7

Fourier and Laplace Transforms

In this chapter we will turn to the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can represent analog signals. Analog signals are continuous signals which may be sums over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent functions defined over a finite domain in Chapter 3.

We will also discuss a related integral transform, the Laplace transform. Laplace transforms are useful in solving initial value problems in differential equations and can be used to relate the input to the output of a linear system. Both transforms provide an introduction to a more general theory of transforms, which are used to transform specific problems to simpler ones.

We will begin by introducing the Fourier transform. First, we need to see how one can rewrite a trigonometric Fourier series as complex exponential series. Then we can extend the new representation of such series to analog signals, which typically have infinite periods. In later chapters we will highlight the connection between these analog signals and related digital signals.

7.1 Complex Exponential Fourier Series

We first recall from Chapter 3 the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right), \tag{7.1}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, \dots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots. \end{aligned} \tag{7.2}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms. This gives

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-inx}. \end{aligned} \tag{7.3}$$

The coefficients can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \tag{7.4}$$

This implies that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \tag{7.5}$$

So far the representation is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Reindexing the first sum, by introducing $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Since k is a dummy index, we replace it with a new n as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=-1}^{-\infty} \bar{c}_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

We can now combine all of the terms into a simple sum. We first define c_n for negative n's by

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Letting $c_0 = \frac{a_0}{2}$, we can write the *complex exponential Fourier series* representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{7.6}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\ c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\ c_0 &= \frac{a_0}{2}. \end{aligned} \tag{7.7}$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots$.

$$\begin{aligned}
c_n &= \frac{1}{2}(a_n + ib_n) \\
&= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}+e^{-inx}}{2}\right)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}-e^{-inx}}{2i}\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \qquad (7.8)
\end{aligned}$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, for all n we have obtained the complex exponential series for $f(x)$ defined on $[-\pi, \pi]$.

Complex Exponential Series for $f(x)$ defined on $[-\pi, \pi]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \qquad (7.9)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \qquad (7.10)$$

We have converted our trigonometric series for functions defined on $[-\pi, \pi]$ to a complex exponential series in Equation (7.11) with Fourier coefficients given by (7.10). We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

This can be rewritten as an exponential Fourier series of the form

Complex Exponential Series for $f(x)$ defined on $[-L, L]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \tag{7.11}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx. \tag{7.12}$$

7.2 Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions of finite period in term of sums over a discrete set of frequencies. In particular, for functions defined on $x \in [-L, L]$, the period of the Fourier series representation is $2L$. We can write the arguments in the exponentials, $e^{-in\pi x/L}$, in terms of the angular frequency, $\omega_n = n\pi/L$, as $e^{-i\omega_n x}$. We note that the frequencies, ν_n, are then defined through $\omega_n = 2\pi\nu_n = \frac{n\pi}{L}$. Therefore, the complex exponential series is seen to be a sum over a discrete, or countable, set of frequencies.

We would now like to extend our finite interval to $x \in (-\infty, \infty)$ and to extend the discrete set of frequencies to a continuous range of frequencies. One can do this rigorously. It amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed. We first define $\Delta\omega = \frac{\pi}{L}$, so that $\omega_n = n\Delta\omega$. Inserting the Fourier coefficients (7.12) into Equation (7.11), we have

$$\begin{aligned}
f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L}\int_{-L}^{L} f(\xi)e^{in\pi\xi/L}\,d\xi\right) e^{-in\pi x/L} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(\xi)e^{i\omega_n\xi}\,d\xi\right) e^{-i\omega_n x}.
\end{aligned} \tag{7.13}$$

Now, we let L get large, so that $\Delta\omega$ becomes small and ω_n approaches the angular frequency ω. Then

$$\begin{aligned}
f(x) &\sim \lim_{\Delta\omega\to 0, L\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi)e^{i\omega_n\xi}\,d\xi\right) e^{-i\omega_n x}\Delta\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}\,d\xi\right) e^{-i\omega x}\,d\omega.
\end{aligned} \tag{7.14}$$

Looking at this last result, we formally arrive at the definition of the *Fourier transform*

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{7.15}$$

This is a generalization of the Fourier coefficients (7.10). Once we know the Fourier transform, $\hat{f}(\omega)$, then we can *reconstruct* the original function, $f(x)$, using the *inverse Fourier transform*, which is given by

$$F^{-1}[f] = f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{7.16}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Such functions are said to be L_1.

The Fourier transform and inverse Fourier transform are inverse operations. This means that

$$F^{-1}[F[f]] = f(x) \tag{7.17}$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega). \tag{7.18}$$

We will now prove the first of these equations, (7.17). [The second equation, (7.18), follows in a similar way.]

The proof is carried out by inserting the definition of the Fourier transform, (7.15), into the inverse transform definition, (7.16), and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F[f] e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi \right] e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi - x)}\, d\xi d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega \right] f(\xi)\, d\xi. \end{aligned} \tag{7.19}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we first define

$$D_L(x) = \int_{-L}^{L} e^{i\omega x}\, d\omega$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega = \lim_{L\to\infty} D_L(\xi - x).$$

We can compute $D_L(x)$. A simple evaluation yields

$$\begin{aligned} D_L(x) &= \int_{-L}^{L} e^{i\omega x}\, d\omega \\ &= \frac{e^{i\omega x}}{ix}\Big|_{-L}^{L} \\ &= \frac{e^{ixL} - e^{-ixL}}{2ix} \\ &= \frac{2\sin xL}{x}. \end{aligned} \tag{7.20}$$

A plot of this function is in Figure 7.1 for $L = 4$. For large L the peak grows and the values of $D_L(x)$ for $x \neq 0$ tend to zero as show in Figure 7.2. In fact, as x approaches 0, $D_L(x)$ approaches $2L$. For $x \neq 0$, $D_L(x)$ function tends to zero.

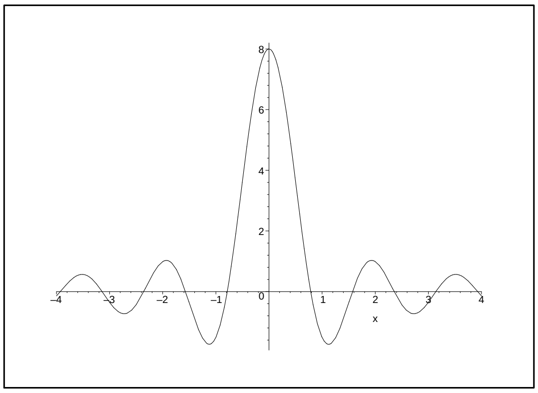

Fig. 7.1. A plot of the function $D_L(x)$ for $L = 4$.

We further note that

$$\lim_{L\to\infty} D_L(x) = 0 \quad x \neq 0$$

and $\lim_{L\to\infty} D_L(x)$ is infinite at $x = 0$. However, the area is constant for each L. In fact,

$$\int_{-\infty}^{\infty} D_L(x)\, dx = 2\pi.$$

We can show this by recalling the computation in Example 6.38,

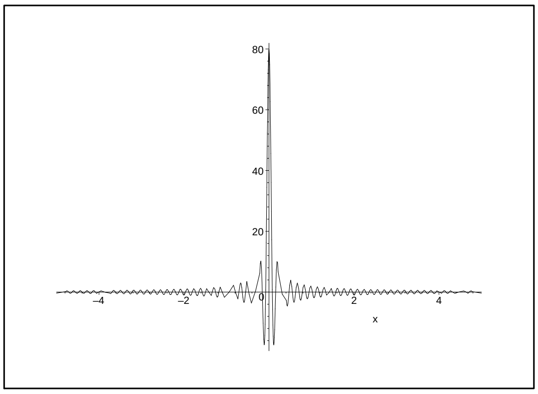

Fig. 7.2. A plot of the function $D_L(x)$ for $L = 40$.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Then,

$$\begin{aligned}\int_{-\infty}^{\infty} D_L(x)\, dx &= \int_{-\infty}^{\infty} \frac{2\sin xL}{x}\, dx \\ &= \int_{-\infty}^{\infty} 2\frac{\sin y}{y}\, dy \\ &= 2\pi. \end{aligned} \tag{7.21}$$

Another way to look at $D_L(x)$ is to consider the sequence of functions $f_n(x) = \frac{\sin nx}{\pi x}$, $n = 1, 2, \dots$. Then we have shown that this sequence of functions satisfies the two properties,

$$\lim_{n\to\infty} f_n(x) = 0,$$

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1.$$

This is a key representation of such generalized functions. The limiting value vanishes at all but one point, but the area is finite.

Such behavior can be seen for the limit of other sequences of functions. For example, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n}, \\ \frac{n}{2}, & |x| < \frac{1}{n}. \end{cases}$$

This is a sequence of functions as shown in Figure 7.4. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

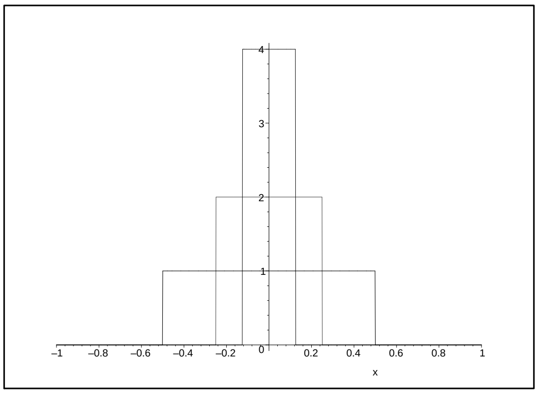

Fig. 7.3. A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a *generalized function.* It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

Before returning to the proof that the inverse Fourier transform of the Fourier transform is the identity, we state one more property of the Dirac delta function, which we will prove in the next section. Namely, we will show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\,d\omega = \lim_{L\to\infty} D_L(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (7.19), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\,d\omega\right] f(\xi)\,d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\,d\xi. \\ &= f(x). \end{aligned} \tag{7.22}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f.

7.3 The Dirac Delta Function

In the last section we introduced the Dirac delta function, $\delta(x)$. This is one example of what is known as a *generalized function*, or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral. [Note: The Dirac delta function was also discussed in the optional Section 4.4.]

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Integration over more general intervals gives

$$\int_{a}^{b} \delta(x)\, dx = \begin{cases} 1, \ 0 \in [a,b], \\ 0, \ 0 \notin [a,b]. \end{cases} \tag{7.23}$$

The other property that was used was the *sifting property*:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a).$$

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1} \delta(x). \tag{7.24}$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(ax)\, dx &= \lim_{L\to\infty} \int_{-L}^{L} \delta(ax)\, dx \\ &= \lim_{L\to\infty} \frac{1}{a} \int_{-aL}^{aL} \delta(y)\, dy. \end{aligned} \tag{7.25}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{\infty}^{-\infty} \delta(y)\, dy = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy. \tag{7.26}$$

Example 7.1. Evaluate $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx$. This is a straight forward integration:

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{11}{4}.$$

A more general scaling of the argument takes the form $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|} \delta(x - x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|} \delta(x - x_1).$$

Example 7.2. Evaluate $\int_{-\infty}^{\infty} \delta(3x-2)x^2\, dx$.

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x) = 3x-2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\, dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(x - \frac{2}{3})x^2\, dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y = 3x - 2$. Then, $dy = 3dx$ and $x = (y+2)/3$. This gives

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\, dx = \frac{1}{3} \int_{-\infty}^{\infty} \delta(y) \left(\frac{y+2}{3}\right)^2 dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for x_j, $j = 1, 2, \ldots, n$, (i.e.; when one has n simple zeros), then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|} \delta(x - x_j).$$

Example 7.3. Evaluate $\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx$.

In this case the argument of the delta function has two simple roots. Namely, $f(x) = x^2 - \pi^2 = 0$ when $x = \pm\pi$. Furthermore, $f'(x) = 2x$. Therefore, $|f'(\pm\pi)| = 2\pi$. This gives

$$\delta(x^2-\pi^2) = \frac{1}{2\pi}[\delta(x-\pi)+\delta(x+\pi)].$$

Inserting this expression into the integral and noting that $x=-\pi$ is not in the integration interval, we have

$$\begin{aligned}\int_0^{2\pi} \cos x\, \delta(x^2-\pi^2)\, dx &= \frac{1}{2\pi}\int_0^{2\pi} \cos x\, [\delta(x-\pi)+\delta(x+\pi)]\, dx \\ &= \frac{1}{2\pi}\cos\pi = -\frac{1}{2\pi}. \end{aligned} \tag{7.27}$$

Finally, one can show that there is a relationship between the Heaviside function (or, step function) and the Dirac delta function. We define the Heaviside function as

$$H(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$$

Then, it is easy to see that $H'(x)=\delta(x)$. In some texts the notation $\theta(x)$ is used for the step function.

7.4 Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt, \tag{7.28}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\, dx. \tag{7.29}$$

ω is called the *angular frequency* and is related to the *frequency* ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the *wavenumber*. It has units of inverse length and is related to the wavelength, λ, by $k=\frac{2\pi}{\lambda}$.

1. **Linearity.** For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

   $$F[f+g] = F[f]+F[g]$$

   and

$$F[af] = aF[f].$$

These simply follow from the properties of integration and establish the linearity of the Fourier transform.

2. **Transform of a Derivative.** $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

   Here we simple compute the Fourier transform (7.15) of the derivative by inserting the derivative in the Fourier integral and using integration by parts.

$$\begin{aligned} F\left[\frac{df}{dx}\right] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx \\ &= \lim_{L\to\infty} \left(f(x)e^{ikx}\right)\Big|_{-L}^{L} - ik\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx. \end{aligned} \tag{7.30}$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The last integral is recognized as the Fourier transform of f, proving the given property.

3. **Higher Order Derivatives.** $F\left[\frac{d^n f}{dx^n}\right] = (-ik)^n\hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx}f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2\hat{f}(k). \end{aligned} \tag{7.31}$$

   This result will be true if both $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to\pm\infty} f'(x) = 0$. The generalization to the transform of the nth derivative easily follows.

4. $F\left[xf(x)\right] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and the ability to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)\, dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \tag{7.32}$$

5. **Shifting Properties.** For constant a, we have the following shifting properties:

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{7.33}$$

$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{7.34}$$

Here we have denoted the Fourier transform pairs as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform (7.15), or inverse Fourier transform (7.16). The first shift property (7.33) is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

Now perform the substitution $y = x - a$. Then,

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy = e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy = e^{ika}\hat{f}(k).$$

The second shift property (7.34) follows in a similar way.

6. **Convolution** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dx. \tag{7.35}$$

Then the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions:

$$F[f*g] = \hat{f}(k)\hat{g}(k). \tag{7.36}$$

We will return to the proof of this property in Section 7.5.

7.4.1 Fourier Transform Examples

In this section we will compute the Fourier transforms of several functions.

Example 7.4. **Gaussian Functions.** $f(x) = e^{-ax^2/2}$.

This function is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{7.37}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\, dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned}
-\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\
&= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\
&= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}. \qquad (7.38)
\end{aligned}$$

We now substitute this expression in the integral and make the substitution $y = x - \frac{ik}{a}$.

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\, dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2}\, dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2}\, dy. \qquad (7.39)
\end{aligned}$$

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane. In this case we can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we now safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Define I by

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Note that we needed to change the integration variable so that we can write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-\beta r^2}\, r dr d\theta \\ &= 2\pi \int_0^{\infty} e^{-\beta r^2}\, r dr \\ &= -\frac{\pi}{\beta}\left[e^{-\beta r^2}\right]_0^{\infty} = \frac{\pi}{\beta}. \end{aligned} \tag{7.40}$$

The final result is gotten by taking the square root, yielding

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{7.41}$$

Example 7.5. **Box or Gate Function.** $f(x) = \begin{cases} b, & |x| \leq a \\ 0, & |x| > a \end{cases}.$

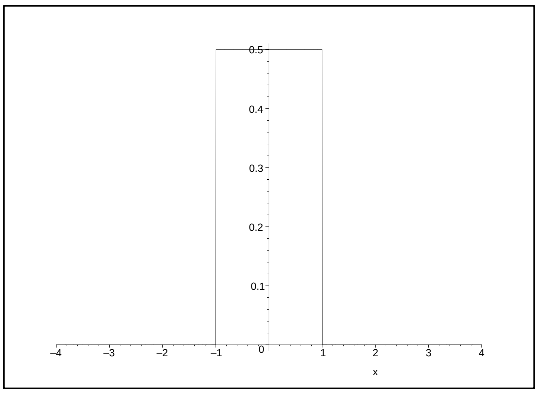

Fig. 7.4. A plot of the box function in Example 7.5.

This function is called the box function, or gate function. It is shown in Figure 7.4. The Fourier transform of the box function is relatively easy to compute. It is given by

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\ &= \int_{-a}^{a} b e^{ikx}\, dx \end{aligned}$$

$$= \frac{b}{ik} e^{ikx} \Big|_{-a}^{a}$$
$$= \frac{2b}{k} \sin ka. \tag{7.42}$$

We can rewrite this as

$$\hat{f}(k) = 2ab \frac{\sin ka}{ka} \equiv 2ab \,\text{sinc}\; ka.$$

Here we introduced the sinc function,

$$\text{sinc}\; x = \frac{\sin x}{x}.$$

A plot of this function is shown in Figure 7.5.

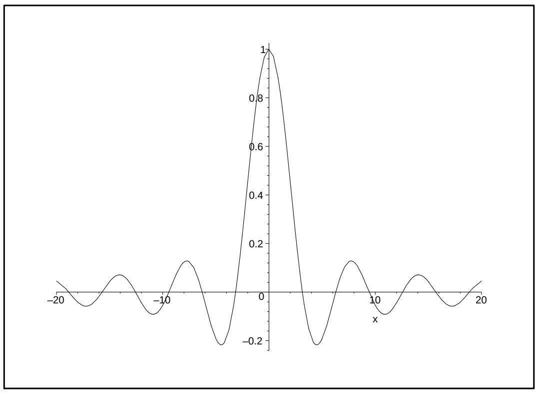

Fig. 7.5. A plot of the Fourier transform of the box function in Example 7.5. This is the general shape of the sinc function.

We will now consider special limiting values for the box function and its transform. This will lead us to the Uncertainty Principle for signals, connecting the relationship between the localization properties of a signal and its transform.

1. $a \to \infty$ and b fixed.
   In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 7.5 with Figure 7.1. In fact, $\hat{f}(k) = bD_a(k)$. [Recall the definition of $D_L(x)$ in Equation (7.20).] So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit implies fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of

the box becomes wider, the Fourier transform becomes more localized. In fact, we have arrived at the result that

$$\int_{-\infty}^{\infty} e^{ikx} = 2\pi\delta(k). \tag{7.43}$$

2. $b \to \infty$, $a \to 0$, and $2ab = 1$.
   In this case our box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

   $$\int_{-\infty}^{\infty} \delta(x)e^{ikx} = 1. \tag{7.44}$$

   In this case we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform, $\hat{f}(k)$, is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.
3. **The Uncertainty Principle**
   The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx of the box function as

   $$\Delta x = 2a.$$

   The width of the Fourier transform is a little trickier. This function actually extends the entire k-axis. However, as $\hat{f}(k)$ became more localized, the central peak in Figure 7.5 became narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe. This gives

   $$\Delta k = \frac{2\pi}{a}.$$

   Combining these two relations, we find that

   $$\Delta x \Delta k = 4\pi.$$

   Thus, the more localized a signal, the less localized its transform. This notion is referred to as the Uncertainty Principle. For general signals, one needs to define the effective widths more carefully, but the main idea holds:

   $$\Delta x \Delta k \geq c > 0.$$

We now turn to other examples of Fourier transforms.

Example 7.6. $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \ a > 0.$

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\ &= \int_{o}^{\infty} e^{ikx - ax}\, dx \\ &= \frac{1}{a - ik}. \end{aligned} \tag{7.45}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 7.7. $\hat{f}(k) = \frac{1}{a-ik}$.

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{a - ik}\, dk.$$

This integral can be evaluated using contour integral methods. We recall Jordan's Lemma from the last chapter:

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{ikz}\, dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$. A similar result applies for $k < 0$, but one closes the contour in the lower half plane.

In this example, we have to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a - iz}\, dz.$$

According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$. The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{a - ik}\, dk \\ &= \pm \frac{1}{2\pi} \oint_C \frac{e^{-ixz}}{a - iz}\, dz \\ &= \begin{cases} 0, & x < 0 \\ -\frac{1}{2\pi} 2\pi i \text{ Res } [z = -ia], & x > 0 \end{cases} \\ &= \begin{cases} 0, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}. \end{aligned} \tag{7.46}$$

Example 7.8. $\hat{f}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega - \omega_0)$. This is a shifted function. From the shift theorems in Equations (7.33)-(7.34) we have

$$e^{i\omega_0 t} f(t) \leftrightarrow \hat{f}(\omega - \omega_0).$$

Recalling from a previous example that

$$\int_{-\infty}^{\infty} e^{i\omega t}\, dt = 2\pi\delta(\omega),$$

we have

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} e^{-i\omega_0 t}.$$

The second term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0] = \frac{1}{2} e^{i\omega_0 t} + \frac{1}{2} e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 7.9. **The Finite Wave Train.** $f(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}$.

For our last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. A straight forward computation gives

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \\ &= \int_{-a}^{a} \cos\omega_0 t e^{i\omega t}\, dt \\ &= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\, dt \\ &= \frac{1}{2} \int_{-a}^{a} \left[\cos(\omega + \omega_0)t + \cos(\omega - \omega_0)t\right]\, dt \\ &= \frac{\sin(\omega + \omega_0)a}{\omega + \omega_0} + \frac{\sin(\omega - \omega_0)a}{\omega - \omega_0}. \end{aligned} \tag{7.47}$$

7.5 The Convolution Theorem

In our list of properties, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t)\, dt. \tag{7.48}$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is *faltung*, which means "folding".

First, we note that the convolution is commutative: $f * g = g * f$. This is easily shown by replacing $x - t$ with a new variable, y.

$$\begin{aligned}(g * f)(x) &= \int_{-\infty}^{\infty} g(t) f(x-t)\, dt \\ &= -\int_{\infty}^{-\infty} g(x-y) f(y)\, dy \\ &= \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \\ &= (f * g)(x). \end{aligned} \tag{7.49}$$

Example 7.10. Graphical Convolution.

In order to understand the convolution operation, we need to apply it to several functions. We will do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

as shown in Figures 7.6 and 7.7.

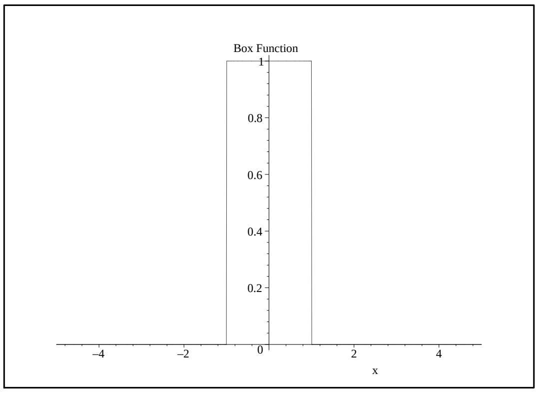

Fig. 7.6. A plot of the box function $f(x)$.

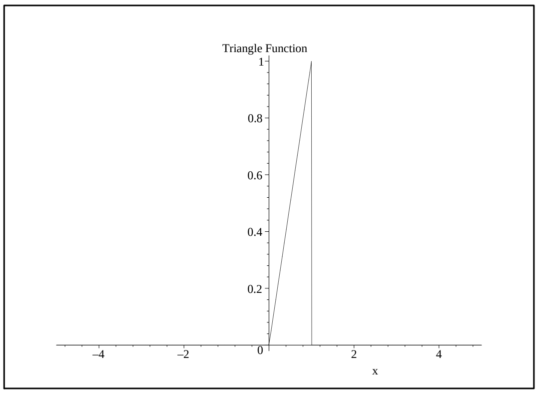

Fig. 7.7. A plot of the triangle function.

In order to determine the contributions to the integrand, we look at the shifted and reflected function $g(t-x)$ for various values of t. For $t=0$, we have $g(-x)$. This is a reflection of the triangle function as shown in Figure 7.8.

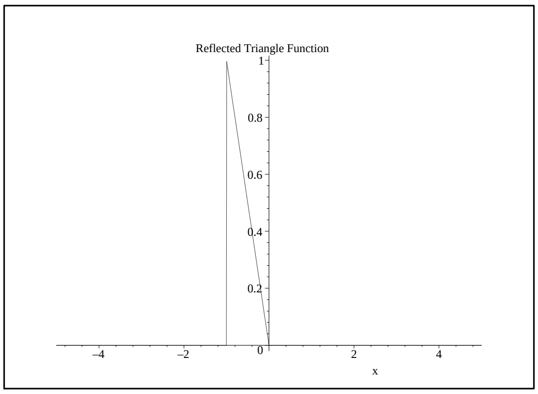

Fig. 7.8. A plot of the reflected triangle function.

We then translate this function performing horizontal shifts by t. In Figure 7.9 we show such a shifted and reflected $g(x)$ for $t=2$. The following figures show other shifts superimposed on $f(x)$. The integrand is the product of $f(x)$ and $g(t-x)$ and the convolution evaluated at t is given by the shaded areas. In Figures 7.10 and 7.14 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in Figures 7.11-7.13 and the convolution is shown by the area under the product of the two functions.

Next we would like to compute the Fourier transform of the convolution integral. First, we use the definitions of Fourier transform and convolution to

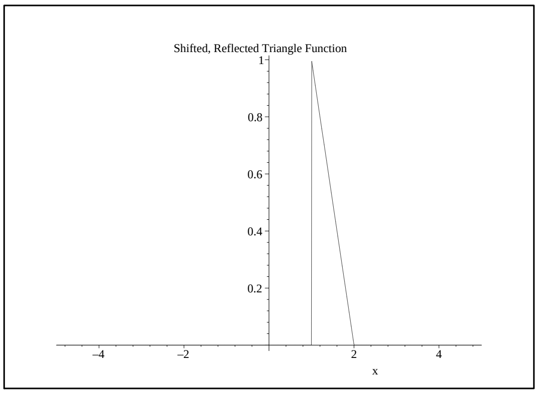

Fig. 7.9. A plot of the reflected triangle function shifted by 2 units.

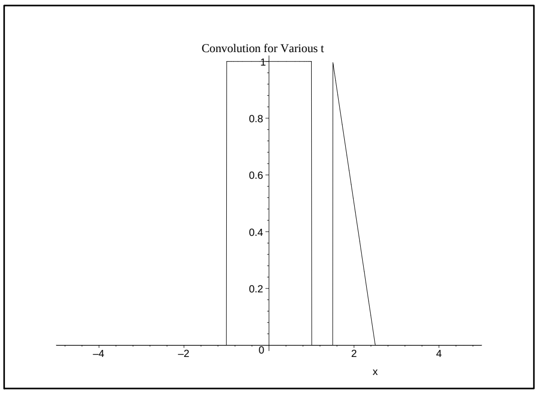

Fig. 7.10. A plot of the box and triangle functions with the overlap indicated by the shaded area.

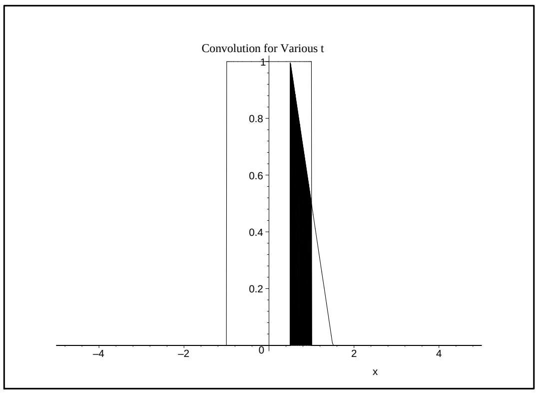

Fig. 7.11. A plot of the box and triangle functions with the overlap indicated by the shaded area.

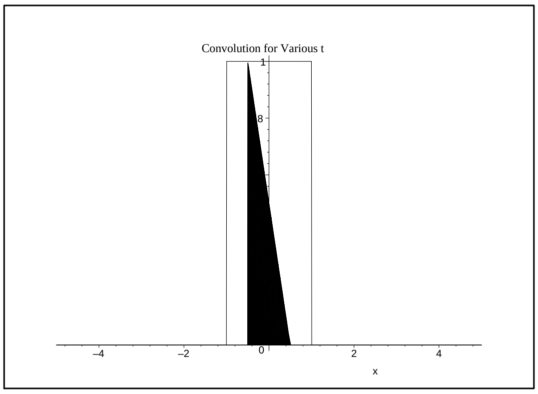

Fig. 7.12. A plot of the box and triangle functions with the overlap indicated by the shaded area.

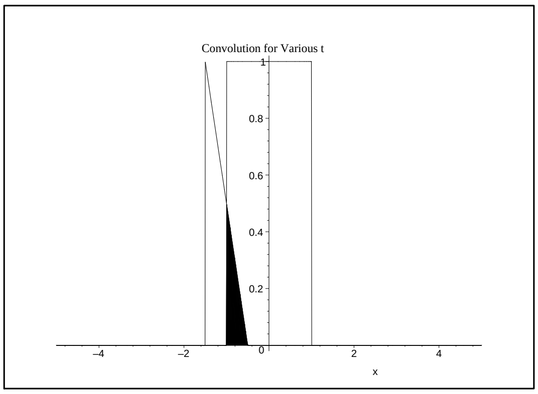

Fig. 7.13. A plot of the box and triangle functions with the overlap indicated by the shaded area.

write the transform as

$$\begin{aligned} F[f * g] &= \int_{-\infty}^{\infty} (f * g)(x) e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) g(x-t)\, dt \right) e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt. \end{aligned} \tag{7.50}$$

Next, we substitute $y = x - t$ on the inside integral and separate the integrals:

$$F[f * g] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{ik(y+t)}\, dy \right) f(t)\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right) f(t) e^{ikt}\, dt. \qquad (7.51)
\end{aligned}
$$

We see that the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f * g] = \hat{f}(k)\hat{g}(k). \qquad (7.52)$$

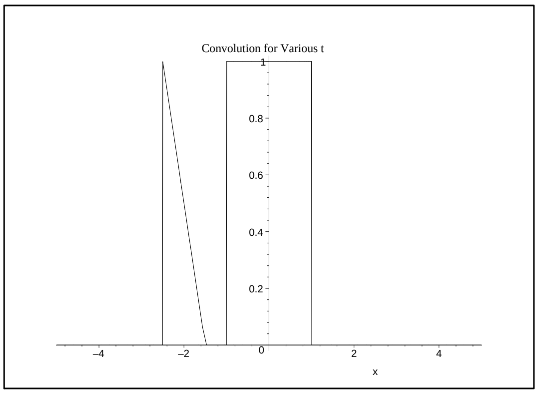

Fig. 7.14. A plot of the box and triangle functions with the convolution indicated by the shaded area.

We see that the value of the convolution integral builds up and then quickly drops to zero. The plot of the convolution of the box and triangle functions is given in Figure 7.15.

Example 7.11. Convolution of two Gaussian functions $f(x) = e^{-ax^2}$.

In this example we will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt = \int_{-\infty}^{\infty} e^{-at^2 - b(x-t)^2}\, dt.$$

This integral can be rewritten as

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2bxt}\, dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution. Recalling the Fourier transform of a Gaussian, we have

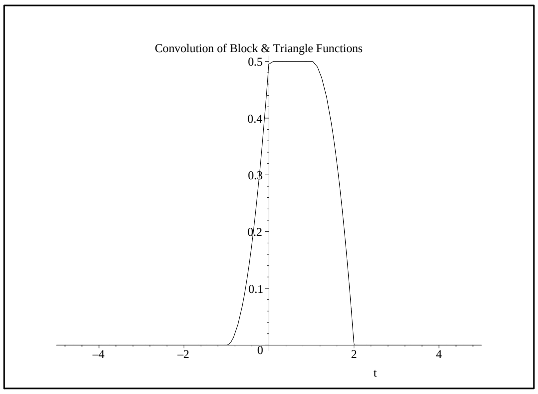

Fig. 7.15. A plot of the overlap of the box and triangle functions.

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a} \tag{7.53}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{2\pi}{b}} e^{-k^2/2b}.$$

Denoting the convolution function by $h(x) = (f * g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{2\pi}{\sqrt{ab}} e^{-k^2/2a} e^{-k^2/2b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{2\pi}{\sqrt{ab}} e^{-\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{2\pi}{\sqrt{ab}} e^{-\frac{a+b}{2ab}k^2}. \tag{7.54}$$

In order to complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (7.54). We can do this by looking at Equation (7.53). We have first that

$$F^{-1}\left[\sqrt{\frac{2\pi}{a}} e^{-k^2/2a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/2a}] = \sqrt{\frac{a}{2\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{2a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/2\alpha}.$$

This is in the form needed to invert (7.54). Thus, for $\alpha = \frac{a+b}{2ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{2\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

7.6 Applications of the Convolution Theorem

There are many applications of the convolution operation. In this section we will describe a few of the applications, which are useful in studying signal analysis.

The first application is filtering. For a given signal there might be some noise in the signal, or some undesirable high frequencies. Or, the device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal. An example is provided in Figure 7.16. Recall that the Fourier transform gives the frequency content of the signal and that $\omega = 2\pi\nu$, where ν is the frequency in Hertz, or cycles per second (cps).

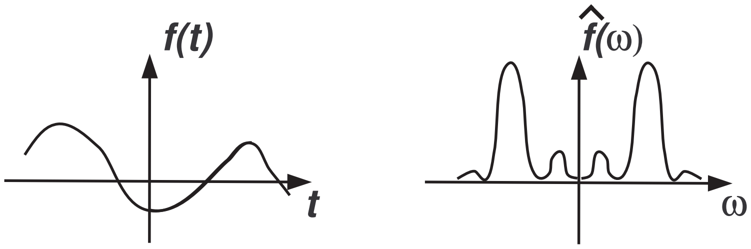

Fig. 7.16. Plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high frequencies, $|\omega| > \omega_0$ for some cutoff frequency ω_0. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could consider the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}. \tag{7.55}$$

Figure 7.17 shows how the gate function is used to filter the signal.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

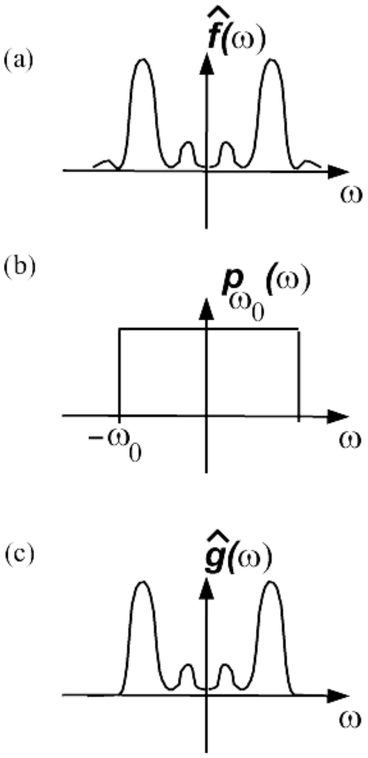

Fig. 7.17. (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b).

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{7.56}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the *impulse response.* This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\,d\tau = h(t).$$

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

We will later see that the effect of windowing would be to change the spectral content of the signal we are trying to analyze. We will study these natural windowing and filtering effects from recording data in the last chapter.

Example 7.12. **Finite Wave Train, Revisited.**

We return to the finite wave train in Example 7.9 given by

$$h(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}.$$

We can view this as a windowed version of $f(t) = \cos\omega_0 t$ obtained by multiplying $f(t)$ by the gate function

$$g_a(t) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}. \tag{7.57}$$

Then, the Fourier transform is given as

$$\begin{aligned} \hat{h}(\omega) &= (\hat{f} * \hat{g}_a)(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\,d\nu. \end{aligned} \tag{7.58}$$

Note that the convolution in frequency space requires the extra factor of $1/(2\pi)$.

We need the Fourier transforms of f and g in order to finish the computation. The Fourier transform of the box function was already found as

$$\hat{g}_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

The Fourier transform of the cosine function is

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t)e^{i\omega t}\,dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)e^{i\omega t}\,dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right)\,dt \\ &= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]. \end{aligned} \tag{7.59}$$

Note that we had earlier computed the inverse Fourier transform of this function.

Inserting these results in the convolution integral, we have

$$\begin{aligned}\hat{h}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega-\nu)\hat{g}_a(\nu)\,d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\left[\delta(\omega-\nu+\omega_0)+\delta(\omega-\nu-\omega_0)\right]\frac{2}{\nu}\sin\nu a\,d\nu \\ &= \frac{\sin(\omega+\omega_0)a}{\omega+\omega_0}+\frac{\sin(\omega-\omega_0)a}{\omega-\omega_0}. \qquad (7.60)\end{aligned}$$

This is the same result we had obtained in Example 7.9.

7.6.1 Parseval's Equality

As another example of the convolution theorem, we derive **Parseval's Equality**:

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega. \qquad (7.61)$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, sometimes referred as Plancherel's formula, is simply a statement that the energy is invariant under the transform.

Let's rewrite the Convolution Theorem in the form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f*g)(t). \qquad (7.62)$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty}f(t-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Setting $t=0$,

$$\int_{-\infty}^{\infty}f(-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)\,d\omega. \qquad (7.63)$$

Now, let $g(t)=\overline{f(-t)}$, or $f(-t)=\overline{g(t)}$. Then, the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty}\overline{f(-t)}e^{i\omega t}\,dt \\ &= -\int_{\infty}^{-\infty}\overline{f(\tau)}e^{-i\omega\tau}\,d\tau \\ &= \overline{\int_{-\infty}^{\infty}f(\tau)e^{i\omega\tau}\,d\tau} = \overline{\hat{f}(\omega)}. \qquad (7.64)\end{aligned}$$

So, inserting this result into Equation (7.63), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

which implies Parseval's Equality in the form (7.61).

The forms in Equations (7.61) and (7.63) are often referred to as the bf Plancheral formula or Paresval formula. A more commonly defined Paresval equation is that given for Fourier series. For example, for a function $f(x)$ defined on $[-\pi,\pi]$, which has a Fourier series representation, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx.$$

There is a Parseval identity for functions that can be expanded in a complete sets of orthonormal functions, $\{\phi_n(x)\}$, $n = 1, 2, \ldots$, which is given by

$$\sum_{n=1}^{\infty} < f, \phi_n >^2 = \|f\|^2.$$

Here $\|f\|^2 =< f, f >$. The Fourier series example is just a special case of this formula.

7.7 The Laplace Transform

Up until this point we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving ordinary differential equations.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 7.18 it is shown that a signal $x(t)$ is provided as input to a linear system, $h(t)$. One is interested in the output, $y(t)$. By transforming both $x(t)$ and $y(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to that we saw earlier in the chapter for Fourier transforms. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-st}\, dt, \quad s > 0. \tag{7.65}$$

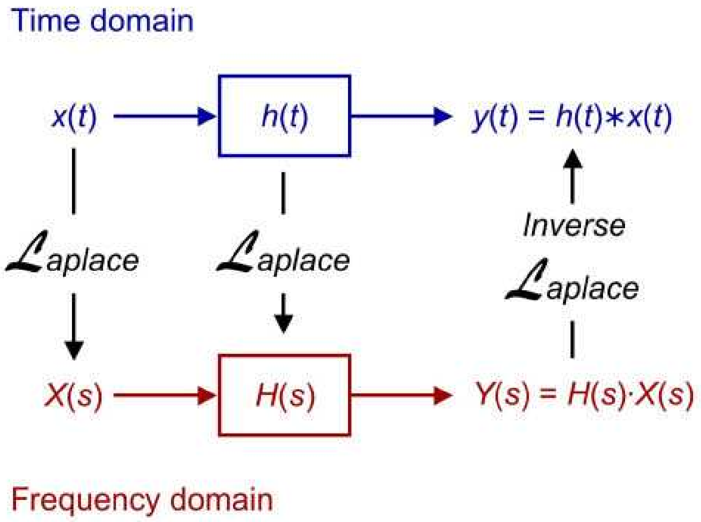

Fig. 7.18. A schematic depicting the use of Laplace transforms in systems theory. (current version from Wikipedia).

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

to guarantee convergence.

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 7.7. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 7.7, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these can be used to solve ordinary differential equations.

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 7.13. $\mathcal{L}[1]$

For this example, we insert $f(t) = 1$ into the integral transform:

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a \to \infty$. We will not always write this

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a}, \quad s>a$
t^n	$\frac{n!}{s^{n+1}}, \quad s>0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s>0$	$\delta(t-a)$	$e^{-as}, \quad a\geq 0, s>0$

Table 7.1. Table of selected Laplace transform pairs.

limit, but it will be understood that this is how one computes such improper integrals. Proceeding with the computation, we have

$$\begin{aligned}
\mathcal{L}[1] &= \int_0^\infty e^{-st}\,dt \\
&= \lim_{a\to\infty}\int_0^a e^{-st}\,dt \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-st}\right)_0^a \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-sa}+\frac{1}{s}\right) = \frac{1}{s}. \qquad (7.66)
\end{aligned}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant c, using the linearity of the transform. Since the Laplace transform is simply an integral, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 7.14. $\mathcal{L}[e^{at}]$,

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty
\end{aligned}$$

$$= \lim_{t\to\infty} \frac{1}{a-s} e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}. \tag{7.67}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$. This is only true if $a - s < 0$, or $s > a$. [Actually, a could be complex. In this case we would only need that s is greater than the real part of a.]

Example 7.15. $\mathcal{L}[\cos at]$ and $\mathcal{L}[\sin at]$

In these cases, we could again insert the functions directly into the transform. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st} \cos at \, dt.$$

Recall how one does such integrals involving both the trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one can arrive at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i \sin at$. Making use of the linearity of the Laplace transform, we have

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential and looking at the real and imaginary parts of the results will give both transforms at the same time! The transform is simply computed as

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat} e^{-st} \, dt = \int_0^\infty e^{-(s-ia)t} \, dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia} \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts gives

$$\begin{aligned} \mathcal{L}[\cos at] &= \frac{s}{s^2+a^2} \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}. \end{aligned} \tag{7.68}$$

Example 7.16. $\mathcal{L}[t]$

For this example we need to evaluate

$$\mathcal{L}[t] = \int_0^\infty t e^{-st} \, dt.$$

This integration can be done using integration by parts. (Pick $u = t$ and $dv = e^{-st}\,dt$. Then, $du = dt$ and $v = -\frac{1}{s}e^{-st}$.) So, we have

$$\begin{aligned}\int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}.\end{aligned} \tag{7.69}$$

Example 7.17. $\mathcal{L}[t^n]$

We can generalize the last example to integer powers of t greater than $n = 1$. In this case we have to do the integral

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

Following the previous example, we integrate by parts:

$$\begin{aligned}\int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + n\frac{1}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= n\frac{1}{s}\int_0^\infty t^{-n}e^{-st}\,dt.\end{aligned} \tag{7.70}$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

This is an example of a recursive definition of a sequence. In this case we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I[0] = \mathcal{L}[1] = \frac{1}{s}$, we have the following:

$$I_n = \frac{n}{s}I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{7.71}$$

This is also what is called a difference equation. It is a *first order difference equation* with an “initial condition", I_0. There is a whole theory of difference equations, which we will not get into here.

Our goal is to solve the above difference equation. It is easy to do by simple iteration. Note that replacing n with $n-1$, we have

$$I_{n-1} = \frac{n-1}{s}I_{n-2}.$$

So, repeating the process we find

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s} I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2} I_{n-2}. \end{aligned} \tag{7.72}$$

We can repeat this process until we get to I_0, which we know. In some cases you need to be careful so that you can count the number of iterations of the process. So, we first ask what the result is after k steps. This can be seen by watching for patterns. Continuing the iteration process, we have

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \ldots \\ &= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}. \end{aligned} \tag{7.73}$$

Since we know I_0, we choose to stop at $k = n$ obtaining

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$. [Such iterative techniques are useful in obtaining a variety of of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$.]

As a final note, one can extend this result to cases when n is not an integer. To do this, one introduces what is called the Gamma function. This function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{7.74}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x - 1$ an integer and $s = 1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function seems to be a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Laplace Transform Properties
$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}[\frac{dy}{dt}] = sY(s)-y(0)$
$\mathcal{L}[\frac{d^2y}{dt^2}] = s^2Y(s)-sy(0)-y'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Table 7.2. Table of selected Laplace transform properties.

Example 7.18. $\mathcal{L}[\frac{df}{dt}]$

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}\,dt.$$

We can move the derivative off of f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have

$$\begin{aligned}\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0)+sF(s). \end{aligned} \tag{7.75}$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

Example 6: $\mathcal{L}[\frac{d^2f}{dt^2}]$

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s)-g(0) = sG(s)-f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

So,

$$\begin{aligned}\mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0)\\ &= s\left[sF(s) - f(0)\right] - f'(0)\\ &= s^2F(s) - sf(0) - f'(0).\end{aligned} \tag{7.76}$$

7.8 Further Uses of Laplace Transforms

The Laplace transform is a very useful transform and is often encountered as just a method for solving initial value problems. In this section we will show how this is possible, introduce Fourier transforms of other functions and present the Laplace transform of a convolution. We will also derive the Bromwich integral, which is the inverse Laplace transform integral. The inverse transform is not usually covered in differential equations courses because the integration takes place in the complex plane.

Another application of Laplace transforms is in finding sums of infinite series. We explore this topic first. Generally, many of the topics in this section are optional and not needed in the rest of the text.

7.8.1 Series Summation Using Laplace Transforms

We saw in Chapter 3 that Fourier series can be used to sum series. For example, in Problem 5.3, one gets to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this section we will show how Laplace transforms can be used to sum series. [See Wheelon's book[1].] There is an interesting history of using integral transforms to sum series. For example, Richard Feynman[2] described how one can use the convolution theorem for Laplace transforms to sum series with denominators that involved products. We will describe this and simpler sums in this section.

We begin by considering the Laplace transform of a known function,

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Inserting this expression into the sum $\sum_n F(n)$ and interchanging the sum and integral, we find

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[1] Albert D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, 1968.

[2] R. P. Feynman, 1949, *Phys. Rev.* **76**, p. 769

$$\begin{aligned} \sum_{n=0}^{\infty} F(n) &= \sum_{n=0}^{\infty} \int_0^{\infty} f(t) e^{-nt}\, dt \\ &= \int_0^{\infty} f(t) \sum_{n=0}^{\infty} \left(e^{-t}\right)^n\, dt \\ &= \int_0^{\infty} f(t) \frac{1}{1-e^{-t}}\, dt. \end{aligned} \tag{7.77}$$

The last step was obtained using the sum of a geometric series. The key is being able to carry out the final integral as we show in the next example.

Example 7.19. Evaluate the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Since, $\mathcal{L}[1] = 1/s$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} &= \sum_{n=1}^{\infty} \int_0^{\infty} (-1)^n e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{-e^t}{1+e^t}\, dt = \ln 2. \end{aligned} \tag{7.78}$$

Example 7.20. Evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This is a special case of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7.79}$$

This function is important in the study of prime numbers and more recently has seen applications in the study of dynamical systems. The series in this example is $\zeta(2)$. We have already seen that

$$\zeta(2) = \frac{\pi^2}{6}.$$

Using Laplace transforms, we can provide an integral representation of $\zeta(2)$.

The first step is to find the correct Laplace transform pair. The sum involves the function $F(n) = 1/n^2$. So, we look for a function $f(t)$ whose Laplace transform is $F(s) = 1/s^2$. We know by now that The inverse Laplace transform of $F(s) = 1/s^2$ is $f(t) = t$. As before, we replace each term in the series by a Laplace transform integral:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{t}{e^t - 1}\, dt. \end{aligned} \tag{7.80}$$

So, we have that

$$\int_0^\infty \frac{t}{e^t - 1}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Integrals of this type occur often in statistical mechanics in the form of Bose-Einstein integrals. These are of the form

$$G_n(z) = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx.$$

Note that $G_n(1) = \Gamma(n)\zeta(n)$.

In general the Riemann zeta function has to be tabulated through other means. In some special cases, one can closed form expressions. For example,

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n,$$

where the B_n's are the Bernoulli numbers. Bernoulli numbers are defined through the Maclaurin series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The first few zeta functions are,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

We can extend this method of using Laplace transforms to summing series whose terms take special general forms. For example we note that

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a} \int_0^\infty e^{-s(a+bn)}\, ds.$$

This can be shown easily by first noting

$$\int_0^\infty e^{-s(a+bn)}\, ds = \left[\frac{-e^{-s(a+bn)}}{a+bn} \right]_0^\infty = \frac{1}{a+bn}.$$

Now, differentiate this with respect to a and the result follows.

This can be generalized further as

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial a^k} \int_0^\infty e^{-s(a+bn)}\, ds.$$

In Feynman's 1949 paper, he develops methods for handling several other general sums using the convolution theorem. Wheelon gives more examples of these. We will just provide one such result and an example. First, we note that

$$\frac{1}{ab} = \int_0^1 \frac{du}{[a(1-u)+bu]^2}.$$

However,

$$\frac{1}{[a(1-u)+bu]^2} = \int_0^\infty te^{-t[a(1-u)+bu]}\,dt.$$

So, we have

$$\frac{1}{ab} = \int_0^1 du \int_0^\infty te^{-t[a(1-u)+bu]}\,dt.$$

We see in the next example how this representation can be useful.

Example 7.21. Evaluate $\sum_{n=0}^\infty \frac{1}{(2n+1)(2n+2)}$. We compute this as follows:

$$\begin{aligned}
\sum_{n=0}^\infty \frac{1}{(2n+1)(2n+2)} &= \sum_{n=0}^\infty \int_0^1 \frac{du}{[(2n+1)(1-u)+(2n+2)u]^2} \\
&= \sum_{n=0}^\infty \int_0^1 du \int_0^\infty te^{-t(2n+1+u)}\,dt \\
&= \int_0^\infty \frac{e^{-t}}{1-e^{-2t}} \int_0^1 e^{-tu}\,du\,dt \\
&= \int_0^\infty \frac{te^{-t}}{1-e^{-2t}} \frac{1-e^{-t}}{t}\,dt \\
&= \int_0^\infty \frac{e^{-t}}{1+e^{-t}}\,dt \\
&= -\ln(1+e^{-t})\Big|_0^\infty = \ln 2. \qquad (7.81)
\end{aligned}$$

7.8.2 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 7.19.

Later we will see that there is an integral form for the inverse transform. This is typically not covered in introductory differential equations classes as one needs carry out integrations in the complex plane.

Example 7.22. Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

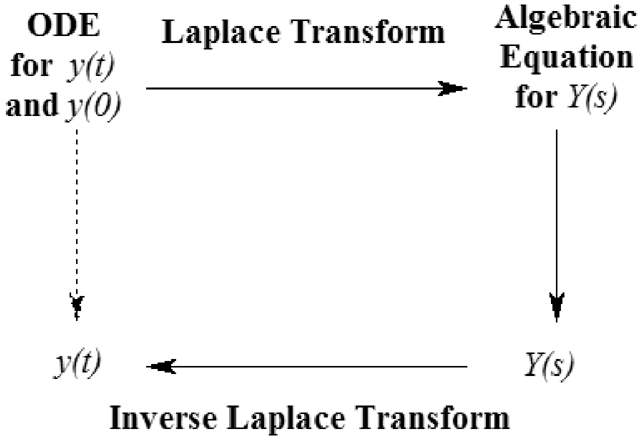

Fig. 7.19. The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform give the solution to the initial value problem.

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the right hand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

Combining these, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now, we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. It is easy to do if we only had the first term. The inverse transform of the first term is e^{-3t}.

We have not seen anything that looks like the second form in the table of transforms that we have compiled so far. However, we are not stuck. We know that we can rewrite the second term by using a *partial fraction decomposition.* Let's recall how to do this. The goal is to find constants, A and B, such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}.$$

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterwards. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3)+B(s-2).$$

This has to be true for all s. Rewriting the equation by gathering terms with common powers of s, we have

$$(A+B)s + 3A - 2B = 1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

$$\begin{aligned} A+B &= 0 \\ 3A-2B &= 1. \end{aligned} \tag{7.82}$$

The first equation gives $A = -B$, so the second equation becomes $-5B = 1$. The solution is then $A = -B = \frac{1}{5}$.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

[Of course, we could have tried to guess the form of the partial fraction decomposition as we had done earlier when talking about Laurent series.] In order to finish the problem at hand, we find a function whose Laplace transform is of this form. We easily see that

$$y(t) = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

Example 7.23. Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{7.83}$$

Solving for Y, we have

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

The first term is now recognizable as the transform of $\cos 2t$. The second term is not the transform of $\sin 2t$. It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

The solution is then found as

$$y(t) = \mathcal{L}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

7.8.3 Step and Impulse Functions

Often the initial value problems that one faces in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple of more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{7.84}$$

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 7.20. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \frac{e^{-st}}{s}\bigg|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{7.85}$$

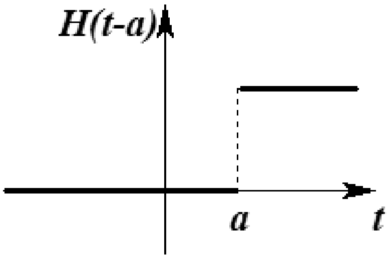

Fig. 7.20. A shifted Heaviside function, $H(t-a)$.

Just like the Fourier transform, the Laplace transform has two shift theorems involving multiplication of $f(t)$ or $F(s)$ by exponentials. These are given by

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \tag{7.86}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s). \tag{7.87}$$

We prove the first shift theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned}\mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a).\end{aligned} \tag{7.88}$$

Example 7.24. Compute the Laplace transform of $e^{-at}\sin\omega t$.

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The shift theorem tells us that we need the transform of the sine function. So,

$$F(s) = \frac{\omega}{s^2+\omega^2}.$$

Using this transform, we can obtain the solution to our problem as

$$\mathcal{L}[e^{-at}\sin\omega t] = F(s+a) = \frac{\omega}{(s+a)^2+\omega^2}.$$

More interesting examples can be found in piecewise functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$ both terms are zero. In the interval $[0, a]$ the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0, a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. This function is shown in Figure 7.21.

We now consider the piecewise defined function

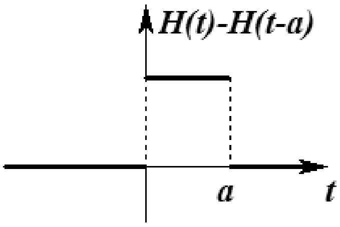

Fig. 7.21. The box function, $H(t) - H(t-a)$.

$$g(t) = \begin{cases} f(t), & 0 \leq t \leq a, \\ 0, & t < 0, t > a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function, $g(t) = f(t)[H(t) - H(t-a)]$. We depict this in Figure 7.22.

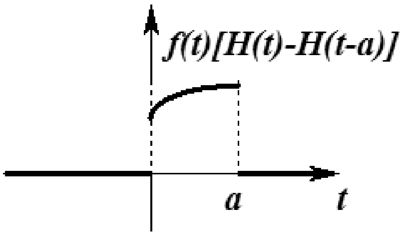

Fig. 7.22. Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

Even more complicated functions can be written out in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a) - H(t-b)]$ for $b > a$. This is just a box between a and b of height $f(t)$. An example of a square wave function is shown in Figure 7.23. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty} [H(t-2na) - H(t-(2n+1)a)].$$

Example 7.25. Laplace Transform of a square wave turned on at $t = 0$,

$$f(t) = \sum_{n=0}^{\infty} [H(t-2na) - H(t-(2n+1)a)].$$

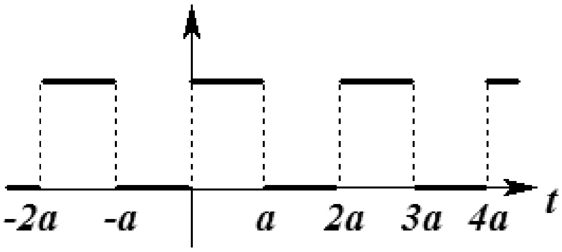

Fig. 7.23. A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)]$.

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty} \left[\mathcal{L}[H(t-2na)] - \mathcal{L}[H(t-(2n+1)a)]\right] \\
&= \sum_{n=0}^{\infty} \left[\frac{e^{-2nas}}{s} - \frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s} \sum_{n=0}^{\infty} \left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s} \left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (7.89)
\end{aligned}$$

Note that the third line in the derivation is a geometric series. We summed this series to get our answer in a compact form.

Another interesting example is the delta function. The delta function represents a point impulse, or point driving force. For example, while a mass on a spring is undergoing simple harmonic motion, one could hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$. One would then need the Laplace transform of the delta function to solve the associated initial value problem.

We find that for $a > 0$

$$\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \int_0^{\infty} \delta(t-a)e^{-st}\,dt \\
&= \int_{-\infty}^{\infty} \delta(t-a)e^{-st}\,dt \\
&= e^{-as}. \qquad (7.90)
\end{aligned}$$

Example 7.26. Solve the initial value problem $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.

First, transform the differential equation:

$$s^2 Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some $F(s)$. Thus, we need the second shift theorem. First we need to find the $f(t)$ corresponding to

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

The denominator suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi} \frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1} \sin 2\pi t$.

We now apply the second shift theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$.

$$\begin{aligned} y(t) &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi} H(t-2) \sin 2\pi(t-2). \end{aligned} \tag{7.91}$$

This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 7.24

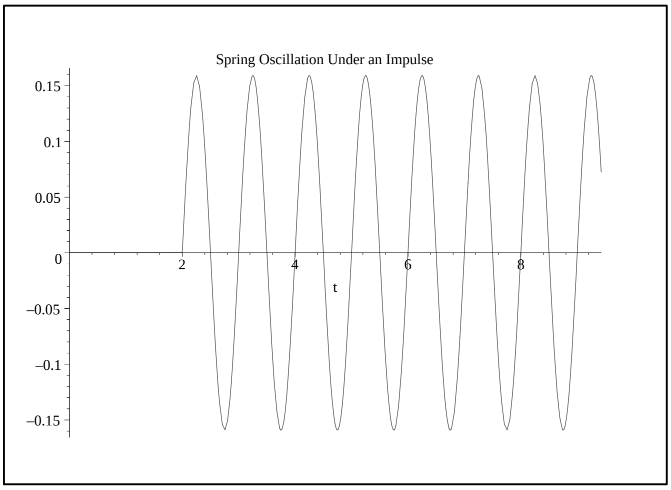

Fig. 7.24. A plot of the solution to Exercise in which a spring at rest experiences an impulse force at $t = 2$.

7.8.4 The Convolution Theorem

Finally, we consider the convolution of two functions. Often we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say you end up with $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve a differential equation. We know how to do this if we only have one of the denominators present. Of course, we could do a partial fraction decomposition. But, there is another way to find the inverse transform, especially if we cannot perform a partial fraction decomposition.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The *convolution* $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du.$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

1. The convolution is commutative: $f * g = g * f$
   Proof: The key is to make a substitution $y = t - u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned}(g * f)(t) &= \int_0^t g(u)f(t-u)\,du \\ &= -\int_t^0 g(t-y)f(y)\,dy\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t f(y)g(t-y)\,dy \\
&= (f*g)(t). \qquad (7.92)
\end{aligned}
$$

2. **The Convolution Theorem:** The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f*g] = F(s)G(s)$$

Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that our functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Secondly, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

$$
\begin{aligned}
\mathcal{L}[f*g] &= \int_0^\infty \left(\int_0^t f(u)g(t-u)\,du\right) e^{-st}\,dt \\
&= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)\,du\right) e^{-st}\,dt \\
&= \int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt\right) du \\
&= \int_0^\infty f(u) \left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\,d\tau\right) du \\
&= \int_0^\infty f(u)e^{-su} \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right) du \\
&= \left(\int_0^\infty f(u)e^{-su}\,du\right) \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right) \\
&= F(s)G(s). \qquad (7.93)
\end{aligned}
$$

We make use of the Convolution Theorem to do the following example.

Example 7.27. $y(t) = \mathcal{L}^{-1}[\frac{1}{(s-1)(s-2)}]$.

We note that this is a product of two functions

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors: $f(t) = e^t$ and $g(t) = e^{2t}$.

Using the Convolution Theorem, we find $y(t) = (f*g)(t)$. We compute the convolution:

$$\begin{aligned} y(t) &= \int_0^t f(u)g(t-u)\,du \\ &= \int_0^t e^u e^{2(t-u)}\,du \\ &= e^{2t}\int_0^t e^{-u}\,du \\ &= e^{2t}[-e^t+1] = e^{2t}-e^t. \end{aligned} \tag{7.94}$$

One can also confirm this by carrying out a partial fraction decomposition.

7.8.5 The Inverse Laplace Transform

Up until this point we have seen that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used. One can do the same for Fourier transforms. However, in that case we introduced an inverse transform in the form of an integral. Does such an inverse exist for the Laplace transform? Yes, it does! In this section we will introduce the inverse Laplace transform integral and show how it is used.

We begin by considering a function $f(t)$ which vanishes for $t < 0$ and define the function $g(t) = f(t)e^{-ct}$. For $g(t)$ absolutely integrable,

$$\int_{-\infty}^{\infty} |g(t)|\,dt = \int_0^{\infty} |f(t)|e^{-ct}\,dt < \infty,$$

we can write the Fourier transform,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \int_0^{\infty} f(t)e^{i\omega t-ct}dt$$

and the inverse Fourier transform,

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\tau)e^{(i\omega-c)\tau}d\tau e^{-(i\omega-c)t}\,d\omega.$$

Letting $s = c - i\omega$ (so $d\omega = ids$), we have

$$f(t) = \frac{i}{2\pi}\int_{c+i\infty}^{c-i\infty}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau e^{st}\,ds.$$

Note that the inside integral is simply $F(s)$. So, we have

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st}\, ds.$$

This is the inverse Laplace transform, called the *Bromwich integral.* This integral is evaluated along a path in the complex plane. The typical way to compute this integral is to chose c so that all poles are to the left of the contour and to close the contour with a semicircle enclosing the poles. One then relies on Jordan's lemma extended into the second and third quadrants.

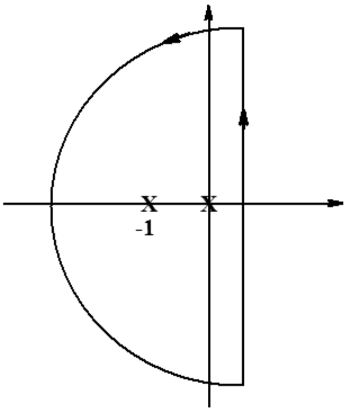

Fig. 7.25. The contour used for applying the Bromwich integral to $F(s) = \frac{1}{s(s+1)}$.

Example 7.28. Find the inverse Laplace transform of $F(s) = \frac{1}{s(s+1)}$.

The integral we have to compute is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+1)}\, ds.$$

This integral has poles at $s = 0$ and $s = -1$. The contour we will use is shown in Figure 7.25. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as $2\pi i$ times the sum of the residues. The residues in this case are:

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = 0\right] = \lim_{z\to 0} \frac{e^{zt}}{(z+1)} = 1$$

and

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = -1\right] = \lim_{z\to -1} \frac{e^{zt}}{z} = -e^{-t}.$$

Therefore, we have

$$f(t) = 2\pi i \left[\frac{1}{2\pi i}(1) + \frac{1}{2\pi i}(-e^{-t}) \right] = 1 - e^{-t}.$$

We can verify this result using the Convolution Theorem or using a partial fraction decomposition. The decomposition is simplest:

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives an e^{-t}. Thus, we have verified the result from doing contour integration.

Problems

7.1. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi} \left(\frac{1}{1+n^2x^2} \right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.
b. Show that the area under each function is one.

7.2. Evaluate the following integrals:

a. $\int_0^\pi \sin x \delta\left(x - \frac{\pi}{2}\right)\, dx$.
b. $\int_{-\infty}^\infty \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\, dx$.
c. $\int_0^\pi x^2 \delta\left(x + \frac{\pi}{2}\right)\, dx$.
d. $\int_0^\infty e^{-2x}\delta(x^2 - 5x + 6)\, dx$. [See Problem 7.3.]
e. $\int_{-\infty}^\infty (x^2 - 2x + 3)\delta(x^2 - 9)\, dx$. [See Problem 7.3.]

7.3. For the case that a function has multiple roots, $f(x_i) = 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^\infty \delta(x^2 - 5x + 6)(3x^2 - 7x + 2)\, dx$.

7.4. For $a > 0$, find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-a|x|}$.

7.5. If $f(x) = g(x + a)$, show that $\hat{f}(k) = e^{-iak}\hat{g}(k)$.

7.6. A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t}e^{i\omega_0 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

.

a. Find $\hat{f}(\omega)$ and
b. the frequency distribution $|\hat{f}(\omega)|^2$.
c. Sketch the frequency distribution.

7.7. Show that the convolution operation is associative: $(f * (g * h))(t) = ((f * g) * h)(t)$.

7.8. You will compute the convolution of two box functions of the same width. Recall the box function is given by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

Consider $(f_a * f_a)(x)$ for different intervals of x. A few preliminary sketches would help. In Figure 7.26 the factors in the convolution integrand are show for one value of x. The integrand is the product of the first two functions. The convolution at x is the area of the overlap in the third figure. Think about how these pictures change as you vary x. Plot the resulting areas as a function of x. This is the graph of the desired convolution.

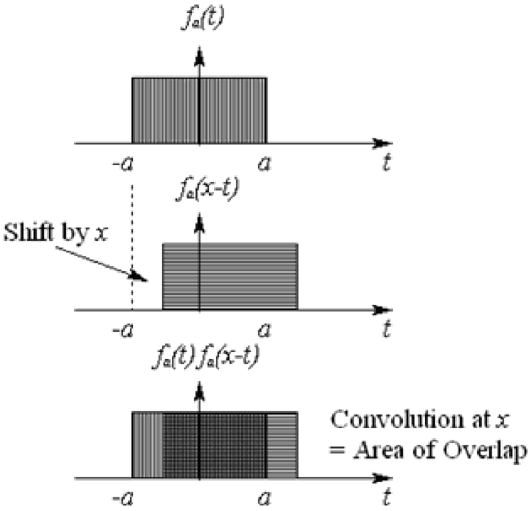

Fig. 7.26. Sketch used to compute the convolution of the box function with itself. In the top figure is the box function. The second figure shows the box shifted by x. The last figure indicates the overlap of the functions.

7.9. Define the integrals $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$. Noting that $I_0 = \sqrt{\pi}$,

a. Find a recursive relation between I_n and I_{n-1}.
b. Use this relation to determine I_1, I_2 and I_3.

c. Find an expression in terms of n for I_n.

7.10. Find the Laplace transform of the following functions.

a. $f(t) = 9t^2 - 7$.
b. $f(t) = e^{5t-3}$.
c. $f(t) = \cos 7t$.
d. $f(t) = e^{4t} \sin 2t$.
e. $f(t) = e^{2t}(t + \cosh t)$.
f. $f(t) = t^2 H(t-1)$.
g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi \end{cases}$.
h. $f(t) = \int_0^t (t-u)^2 \sin u \, du$.

7.11. Find the inverse Laplace transform of the following functions using table of Laplace transform pairs and the properties of Laplace transforms.

a. $F(s) = \frac{18}{s^3} + \frac{7}{s}$.
b. $F(s) = \frac{1}{s-5} - \frac{2}{s^2+4}$.
c. $F(s) = \frac{s+1}{s^2+1}$.
d. $F(s) = \frac{3}{s^2+2s+2}$.
e. $F(s) = \frac{1}{(s-1)^2}$.
f. $F(s) = \frac{e^{-3s}}{s^2-1}$.

7.12. Use the convolution theorem to compute the inverse transform of the following:

a. $F(s) = \frac{2}{s^2(s^2+1)}$.
b. $F(s) = \frac{e^{-3s}}{s^2}$.

7.13. Find the inverse Laplace transform two different ways: i) Use Tables. ii) Use the Bromwich Integral.

a. $F(s) = \frac{1}{s^2-4s-5}$.
b. $F(s) = \frac{s+3}{s^2+8s+17}$.

7.14. Use Laplace transforms to solve the following initial value problems.

a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.
b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.
c. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.

7.15. Use Laplace transforms sum the following series.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2n}$.
b. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.
c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+3)}$.

d. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2-a^2}$.
e. $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2-a^2}$.
f. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-an}$.

7.16. Do the following.

a. Find the first four nonvanishing terms of the Maclaurin series expansion of $f(x) = \frac{x}{e^x-1}$.
b. Use the result in part a. to determine the first four nonvanishing Bernoulli numbers, B_n.
c. Use these results to compute $\zeta(2n)$ for $n = 1, 2, 3, 4$.

7.17. The Gamma function was defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt.$$

a. Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$.
b. Show that $\Gamma(1) = 1$.
c. Use parts a and b to show that $\Gamma(n+1) = n!$ for n a positive integer.
d. Derive that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ by making the change of variables $t = y^2$ in the integral for $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
e. Find $\Gamma\left(\frac{5}{2}\right)$ without integration.