converges pointwise on D . So, let $\sum_{n=1}^{\infty} f_n = g$.

We now want to prove that this convergence is in fact uniform. So, given $\epsilon > 0$ we need to find an N such that

$$|g(x) - \sum_{j=1}^n f_j(x)| < \epsilon$$

if $n \geq N$ for all $x \in D$.

So, for any $x \in D$,

$$\begin{aligned} |g(x) - \sum_{j=1}^n f_j(x)| &= \left| \sum_{j=1}^{\infty} f_j(x) - \sum_{j=1}^n f_j(x) \right| \\ &= \left| \sum_{j=n+1}^{\infty} f_j(x) \right| \\ &\leq \sum_{j=n+1}^{\infty} |f_j(x)|, \quad \text{by the triangle inequality} \\ &\leq \sum_{j=n+1}^{\infty} M_j. \end{aligned} \tag{2.14}$$

Now, the sum over the M_j 's is convergent, so we can choose our N such that

$$\sum_{j=n+1}^{\infty} M_j < \epsilon, \quad n \geq N.$$

Then, we have from above that

$$|g(x) - \sum_{j=1}^n f_j(x)| \leq \sum_{j=n+1}^{\infty} M_j < \epsilon$$

for all $n \geq N$ and $x \in D$. Thus, $\sum f_j \rightarrow g$ uniformly on D . **QED**

We now give an example of how to use the M-Test.

Example 2.37.

Example We consider the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ defined on $[-\pi, \pi]$. Each term is bounded by $|\frac{\cos nx}{n^2}| = \frac{1}{n^2} \equiv M_n$. We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Thus, we can conclude that the original series converges uniformly, as it satisfies the conditions of the Weierstrass M-Test.

2.10 Power Series

A typical example of a series of functions that the student has encountered in previous courses is the power series, with examples being provided by Taylor and MacLaurin series.

Definition 2.38. A power series expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n(x - a)^n$.

For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

An example of such a power series is the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \quad (2.15)$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to what our infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (2.10), we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \quad |x| < 1.$$

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide approximations to the function $(1 - x)^{-1}$. If x is small, we could write

$$(1 - x)^{-1} = 1 + x + O(x^2).$$

In Figure 2.12 we see that for small values of x these functions do agree. Of course, if we want better agreement, we select more terms. In Figure 2.13 we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure 2.14 shows that keeping only quadratic terms may not be good enough. Keeping the cubic terms is better. Finally, in Figure 2.15 we show the sum of the first 21 terms over the entire interval $[-1, 1]$.

With this example we see how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with the function and then find a series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and MacLaurin series.

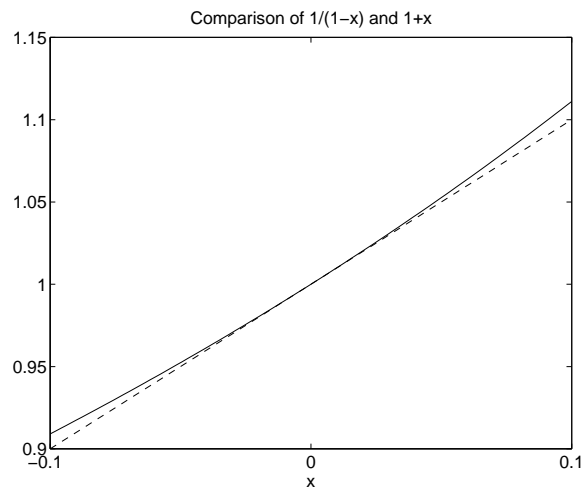


Fig. 2.12. Comparison of $\frac{1}{1-x}$ (solid) to $1+x$ (dashed) for $x \in [-0.1, 0.1]$.

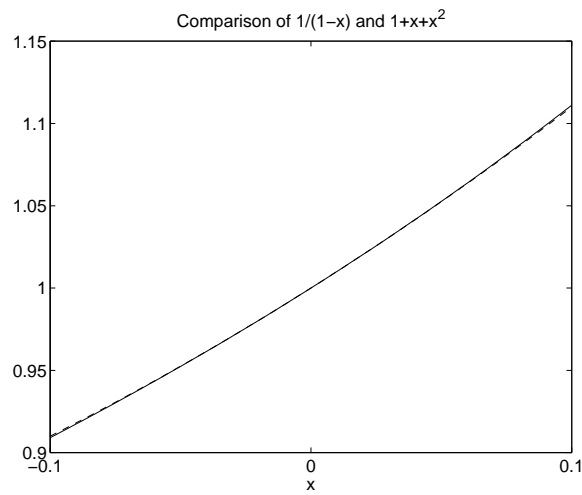


Fig. 2.13. Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) for $x \in [-0.1, 0.1]$.

Definition 2.39. A Taylor series expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \quad (2.16)$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \quad (2.17)$$

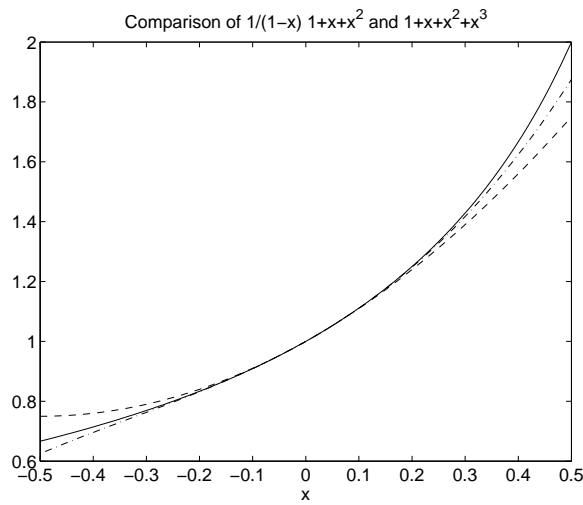


Fig. 2.14. Comparison of $\frac{1}{1-x}$ (solid) to $1+x+x^2$ (dashed) and $1+x+x^2+x^3$ (dash-dot) for $x \in [-0.5, 0.5]$.

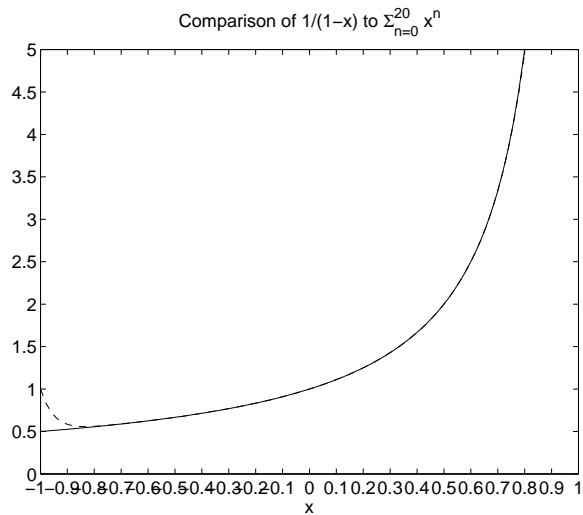


Fig. 2.15. Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{\infty} x^n$ for $x \in [-1, 1]$.

Note that we use \sim to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those for which the expansion is about $x = 0$.

Definition 2.40. A MacLaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \quad (2.18)$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \quad (2.19)$$

Example 2.41. Expand $f(x) = e^x$ about $x = 0$.

We begin by creating a table. We will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then one only needs to evaluate the second column at $x = 0$ and divide by $n!$.

n	$f^{(n)}(x)$	c_n
0	e^x	$\frac{e^0}{0!} = 1$
1	e^x	$\frac{e^0}{1!} = 1$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$

Next, one looks at the last column and tries to determine some pattern so as to write down the general term of the series. If there is only a need to get an approximation, then the first few terms may be sufficient.

In this case, we have that the pattern is obvious: $c_n = \frac{1}{n!}$. So, $e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Example 2.42. Expand $f(x) = e^x$ about $x = 1$.

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. We could create a table like the last example. In fact, the last column would have values of the form $\frac{e}{n!}$. (You should confirm this.) However, we could make use of the MacLaurin series expansion for e^x and get the result quicker. Note that $e^x = e^{x-1+1} = ee^{x-1}$. Now, apply the known expansion for e^x :

$$e^x = e \left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \dots \right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

Example 2.43. Expand $f(x) = \frac{1}{1-x}$ about $x = 0$.

This is the example with which we started our discussion. We set up a table again. We see from the last column that we get back our geometric series (2.15).

n	$f^{(n)}(x)$	c_n
0	$\frac{1}{1-x}$	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$\frac{3!}{3!} = 1$

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided below.

Series Expansions You Should Know	
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	(2.20)
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	(2.21)
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	(2.22)
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$	(2.23)
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	(2.24)
$\ln(1+x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \dots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$	(2.25)

What is still left to be determined is for what values do such power series converge. The first three of the above expansions converge for all reals, but the other three only converge for $|x| < 1$.

We consider the convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

Theorem 2.44. If $\sum_{n=0}^{\infty} c_n(b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely for all x satisfying $|x-a| < |b-a|$.

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may only converge at $x = a$.
2. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.

The number R is called the *radius of convergence* of the power series and $(a-R, a+R)$ is called the *interval of convergence*. Convergence at the endpoints of this interval has to be tested for each power series.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty} |c_n(x-a)^n|$. This is easily done using either the ratio test or the n th root test. We first identify our nonnegative terms $a_n = |c_n||x-a|^n$. Then we apply our tests.

For example, the n th root test gives the convergence condition

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{|c_n||x-a|} < 1.$$

Thus,

$$|x-a| < \left(\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} \right)^{-1} \equiv R.$$

Similarly, we can apply the ratio test.

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} |x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} \right)^{-1} \equiv R.$$

Example 2.45. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Since there is a factorial, we will use the ratio test with $a = 0$.

$$\rho = \lim_{n \rightarrow \infty} \frac{|n!|}{|(n+1)!|} |x| = \lim_{n \rightarrow \infty} \frac{1}{n+1} |x| = 0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x . We also can say that the radius of convergence is infinite.

Example 2.46. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

In this example we will use the n th root test with $a = 0$.

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.

Example 2.47. $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.

In this example, we have an expansion about $x = 2$. Using the n th root test we find that

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{3^n}{n}} |x - 2| = 3|x - 2| < 1.$$

Solving for $|x - 2|$ in this inequality, we find $|x - 2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $(\frac{1}{3} - 2, \frac{1}{3} + 2) = (-\frac{5}{3}, \frac{7}{3})$.

As for the endpoints, we need to first test at $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = -\frac{5}{3}$ we get the alternating harmonic series, which does converge. So, we have convergence on $[-\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the one endpoint, $x = -\frac{5}{3}$.

Example 2.48. Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x - 1$, since we are expanding about $x = 1$. This is easily done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. Factoring out a 3, we can rewrite this as a sum of a geometric series. Namely, we use the expansion for

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \dots \end{aligned} \tag{2.26}$$

Factoring a 3 from the denominator, we have

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3[1 + \frac{1}{3}(x-1)]} \\ &= \frac{1}{3} \frac{1}{1 + \frac{1}{3}(x-1)}. \end{aligned} \tag{2.27}$$

Now we note that $f(x) = \frac{1}{3}g(\frac{1}{3}(x-1))$. So,

$$f(x) = \frac{1}{3} \left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \dots \right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \dots$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $|\frac{1}{3}(x-1)| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for both $x = -2$ and $x = 4$. You should verify this!

As a final application, we can derive Euler's Formula,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

Next we note that each term has a power of i . The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots\}$. See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n :

$$e^{in\theta} = \cos(n\theta) + i \sin(n\theta).$$

We also have

$$e^{in\theta} = (e^{i\theta})^n = (\cos \theta + i \sin \theta)^n.$$

Equating these two expressions, we are lead to DeMoivre's Formula

$$e^{in\theta} = (\cos \theta + i \sin \theta)^n.$$

This formula is useful for deriving needed identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$, we find

$$\cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta.$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \quad \sin 2\theta = 2 \sin \theta \cos \theta.$$

Replacing $\cos^2 \theta = 1 - \sin^2 \theta$ or $\sin^2 \theta = 1 - \cos^2 \theta$ leads to the half angle formulae:

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos \theta - i \sin \theta.$$

Combining this with Euler's Formula, we have that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos \theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i \sin \theta$ and $\sin(ix) = -i \sinh x$.

2.11 The Binomial Expansion

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a + b)^p$ in powers of a and b . We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p . While the binomial expansion can be obtained using Taylor series, we will provide a more interesting derivation here to show that

$$(a + b)^p = \sum_{r=0}^{\infty} C_p^r a^{n-r} b^r, \quad (2.28)$$

where the C_p^r are called the *binomial coefficients*.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a + b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p .

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned} (a + b)^0 &= 1 \\ (a + b)^1 &= a + b \\ (a + b)^2 &= a^2 + 2ab + b^2 \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &\dots \end{aligned} \quad (2.29)$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b . The powers of a are decreasing from n to 0 in the expansion of $(a + b)^n$. Similarly, the powers of b increase from 0 to n . The sums of the exponents in each term is n . So, we can write the $k + 1$ st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a + b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{aligned} n = 0 : & \quad 1 \\ n = 1 : & \quad 1 \quad 1 \\ n = 2 : & \quad 1 \quad 2 \quad 1 \\ n = 3 : & \quad 1 \quad 3 \quad 3 \quad 1 \\ n = 4 : & \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{aligned} \tag{2.30}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n . Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n = 2$ and $n = 3$ that $1 + 2 = 3$ and $2 + 1 = 3$:

$$\begin{array}{ccccccc} n = 2 : & 1 & & 2 & & 1 & \\ & & \searrow & \swarrow & \searrow & \swarrow & \\ n = 3 : & 1 & & 3 & & 3 & & 1 \end{array} \tag{2.31}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{aligned} n = 3 : & \quad 1 \quad 3 \quad 3 \quad 1 \\ n = 4 : & \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ n = 5 : & \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ n = 6 : & \quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{aligned} \tag{2.32}$$

So, $1 + 4 = 5$, $4 + 6 = 10$, etc.

Of course, it would take a while to compute each row up to the desired n . We need a simple expression for computing a specific coefficient. Consider the k th term in the expansion of $(a + b)^n$. Let $r = k - 1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \binom{n}{r}.$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count

the number of ways that we can arrange r products of b with $n - r$ products of a . There are n slots to place the b 's. For example, the $r = 2$ case for $n = 4$ involves the six products: $aabb$, $abab$, $abba$, $baab$, $baba$, and $bbaa$. Thus, it is natural to use this notation. The original problem that concerned Pascal was in gambling. Though, we will not go into that here.

So, we have found that

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r. \quad (2.33)$$

What if $a \gg b$? Can we use this to get an approximation to $(a + b)^n$? If we neglect b then $(a + b)^n \simeq a^n$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big O notation. In order to do this we first divide out a as

$$(a + b)^n = a^n \left(1 + \frac{b}{a}\right)^n.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen above, we can use the binomial expansion to write

$$\left(1 + \frac{b}{a}\right)^n = \sum_{r=0}^n \binom{n}{r} \left(\frac{b}{a}\right)^r. \quad (2.34)$$

Thus, we have a finite sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$\left(1 + \frac{b}{a}\right)^n = 1 + n \frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

note that we have used the observation that the second coefficient in the n th row of Pascal's triangle is n .

Summarizing, this then gives

$$\begin{aligned} (a + b)^n &= a^n \left(1 + \frac{b}{a}\right)^n \\ &= a^n \left(1 + n \frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^n + na^n \frac{b}{a} + a^n O\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \quad (2.35)$$

Therefore, we can approximate $(a + b)^n \simeq a^n + nba^{n-1}$, with an error on the order of $b^2 a^{n-2}$. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a + b)^n \simeq a^n$, but it is not typically good enough in applications because the error in this case is of the order ba^{n-1} .

Now consider the geometric series $1 + x + x^2 + \dots$. We have seen that such a series converges for $|x| < 1$, giving

$$1 + x + x^2 + \dots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$.

This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (2.33).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r. \quad (2.36)$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\dots$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \binom{p}{r} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \quad (2.37)$$

With this in mind we now state the theorem:

General Binomial Expansion	
<p>The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (2.33). For p real, we have that</p>	
$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r.$	(2.38)

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \quad (2.39)$$

2.11.1 Return to the Order of Expansions

We can make use of the binomial expansion to determine the behavior of functions such as $f(x) = (a+x)^b - a^b$. The first terms of the binomial expansion can be written

$$(1+x)^b = 1 + bx + O(x^2)$$

as $x \rightarrow \infty$. We will review the binomial expansion in a later next section. This expansion can also be obtained using Taylor series expansions, which we will also cover later.

Inserting the binomial expression into $f(x)$, we have as $x \rightarrow 0$ that

$$\begin{aligned} f(x) &= (a+x)^b - a^b \\ &= a^b \left[\left(1 + \frac{x}{a}\right)^b - 1 \right] \\ &= a^b \left[\frac{bx}{a} + O\left(\left(\frac{x}{a}\right)^2\right) \right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } x \rightarrow 0. \end{aligned} \quad (2.40)$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^b - a^b = a^{b-1}bx + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } x \rightarrow 0.$$

As an example, we could compute $\sqrt{R^2 + h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. Inserting into a calculator, one finds that

$$\sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7}.$$

How accurate is this? We could use our approximation with $a = R^2$, $x = 1$ and $b = \frac{1}{2}$. Then, our approximation would be of order

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2 + 1} - 6378164 \approx a^{b-1}bx = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

This is a better approximation. Of course, you should verify how many digits should be kept in reporting the result.

Problems

2.1. For those sequences that converge, find the limit $\lim_{n \rightarrow \infty} a_n$.

- $a_n = \frac{n^2+1}{n^3+1}$.
- $a_n = \frac{3n+1}{n+2}$.
- $a_n = \frac{(2n+3)!}{(n+1)!}$.
- $a_n = \left(\frac{3}{n}\right)^{1/n}$.
- $a_n = n \ln\left(1 + \frac{1}{n}\right)$.

2.2. Find the sum of the series

- $\sum_{n=0}^{\infty} \frac{(-1)^n 3}{4^n}$.
- $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$.

2.3. Determine if the following converge, or diverge, using one of the convergence tests:

- $\sum_{n=1}^{\infty} \frac{n+4}{2n^3+1}$.
- $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.
- $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.
- $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2n^2-3}$.

2.4. Determine the radius and interval of convergence of the following infinite series:

- $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n}$.
- $\sum_{n=1}^{\infty} \frac{x^n}{2^n n!}$.

2.5. Consider the sum $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)}$.

- Use an appropriate convergence test to show that this series converges.
- Verify that

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right).$$

- Find the n th partial sum of the series $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

2.6. Find the Taylor series centered at $x = a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

- $f(x) = \sinh x$, $a = 0$.
- $f(x) = \sqrt{1+x}$, $a = 0$.

- c. $f(x) = xe^x$, $a = 1$.
 d. $f(x) = \frac{x-1}{2+x}$, $a = 1$.

2.7. Test for pointwise and uniform convergence on the given set. [The Weierstrass M-Test might be helpful.]

- a. $f(x) = \sum_{n=1}^{\infty} \frac{\ln nx}{n^2}$, $x \in [1, 2]$.
 b. $f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \cos \frac{x}{2^n}$ on R .

2.8. Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

- a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

- b. From this result, derive Gregory's series for π by inserting an appropriate value for x in the above series expansion.

2.9. Use deMoivre's Theorem to write $\sin^3 \theta$ in terms of $\sin \theta$ and $\sin 3\theta$. Hint: Focus on the imaginary part of $e^{3i\theta}$.

2.10. Do the following:

- a. Compute: $\lim_{n \rightarrow \infty} n \ln \left(1 - \frac{3}{n}\right)$.
 b. Use L'Hopital's Rule to evaluate $L = \lim_{x \rightarrow \infty} \left(1 - \frac{4}{x}\right)^x$. Hint: Consider $\ln L$.
 c. Determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{3n+2}\right)^{n^2}$.
 d. Sum the series $\sum_{n=1}^{\infty} [\tan^{-1} n - \tan^{-1}(n+1)]$ by first writing the N th partial sum and then computing $\lim_{N \rightarrow \infty} s_N$.

2.11. Determine the order, $O(x^p)$, of the following functions:

- a. $\sqrt{x(1-x)}$ as $x \rightarrow 0$.
 b. $\frac{x^{5/4}}{1-\cos x}$ as $x \rightarrow 0$.
 c. $\frac{x}{x^2-1}$ as $x \rightarrow \infty$.
 d. $\sqrt{x^2+x} - x$ as $x \rightarrow \infty$.