4

Fourier Series - Function Spaces and Convergence

In this chapter we provide a glimpse into more general notions for generalized Fourier series and the convergence of Fourier series. While much of this is marked as optional, it is useful to think about the general context in which one finds oneself. We can view the sine and cosine functions in the Fourier trigonometric series representations as basis vectors in an infinite dimensional function space. A given function in that space may then be represented as a linear combination over this infinite basis. With this in mind, we can wonder if we have enough basis vectors for the function space, if the infinite series expansions are convergent, if there are other bases, and what functions can be represented by such expansions. In this chapter we touch a little on these ideas, leaving some of the deeper results for more advanced texts and courses.

4.1 Vector Spaces and Inner Product Spaces

Much of the discussion and terminology that we will use comes from the theory of vector spaces. Up until now you may only have dealt with finite dimensional vector spaces. Even then, you might only be comfortable with two and three dimensions. We will review a little of what we know about finite dimensional spaces so that we can introduce more general function spaces.

The notion of a vector space is a generalization of three dimensional vectors and operations on them. In three dimensions, we have things called vectors, which are arrows of a specific length and pointing in a given direction. To each vector, we can associate a point in a three dimensional Cartesian system. We just attach the tail of the vector **v** to the origin and the head lands at (x, y, z). We then use unit vectors **i**, **j** and **k** along the coordinate axes to write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Having defined vectors, we then learned how to add vectors and multiply vectors by numbers, or scalars. Under these operations, we expected to get back new vectors. Then we learned that there were two types of multiplication of vectors. We could multiply them to get a scalar or a vector. This lead to the dot and cross products, respectively. The dot product was useful for determining the length of a vector, the angle between two vectors, or if the vectors were orthogonal.

These notions are then generalized to spaces of more than three dimensions in linear algebra courses. The properties outlined roughly above need to be preserved. So, we have to start with a space of vectors and the operations between them. We also need a set of scalars, which generally come from some *field*. However, in our applications the field will either be the set of real numbers or the set of complex numbers.

Definition 4.1. *A vector space V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions: For any $u, v, w \in V$ and $a, b \in F$*

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. *There exists a* 0 *such that* $0 + v = v$.
4. *There exists a* $-v$ *such that* $v + (-v) = 0$.
5. $a(bv) = (ab)v$.
6. $(a + b)v = av + bv$.
7. $a(u + v) = au + bv$.
8. $1(v) = v$.

Now, for an n-dimensional vector space, we have the idea that any vector in the space can be represented as the sum over n *linearly independent* vectors. Recall that a linearly independent set of vectors $\{\mathbf{v}_j\}_{j=1}^n$ satisfies

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = \mathbf{0} \quad \Leftrightarrow \quad c_j = 0.$$

This leads to the idea of a basis set. The standard basis in an n-dimensional vector space is a generalization of the standard basis in three dimensions ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$). We define

$$\mathbf{e}_k = (0, \ldots, 0, \underbrace{1}_{k\text{th space}}, 0, \ldots, 0), \quad k = 1, \ldots, n. \tag{4.1}$$

Then, we can expand any $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k, \tag{4.2}$$

where the v_k's are called the components of the vector in this basis and one can write $\mathbf{v}$ as an n-tuple $(v_1, v_2, \ldots, v_n)$.

The only other thing we will need at this point is to generalize the dot product, or scalar product. Recall that there are two forms for the dot product in three dimensions. First, one has that

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta, \tag{4.3}$$

where u and v denote the length of the vectors. The other form, is the component form:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{k=1}^{3} u_kv_k. \tag{4.4}$$

Of course, this form is easier to generalize. So, we define the *scalar product* between to n-dimensional vectors as

$$< \mathbf{u}, \mathbf{v} >= \sum_{k=1}^{n} u_kv_k. \tag{4.5}$$

Actually, there are a number of notations that are used in other texts. One can write the scalar product as $(\mathbf{u}, \mathbf{v})$ or even in the Dirac bra-ket notation $< \mathbf{u}|\mathbf{v} >$.

While it does not always make sense to talk about angles between general vectors in higher dimensional vector spaces, there is one concept that is useful. It is that of orthogonality, which in three dimensions another way of say vectors are perpendicular to each other. So, we also say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if and only if $< \mathbf{u}, \mathbf{v} >= 0$. If $\{\mathbf{a}_k\}_{k=1}^n$, is a set of basis vectors such that

$$< \mathbf{a}_j, \mathbf{a}_k >= 0, \quad k \neq j,$$

then it is called an *orthogonal basis.* If in addition each basis vector is a unit vector, then one has an *orthonormal basis*

Let $\{\mathbf{a}_k\}_{k=1}^n$, be a set of basis vectors for vector space V. We know that any vector $\mathbf{v}$ can be represented in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k\mathbf{a}_k$. If we know the basis and vector, can we find the components? The answer is, yes. We can use the scalar product of $\mathbf{v}$ with each basis element $\mathbf{a}_j$. So, we have for $j = 1, \ldots, n$

$$\begin{aligned} < \mathbf{a}_j, \mathbf{v} > &= < \mathbf{a}_j, \sum_{k=1}^{n} v_k\mathbf{a}_k > \\ &= \sum_{k=1}^{n} v_k < \mathbf{a}_j, \mathbf{a}_k > . \end{aligned} \tag{4.6}$$

Since we know the basis elements, we can easily compute the numbers

$$A_{jk} \equiv< \mathbf{a}_j, \mathbf{a}_k >$$

and

$$b_j \equiv < \mathbf{a}_j, \mathbf{v} > .$$

Therefore, the system (4.6) for the v_k's is a linear algebraic system, which takes the form $A\mathbf{v} = \mathbf{b}$. However, if the basis is orthogonal, then the matrix A is diagonal and the system is easily solvable. We have that

$$< \mathbf{a}_j, \mathbf{v} >= v_j < \mathbf{a}_j, \mathbf{a}_j >, \tag{4.7}$$

or

$$v_j = \frac{< \mathbf{a}_j, \mathbf{v} >}{< \mathbf{a}_j, \mathbf{a}_j >}. \tag{4.8}$$

In fact, if the basis is orthonormal, A is the identity matrix and the solution is simpler:

$$v_j =< \mathbf{a}_j, \mathbf{v} > . \tag{4.9}$$

We spent some time looking at this simple case of extracting the components of a vector in a finite dimensional space. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are the key ingredients that we will need in the infinite dimensional case. Recall when we found Fourier trigonometric series representations of functions, we stated with a function (vector?) that we wanted to expand in a set of trigonoetric functions (basis?) and we need to find the Fourier coefficients (components?). So, we need to extend our notions from finite dimensional spaces to infinite dimensional spaces and we will have the needed background linear algebra in which to think about more general Fourier series expansions. Also, this conceptual framework is very important in other areas in mathematics (such as ordinary and partial differential equations) and physics, such as quantum mechanics.

We will consider the space of functions of a certain type. They could be the space of continuous functions on [0,1], or the space of differentiably continuous functions, or the set of functions integrable from a to b. Later, we will specify the types of functions. However, you can see that there are many types of function spaces. We will further need to be able to add functions and multiply them by scalars. So, we can easily obtain a vector space of functions.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. For a real vector space, we define

Definition 4.2. *An inner product* $<,>$ *on a real vector space* V *is a mapping from* $V \times V$ *into* R *such that for* $u, v, w \in V$ *and* $\alpha \in R$ *one has*

1. $< u + v, w >=< u, w > + < v, w >$.
2. $< \alpha v, w >= \alpha < v, w >$.
3. $< v, w >=< w, v >$.
4. $< v, v >\geq 0$ *and* $< v, v >= 0$ *iff* $v = 0$.

A real vector space equipped with the above inner product leads to a real inner product space. A more general definition with the third item replaced with $< v, w >= \overline{< w, v >}$ is needed for complex inner product spaces.

For the time being, we are dealing just with real valued functions. We need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$ and introduce the *weight function* $\sigma(x) > 0$. Then, we define the *inner product*, if the integral exists, as

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\,dx. \tag{4.10}$$

In what follows, we will assume for simplicity that $\sigma(x) = 1$. This is possible to do by using a change of variables.

So, we have functions spaces equipped with an inner product. Can we find a basis for the space? For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will think about those things later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^{\infty}$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

How do we find the c_n's? Does this remind you of the problem we had earlier?

Formally, we take the inner product of f with each ϕ_j, to find

$$\begin{aligned} < \phi_j, f > &= < \phi_j, \sum_{n=1}^{\infty} c_n \phi_n > \\ &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > . \end{aligned} \tag{4.11}$$

If our basis is an *orthogonal basis*, then we have

$$< \phi_j, \phi_n >= N_j \delta_{ij}, \tag{4.12}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{4.13}$$

Thus, we have

$$\begin{aligned} < \phi_j, f > &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{ij} \\ &= c_j N_j. \end{aligned} \tag{4.14}$$

So, the expansion coefficient is

$$c_j = \frac{< \phi_j, f >}{N_j} = \frac{< \phi_j, f >}{< \phi_j, \phi_j >}.$$

This is just the generalization of Fourier series expansions. Let's assume we have a Fourier sine series expansion of $f(x)$, given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi].$$

In the last chapter we already established that the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. Recall that using trigonometric identities, we have for $n \neq m$

$$\begin{aligned} < \phi_n, \phi_m > &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} = 0. \end{aligned} \tag{4.15}$$

So, we have determined that the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[-\pi, \pi]$. Just as with vectors in three dimensions, we can normalize our basis functions to arrive at an *orthonormal basis*. This is simply done by dividing by the *length* of the vector. Recall that the length of a vector was obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ In the same way, we define the *norm* of our functions by

$$\|f\| = \sqrt{< f, f >}.$$

Note, there are many types of norms, but this will be sufficient for us.

For the above basis of sine functions, we want to first compute the norm of each function. Then we would like to find a new basis from this one such that each basis eigenfunction has unit length and is therefore an orthonormal basis. We first compute

$$\begin{aligned} \|\phi_n\|^2 &= \int_{-\pi}^{\pi} \sin^2 nx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [1 - \cos 2nx] \, dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \pi. \end{aligned} \tag{4.16}$$

We have found for our example that

$$< \phi_j, \phi_n >= \pi\delta_{ij} \tag{4.17}$$

and that $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}}\phi_n(x)$, we have *normalized* the ϕ_n's and have obtained an orthonomal basis of functions on $[-\pi, \pi]$.

Now, we can determine the expansion coefficients using

$$b_n = \frac{< \phi_n, f >}{N_j} = \frac{< \phi_n, f >}{< \phi_n, \phi_n >} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Does this result look familiar?

4.2 General Fourier Series Expansions - *optional*

For completeness, we discuss series representations of functions using different bases. Much of the rest of this chapter is optional on first reading. However, the reader should look through these pages to see how the ideas in the last section are generalized.

4.2.1 Classical Orthogonal Polynomials

We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstrass Approximation Theorem this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. We are familiar with being able to expand functions over this basis, since the expansions are just power series representation of the functions,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and $(a, b)=(-1, 1)$. For example,

$$\int_{-1}^{1} x^0 x^2\,dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask if it is possible to obtain an orthogonal basis involving these powers of x. Of course, finite combinations of these basis element are just polynomials!

OK, we will ask. “Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?” The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional

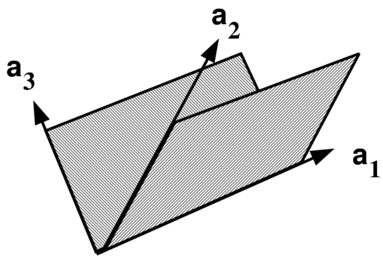

Fig. 4.1. The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$ considered in the text.

vector spaces, that there is a method for carrying this out called the **Gram-Schmidt Orthogonalization Process.** We will review this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 4.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a 'hat':

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 4.2 we see the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. Note that $\mathbf{a}_2$ can be written as a sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{4.18}$$

We recall the projection of one vector onto another from our vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{4.19}$$

Note that this is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (4.18)-(4.19), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{4.20}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

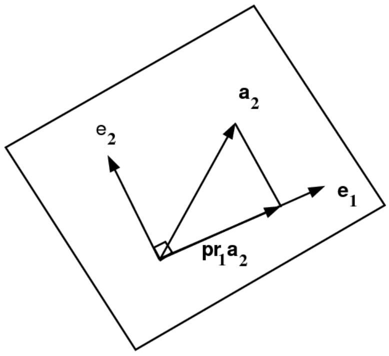

Fig. 4.2. A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2} \mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{4.21}$$

Now, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the new vector. This is shown in Figure 4.3. Then we have,

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2} \mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2} \mathbf{e}_2. \tag{4.22}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

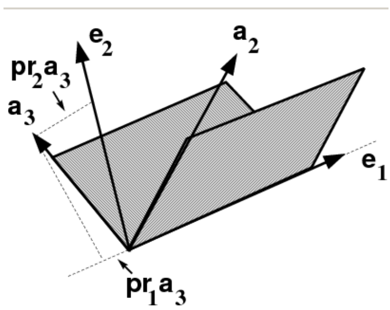

Fig. 4.3. A plot of the vectors and their projections for determining $\mathbf{e}_3$.

We can easily generalize the procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and for $n > 1$,

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2} \mathbf{e}_j. \tag{4.23}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{< f_n, \phi_j >}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \tag{4.24}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$< f, g >= \int_a^b f(x) g(x) \sigma(x)\, dx. \tag{4.25}$$

Note the similarity between the orthogonal basis in (4.24) and the expression for the finite dimensional case in Equation (4.23).

Example 4.3. Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = \frac{1}{2}.$$

We could use this result to fix the normalization of our new basis, but we will hold off on doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{< f_1, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) \\ &= x - \frac{< x, 1 >}{\|1\|^2} 1 = x, \end{aligned} \tag{4.26}$$

since $< x, 1 >$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned} \phi_2(x) &= f_2(x) - \frac{< f_2, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) - \frac{< f_2, \phi_1 >}{\|\phi_1\|^2} \phi_1(x) \\ &= x^2 - \frac{< x^2, 1 >}{\|1\|^2} 1 - \frac{< x^2, x >}{\|x\|^2} x \\ &= x^2 - \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} dx} \\ &= x^2 - \frac{1}{3}. \end{aligned} \tag{4.27}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x) = \phi_1(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\, dx = \frac{2}{5}.$$

The set of Legendre polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 4.2.1.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

Table 4.1. Common classical orthogonal polynomials with the interval and weight function used to define them.

4.2.2 Fourier-Legendre Series

We can see now how a Fourier series expansion is just an expansion over a basis in an infinite dimensional function space. In this subsection and the next one we show examples of non-trigonometric Fourier series.

We first consider series representations in terms of a basis of Legendre polynomials. In this case we have the Fourier-Legendre series expansion for functions $f(x)$ defined on $(-1, 1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{4.28}$$

As with Fourier trigonometric series, we can determine the coefficients by multiplying both sides by $P_m(x)$ and integrating. Orthogonality gives the usual form for the generalized Fourier coefficients. In this case, we have

$$c_n = \frac{< f, P_n >}{\|P_n\|^2}.$$

It can be shown that $\|P_n\|^2 = \frac{2}{2n+1}$. Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{4.29}$$

We can do examples given just a few facts about Legendre polynomials. The first several Legendre polynomials are given in Table 4.2.2. In Figure 4.4 we show plots of these Legendre polynomials.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

Table 4.2. Tabular computation of the Legendre polynomials using the Rodrigues formula.

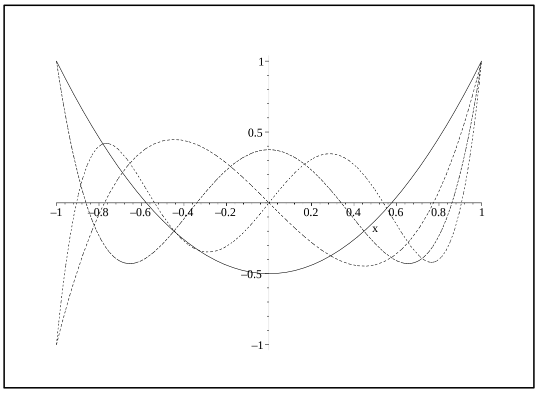

Fig. 4.4. Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Example 4.4. Expand $f(x) = x^3$ in a Fourier-Legendre series.

We simply need to compute

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} x^3 P_n(x)\, dx. \tag{4.30}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m < n.$$

As a result, we will have for this example that $c_n = 0$ for $n > 3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)\, dx$ for $m = 0, 1, 2, \ldots$ outright by looking up Legendre polynomials. But, note that x^3 is an odd function, $c_0 = 0$ and $c_2 = 0$. This leaves us with only two coefficients to compute. We refer to Table 4.2.2 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\, dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3 \left[\frac{1}{2}(5x^3 - 3x)\right] dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 4.2.2:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] = x^3.$$

Well, maybe we could have guessed this without doing any integration. Let's see,

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2(5x^3 - 3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{4.31}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Example 4.5. Expand the Heaviside function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, \ x > 0, \\ 0, \ x < 0. \end{cases} \tag{4.32}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\, dx = \frac{2n+1}{2}\int_{0}^{1} P_n(x)\, dx. \tag{4.33}$$

We can make use of the identity

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 1. \tag{4.34}$$

We have

$$c_n = \frac{1}{2}\int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\, dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

For $n > 0$ we have the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. So, we have

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{2n-2}(0) - P_{2n}(0)]P_{2n-1}(x).$$

We can compute the coefficients using the result

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}. \tag{4.35}$$

$$\begin{aligned}
f(x) &\sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{2n-2}(0) - P_{2n}(0)]P_{2n-1}(x)\\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left[(-1)^{n-1}\frac{(2n-3)!!}{(2n-2)!!} - (-1)^n\frac{(2n-1)!!}{(2n)!!}\right]P_{2n-1}(x)\\
&= \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\left[1 + \frac{2n-1}{2n}\right]P_{2n-1}(x)\\
&= \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x).
\end{aligned} \tag{4.36}$$

The sum of the first 21 terms are shown in Figure 4.5. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$.

4.2.3 Fourier-Bessel Series

Bessel functions are another orthogonal set of eigenfunctions and we can expand square integrable functions in this basis. You might have seen that Bessel functions are solutions of the differential equation

$$x^2y'' + xy' + (\lambda x^2 - p^2)y = 0. \tag{4.37}$$

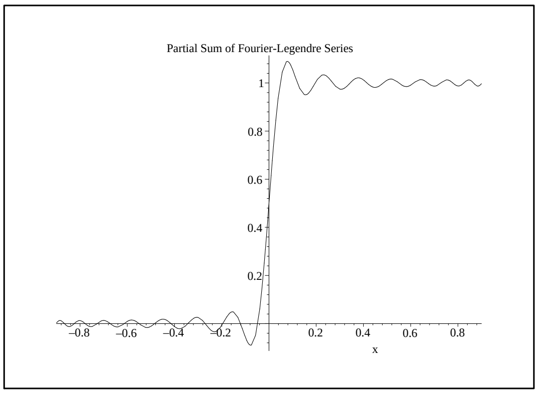

Fig. 4.5. Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

Bessel functions arise in many problems in physics, such as the vibrations of circular drumheads and the radial modes in optical fibers.

Solutions of this differential equation include functions of the form $J_p(\sqrt{\lambda}x)$. One can solve this differential equation with the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. It is found that the solutions $J_p(j_{pn}\frac{x}{a})$ forms a basis and the Fourier-Bessel series expansion of $f(x)$ defined on $0 < x < a$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}) \tag{4.38}$$

where the Fourier-Bessel coefficients are found using an orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a xf(x)J_p(j_{pn}\frac{x}{a})\,dx. \tag{4.39}$$

Example 4.6. Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (4.39):

$$c_n = \frac{2}{\left[J_1(j_{0n})\right]^2}\int_0^1 xJ_0(j_{0n}x)\,dx. \tag{4.40}$$

From the identity

$$\frac{d}{dx}\left[x^pJ_p(x)\right] = x^pJ_{p-1}(x). \tag{4.41}$$

we have

$$\begin{aligned}\int_0^1 xJ_0(j_{0n}x)\,dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} yJ_0(y)\,dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[yJ_1(y)\right]\,dy \\ &= \frac{1}{j_{0n}^2}\left[yJ_1(y)\right]_0^{j_{0n}} \\ &= \frac{1}{j_{0n}}J_1(j_{0n}). \end{aligned} \tag{4.42}$$

As a result, the desired Fourier-Bessel expansion is given as

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \tag{4.43}$$

In Figure 4.6 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

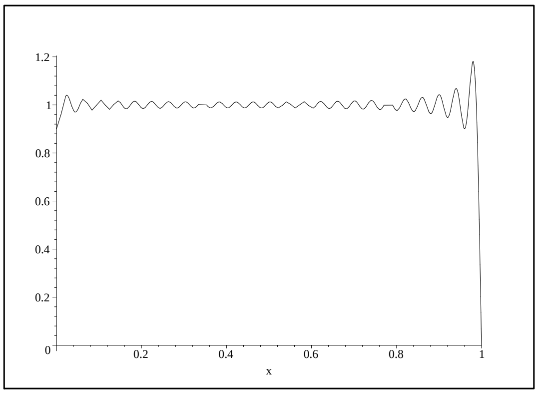

Fig. 4.6. Plot of the first 50 terms of the Fourier-Bessel series in Equation (4.43) for $f(x) = 1$ on $0 < x < 1$.

4.3 The Least Squares Approximation - *optional*

In the last section we found that we can expand functions in a function space over a basis set as

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

and that the generalized Fourier coefficients are given by

$$c_n = \frac{< \phi_n, f >}{< \phi_n, \phi_n >}.$$

In this section we turn to a discussion of approximating $f(x)$ by the partial sums $\sum_{n=1}^{N} c_n \phi_n(x)$ and showing that the Fourier coefficients are the best coefficients minimizing the deviation of the partial sum from $f(x)$. This will lead us to a discussion of the convergence of Fourier series.

More specifically, we set the following goal:

Goal

To find the best approximation of $f(x)$ on $[a, b]$ by $S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$ for a set of fixed functions $\phi_n(x)$; i.e., to find the c_n's such that $S_N(x)$ approximates $f(x)$ in the *least squares sense.*

We want to measure the deviation of the sum from our function, or the error made in the approximation. This is done by introducing the *mean square deviation:*

$$E_N = \int_a^b [f(x) - S_N(x)]^2 \rho(x)\, dx,$$

where we have introduced the weight function $\rho(x) > 0$. It gives us a sense as to how close the Nth partial sum is to $f(x)$. We want to minimize this deviation by choosing the right c_n's.

First, we turn to some definitions. We first define the more general *inner product* of two real functions

$$< \phi, \psi >= \int_a^b \phi(x)\psi(x)\rho(x)\, dx.$$

This then naturally leads to the definition of *orthogonal functions*

$$< \phi, \psi >= \int_a^b \phi(x)\psi(x)\rho(x)\, dx = 0,$$

and the notion of a *mutually orthogonal set of functions*, $\{\phi_n(x)\}_{n=1}^{\infty}$, satisfying $< \phi_n, \phi_m >= 0,\ m \neq n$.

We now turn to the minimization of the mean square deviation, E_N. We insert the partial sums and expand the square in the integrand:

$$\begin{aligned} E_N &= \int_a^b [f(x) - S_N(x)]^2 \rho(x)\, dx \\ &= \int_a^b [f(x) - \sum_{n=1}^{N} c_n \phi_n(x)]^2 \rho(x)\, dx \end{aligned}$$

$$= \int_a^b f^2(x)\rho(x)\,dx - 2\int_a^b f(x)\sum_{n=1}^N c_n\phi_n(x)\rho(x)\,dx$$
$$+ \int_a^b \sum_{n=1}^N c_n\phi_n(x)\sum_{m=1}^N c_m\phi_m(x)\rho(x)\,dx \qquad (4.44)$$

Looking at the three integrals, we see that the first term is just the inner product of f with itself. The other integrations can be rewritten after interchanging the order of integration and summation. The double sum can be reduced to a single sum using the orthogonality of the ϕ's. Thus, we have

$$E_N = < f,f > -2\sum_{n=1}^N c_n < f,\phi_n > + \sum_{n=1}^N\sum_{m=1}^N c_n c_m < \phi_n,\phi_m >$$
$$= < f,f > -2\sum_{n=1}^N c_n < f,\phi_n > + \sum_{n=1}^N c_n^2 < \phi_n,\phi_n > . \qquad (4.45)$$

We are interested in finding the coefficients, so we will complete the square in c_n. Focusing on the last two terms, we have

$$E_N - < f,f > = -2\sum_{n=1}^N c_n < f,\phi_n > + \sum_{n=1}^N c_n^2 < \phi_n,\phi_n >$$
$$= \sum_{n=1}^N < \phi_n,\phi_n > c_n^2 - 2 < f,\phi_n > c_n$$
$$= \sum_{n=1}^N < \phi_n,\phi_n > \left[c_n^2 - \frac{2 < f,\phi_n >}{< \phi_n,\phi_n >}c_n\right]$$
$$= \sum_{n=1}^N < \phi_n,\phi_n > \left[\left(c_n - \frac{< f,\phi_n >}{< \phi_n,\phi_n >}\right)^2 - \left(\frac{< f,\phi_n >}{< \phi_n,\phi_n >}\right)^2\right] \qquad (4.46)$$
$$(4.47)$$

To this point we have shown that the mean square deviation is given as

$$E_N = < f,f > + \sum_{n=1}^N < \phi_n,\phi_n > \left[\left(c_n - \frac{< f,\phi_n >}{< \phi_n,\phi_n >}\right)^2 - \left(\frac{< f,\phi_n >}{< \phi_n,\phi_n >}\right)^2\right].$$

So, E_N is minimized by choosing $c_n = \frac{< f,\phi_n >}{< \phi_n,\phi_n >}$. However, these are the Fourier Coefficients. This minimization is often referred to as **Minimization in Least Squares Sense**.

Inserting the Fourier coefficients into the mean square deviation yields

$$0 \leq E_N =< f, f > - \sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n > .$$

Thus, we obtain *Bessel's Inequality*:

$$< f, f > \geq \sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n > .$$

For convergence, we next let N get large and see if the partial sums converge to the function. In particular, we say that the infinite series *converges in the mean* if

$$\int_a^b [f(x) - S_N(x)]^2 \rho(x)\, dx \ \to 0 \text{ as } N \to \infty.$$

Letting N get large in Bessel's inequality shows that $\sum_{n=1}^{N} c_n^2 < \phi_n, \phi_n >$ converges if

$$(< f, f >= \int_a^b f^2(x)\rho(x)\, dx < \infty.$$

The space of all such f is denoted $L^2_\rho(a, b)$, the space of square integrable functions on (a, b) with weight $\rho(x)$.

From the nth term divergence theorem we know that $\sum a_n$ converges implies that $a_n \to 0$ as $n \to \infty$. Therefore, in this problem the terms $c_n^2 < \phi_n, \phi_n >$ approach zero as n gets large. This is only possible if the c_n's go to zero as n gets large. Thus, if $\sum_{n=1}^{N} c_n\phi_n$ *converges in the mean* to f, then $\int_a^b [f(x) - \sum_{n=1}^{N} c_n\phi_n]^2 \rho(x)\, dx$ approaches zero as $N \to \infty$. This implies from the above derivation of Bessel's inequality that

$$< f, f > - \sum_{n=1}^{N} c_n^2 (\phi_n, \phi_n) \to 0.$$

This leads to *Parseval's equality*:

$$< f, f >= \sum_{n=1}^{\infty} c_n^2 < \phi_n, \phi_n > .$$

Parseval's equality holds if and only if

$$\lim_{N\to\infty} \int\limits_a^b (f(x) - \sum_{n=1}^{N} c_n\phi_n(x))^2 \rho(x)\ dx = 0.$$

If this is true for every square integrable function in $L^2_\rho(a, b)$, then the set of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ is said to be **complete**. One can view these functions as an infinite dimensional basis for the space of square integrable functions on (a, b) with weight $\rho(x) > 0$.

One can extend the above limit $c_n \to 0$ as $n \to \infty$, by assuming that $\frac{\phi_n(x)}{\|\phi_n\|}$ is uniformly bounded and that $\int_a^b |f(x)|\rho(x)\, dx < \infty$. This is the **Riemann-Lebesque Lemma**, but will not be proven now.

4.4 Convergence of Trigonometric Fourier Series - *optional*

In this section we list definitions, lemmas and theorems needed to provide convergence arguments for trigonometric Fourier series.

1. For any nonnegative integer k, a function u is C^k if every k-th order partial derivative of u exists and is continuous.
2. For two functions $f$and g defined on an interval $[a, b]$, we will define the **inner product** as $< f, g >= \int_a^b f(x)g(x)\, dx$.
3. A function f is **periodic with period *p*** if $f(x + p) = f(x)$ for all x.
4. Let f be a function defined on $[-L, L]$ such that $f(-L) = f(L)$. The **periodic extension** $\tilde{f}$of f is the unique periodic function of period $2L$ such that $\tilde{f}(x) = f(x)$for all $x \in [-L, L]$.
5. The expression
$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos\frac{n\pi x}{L}$$
is called the ***N*-th Dirichlet Kernel.** [This will be summed later and the sequences of kernels converges to what is called the **Dirac Delta function**.]
6. A sequence of functions $\{s_1(x), s_2(x), \ldots\}$ is said to **converge pointwise** to $f(x)$ on the interval $[-L, L]$ if for each fixed x in the interval,
$$\lim_{N\to\infty} |f(x) - s_N(x)| = 0.$$
7. A sequence of functions $\{s_1(x), s_2(x), \ldots\}$ is said to **converge uniformly** to $f(x)$ on the interval $[-L, L]$ if
$$\lim_{N\to\infty} \left(\max_{|x|\le L} |f(x) - s_N(x)| \right) = 0.$$
8. **One-sided limits**: $f(x_0^+) = \lim_{x\downarrow x_0} f(x)$ and $f(x_0^-) = \lim_{x\uparrow x_0} f(x)$.
9. A function f is **piecewise continuous** on $[a, b]$ if the function satisfies
   a. f is defined and continuous at all but a finite number of points of $[a, b]$.

b. For all $x \in (a, b)$, the limits $f(x^+)$ and $f(x^-)$ exist.
c. $f(a^+)$ and $f(b^-)$ exist.

10. A function is **piecewise** C^1 on $[a, b]$ if $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$.

Lemmas

1. **Bessel's Inequality:** Let $f(x)$ be defined on $[-L, L]$ and $\int$ $_-L^L f^2(x)\,dx < \infty$. If the trigonometric Fourier coefficients exist, then $a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \le \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx$. This follows from the earlier section on the Least Squares Approximation.
2. **Riemann-Lebesgue Lemma:** Under the conditions of Bessel's Inequality, the Fourier coefficients approach zero as $n \to 0$. This is based upon some earlier convergence results seen in Calculus in which one learns for a series of nonnegative terms, $\sum c_n$with $c_n \ge 0$, *if* c_n*does not approach 0 as* $n \to \infty$, *then* $\sum c_n$*does not converge.* Therefore, the contrapositive holds, if $\sum c_n$ *converges, then* $c_n \to 0$ *as* $n \to \infty$. From Bessel's Inequality, we see that when f is square integrable, the series formed by the sums of squares of the Fourier coefficients converges. Therefore, the Fourier coefficients must go to zero as n increases. This is also referred to in the earlier section on the Least Squares Approximation. However, an extension to absolutely integrable functions exists, which is called the Riemann-Lebesgue Lemma.
3. **Green's Formula:** Let $f$and $g$be C^2 functions on $[a, b]$. Then $< f'', g > - < f, g'' >= [f'(x)g(x) - f(x)g'(x)]|_a^b$. [Note: This is just an iteration of integration by parts.]
4. **Special Case of Green's Formula:** Let $f$and $g$be C^2 functions on $[-L, L]$ and both functions satisfy the conditions $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then $< f'', g >=< f, g'' >$.
5. **Lemma 1:** If g is a periodic function of period $2L$and c any real number, then $\int_{-L+c}^{L+c} g(x)\,dx = \int_{-L}^{L} g(x)\,dx$.
6. **Lemma 2:** Let f be a C^2 function on $[-L, L]$ such that $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Then for $M = \max_{|x|\le L} |f''(x)|$ and $n \ge 1$,

$$|a_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx \right| \le \frac{2L^2 M}{n^2\pi^2} \tag{4.48}$$

$$|b_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx \right| \le \frac{2L^2 M}{n^2\pi^2}. \tag{4.49}$$

7. **Lemma 3:** For any real θ such that $\sin\frac{\theta}{2} \ne 0$,

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin((n + \frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}$$

8. **Lemma 4:** Let $h(x)$ be C^1 on $[-L, L]$. Then

$$\lim_{n\to\infty} \frac{1}{L} \int_{-L}^{L} D_n(x)h(x)\, dx = h(0).$$

Theorems

1. **Theorem 1.** (Pointwise Convergence) Let f be C^1 on $[-L, L]$ with $f(-L) = f(L), f'(-L) = f'(L)$. Then $FS\, f(x) = f(x)$ for all x in $[-L, L]$.
2. **Theorem 2.** (Uniform Convergence) Let f be C^2 on $[-L, L]$ with $f(-L) = f(L), f'(-L) = f'(L)$. Then $FS\, f(x)$ converges uniformly to $f(x)$. In particular,
$$|f(x) - S_N(x)| \leq \frac{4L^2 M}{\pi^2 N}$$
for all x in $[-L, L]$, where $M = \max_{|x|\leq L} |f''(x)|$.
3. **Theorem 3.** (Piecewise C^1 - Pointwise Convergence) Let f be a piecewise C^1 function on $[-L, L]$. Then $FS\, f(x)$ converges to the periodic extension of
$$f(x) = \begin{cases} \frac{1}{2}[f(x^+) + f(x^-)], \ -L < x < L \\ \frac{1}{2}[f(L^+) + f(L^-)], \ x = \pm L \end{cases}$$
for all x in $[-L, L]$.
4. **Theorem 4.** (Piecewise C^1 - Uniform Convergence) Let f be a piecewise C^1 function on $[-L, L]$ such that $f(-L) = f(L)$. Then $FS\, f(x)$ converges uniformly to $f(x)$.

Proof of Convergence

We are considering the Fourier series of $f(x)$:

$$FS\, f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right],$$

where the Fourier coefficients are given by

$$\begin{aligned} a_0 &= \tfrac{1}{2L} \int_{-L}^{L} f(x)\, dx, \\ a_n &= \tfrac{1}{L} \int_{-L}^{L} f(x) \cos\tfrac{n\pi x}{L}\, dx, \\ b_n &= \tfrac{1}{L} \int_{-L}^{L} f(x) \sin\tfrac{n\pi x}{L}\, dx. \end{aligned}$$

We are first interested in the pointwise convergence of the infinite series. Thus, we need to look at the partial sums for each x. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L}\right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy.\right)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy
\end{aligned} \tag{4.50}$$

Here

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L}$$

is called the ***N*-th Dirichlet Kernel**. What we seek to prove is (**Lemma 4**) that

$$\lim_{N\to\infty}\frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy = f(x).$$

[Technically, we need the periodic extension of f.] So, we need to consider the Dirichlet kernel. Then pointwise convergence follows, as $\lim_{N\to\infty} S_N(x) = f(x)$.

Proposition:

$$D_n(x) = \begin{cases} \frac{\sin((n+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0 \\ n + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0 \end{cases}.$$

Proof: Actually, this follows from **Lemma 3**. Let $\theta = \frac{\pi x}{L}$ and multiply $D_n(x)$ by $2\sin\frac{\theta}{2}$ to obtain:

$$\begin{aligned}
2\sin\frac{\theta}{2}D_n(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2} + \cos\theta + \cdots + \cos n\theta\right] \\
&= \sin\frac{\theta}{2} + 2\cos\theta\sin\frac{\theta}{2} + 2\cos 2\theta\sin\frac{\theta}{2} + \cdots + 2\cos n\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2} + \left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right) + \left(\sin\frac{5\theta}{2} - \sin\frac{3\theta}{2}\right) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\left(\sin\left((n+\frac{1}{2})\theta\right)-\sin\left((n-\frac{1}{2})\theta\right)\right)\\
&=\sin\left((n+\frac{1}{2})\theta\right). \qquad (4.51)
\end{aligned}
$$

Thus,

$$2\sin\frac{\theta}{2}D_n(x)=\sin\left((n+\frac{1}{2})\theta\right),$$

or if $\sin\frac{\theta}{2}\neq 0$,

$$D_n(x)=\frac{\sin\left((n+\frac{1}{2})\theta\right)}{2\sin\frac{\theta}{2}},\ \theta=\frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2}=0$, then one needs to apply L'Hospital's Rule:

$$
\begin{aligned}
\lim_{\theta\to 2m\pi}\frac{\sin\left((n+\frac{1}{2})\theta\right)}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{(n+\frac{1}{2})\cos\left((n+\frac{1}{2})\theta\right)}{\cos\frac{\theta}{2}}\\
&=\frac{(n+\frac{1}{2})\cos\left(2mn\pi+m\pi\right)}{\cos m\pi}\\
&=n+\frac{1}{2}. \qquad (4.52)
\end{aligned}
$$

As $n\to\infty$, $D_n(x)\to\delta(x)$, the **Dirac delta function,** on the interval $[-L,L]$. The following are some plots for $L=\pi$and $n=25,50,100$. Note how a central peak grows and the values tend towards zero for nonzero x.

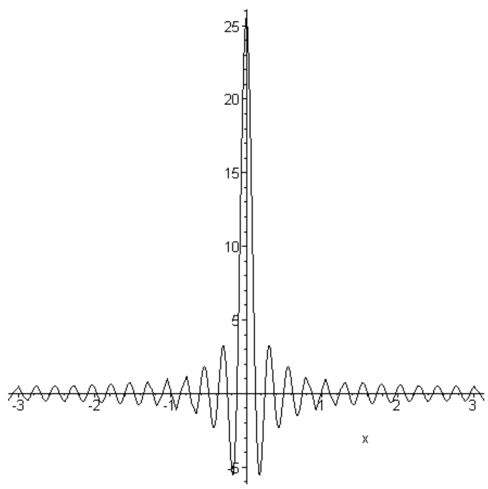

Fig. 4.7. N=25.

The Dirac delta function can be defined as that quantity satisfying

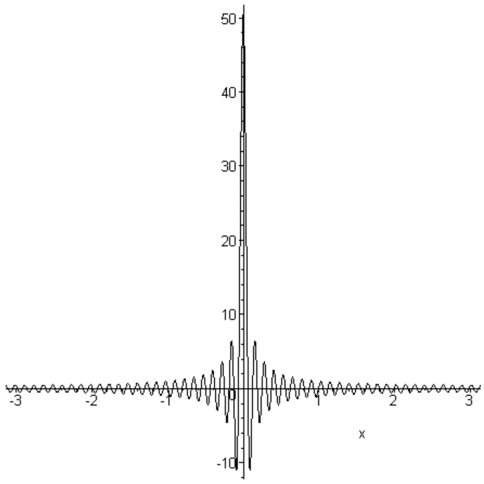

Fig. 4.8. N=50.

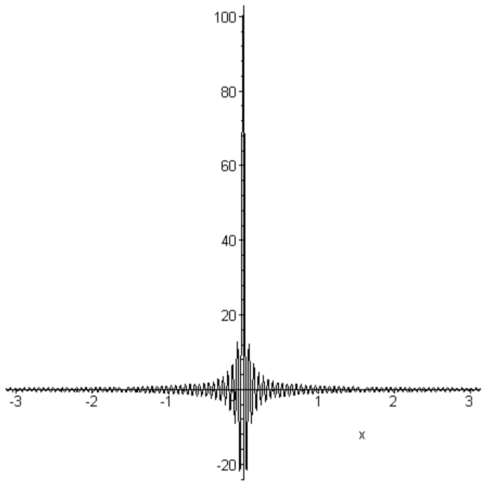

Fig. 4.9. N=100.

a. $\delta(x) = 0,\ x \neq 0$;
b. $\int_{-\infty}^{\infty} \delta(x)\ dx = 1$.

This generalized function, or **distribution**, also has the property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\ dx = f(a).$$

Thus, under the appropriate conditions on f, one can show

$$\lim_{N\to\infty} \frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy = f(x).$$

We need to prove **Lemma 4** first.

Proof: Since $\frac{1}{L}\int_{-L}^{L} D_N(x)\,dx = \frac{1}{2L}\int_{-L}^{L} dx = 1$, we have that

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} D_N(x)h(x)\,dx - h(0) &= \frac{1}{L}\int_{-L}^{L} D_N(x)\left[h(x)-h(0)\right]\,dx \\ &= \frac{1}{2L}\int_{-L}^{L}\left[\cos\frac{n\pi x}{L} + \cot\frac{\pi x}{L}\sin\frac{n\pi x}{L}\right]\left[h(x)-h(0)\right]\,dx.\end{aligned} \tag{4.53}$$

The two terms look like the Fourier coefficients. An application of the Riemann-L:ebesgue Lemma indicates that these coefficients tend to zero as $n\to\infty$, provided the functions being expanded are square integrable and the integrals above exist. The cosine integral follows, but a little work is needed for the sine integral. One can use L'Hospital's Rule with $h \in C^1$.

Now we apply **Lemma 4** to get the convergence from

$$\lim_{N\to\infty} \frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy = f(x).$$

Due to periodicity, we have

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy &= \frac{1}{L}\int_{-L}^{L} D_N(y-x)\tilde{f}(y)\,dy \\ &= \frac{1}{L}\int_{-L+x}^{L+x} D_N(y-x)\tilde{f}(y)\,dy \\ &= \frac{1}{L}\int_{-L}^{L} D_N(z)\tilde{f}(x+z)\,dz.\end{aligned} \tag{4.54}$$

We can apply **Lemma 4** providing $\tilde{f}(z+x)$ is C^1 in z, which is true since f is C^1 and behaves well at $\pm L$.

To prove **Theorem 2** on uniform convergence, we need only combine **Theorem 1** with **Lemma 2**. Then we have,

$$
\begin{aligned}
|f(x) - S_N(x)| &= |f(x) - S_N(x)| \\
&\leq \sum_{n=N+1}^{\infty} \left[\left| a_n \cos \frac{n\pi x}{L} \right| + \left| b_n \sin \frac{n\pi x}{L} \right| \right] \\
&\leq \sum_{n=N+1}^{\infty} [|a_n| + |b_n|] \qquad (4.55) \\
&\leq \frac{4L^2 M}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \\
&\leq \frac{4L^2 M}{\pi^2 N}. \qquad (4.56)
\end{aligned}
$$

This gives the uniform convergence.

These Theorems can be relaxed to include piecewise C^1 functions. **Lemma 4** needs to be changed for such functions to the result that

$$\lim_{n \to \infty} \frac{1}{L} \int_{-L}^{L} D_n(x) h(x)\, dx = \frac{1}{2} [h(0^+) + h(0^-)]$$

by splitting the integral into integrals over $[-L, 0]$, $[0, L]$ and applying a one-sided L'Hospital's Rule. Proving uniform convergence under the conditions in **Theorem 4** takes a little more effort, but it can be done.

Problems