

## Introduction

These are rough notes for a course in applied mathematics taught in the Spring semesters of 2004-2006 at the University of North Carolina Wilmington. This is an introduction to topics in Fourier analysis and complex analysis. It is a course aimed at students majoring in mathematics and science who are at least at their junior level of mathematical maturity. Students are introduced to Fourier series, Fourier transforms, and a basic complex analysis. As motivation we aim for an initial understanding of how analog and digital signals are related through the spectral analysis of time series. There are many applications using spectral analysis.

At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis, Brook Taylor and Jean le Rond d'Alembert.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where  $y$  is the string height and  $c$  is the wave speed. However, his solution led himself and others, like Leonhard Euler and Daniel Bernoulli, to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition  $y(x, 0) = h(x)$  has a discontinuous derivative! (We will see how this led to important questions in analysis.)

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x, t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extends over the interval  $[0, L]$  with fixed ends at  $x = 0$  and  $x = L$ . However, the initial conditions for such superpositions are

$$y(x, 0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x, 0) = \begin{cases} cx, & 0 \leq x \leq L/2 \\ c(L - x), & L/2 \leq x \leq L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's solution of the heat equation. The use of such Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies lead to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems in which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. An example is to look at ocean dynamics. Ocean waves are affected by the gravitational pull of the moon and the sun (and many other forces). These periodic forces lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis. Typically, we can view the tide height  $y(t)$  as a continuous function. One sits at a location and measures the movement of the ocean surface as a function of time. Such a function, or time series, is called an analog function. Another common analog signal is an audio signal, giving the amplitude of a sound (musical note, noise, speech, etc.) as a function of time (or space).

However, in all of these cases, we actually observe a part of the signal. This is because we can only sample a finite amount of data over a finite time interval. Thus, we have only the values  $y_n = y(t_n)$ : Though we are still interested in the spectral (frequency) content of our signals.

For example, in the case of ocean waves we would like to take our discrete signal (the sampled heights) and still try to determine the tidal components. In the case of an audio signal, we may want to save a finite amount of discretized information as a file to play back later on our computer.

So, how are the analog and discrete signals related? By sampling our analog signal, how does this affect the spectral information as compared to the spectral content of the original analog signal? What mathematics do we need to understand these processes? That is what we will look into in this course. We will look at Fourier trigonometric series. Well, we will actually begin with a review of infinite series. We will recall what infinite series are and when they do, or do not, converge. Then we can talk about the convergence of series of sinusoidal functions.

We will see how Fourier series are related to analog signals. A true representation of an analog signal comes from an infinite interval and not a finite interval, such as that the vibrating string lived on. This will lead to Fourier transforms. In order to work with continuous transforms, we will need a little complex analysis. So, we will spend a few sections on an introduction to complex analysis.

Having represented continuous signals and their spectral content by Fourier transforms, we will then see what need to be done to represent discrete signals. We end the course by investigating the connection between these two types of signals and some of the consequences of processing analog data through real measurement and/or storage devices.

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