6

Complex Integration

In the last chapter we introduced functions of a complex variable. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring Fourier transforms in the next chapter.

6.1 Complex Path Integrals

In this section we will investigate the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ, we would like to define the integral of $f(z)$ along Γ,

$$\int_{\Gamma} f(z)\, dz.$$

A natural procedure would be to work in real variables, by writing

$$\int_{\Gamma} f(z)\, dz = \int_{\Gamma} [u(x,y) + iv(x,y)]\,(dx + idy).$$

In order to carry out the integration, we then have to find a parametrization of the path and use methods from a multivariate calculus class.

Before carrying this our with some examples, we first provide some definitions.

Definition 6.1. *A set D is connected if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called disconnected. Examples are shown in Figure 6.2*

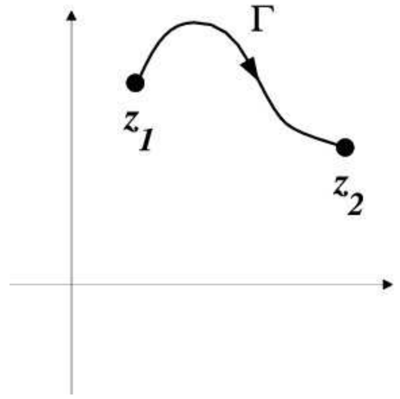

Fig. 6.1. We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

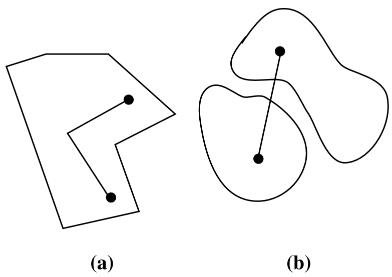

Fig. 6.2. Examples of (a) a connected set and (b) a disconnected set.

Definition 6.2. *A set D is open if and only if for all z_0 in D there exists an open disk $|z - z_0| < \rho$ in D.*

In Figure 6.3 we show a region with two disks. For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

Definition 6.3. *D is called a domain if it is both open and connected.*

Definition 6.4. *Let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \leq t \leq t_1$ and $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$. Then*

$$\int_\Gamma f(z)\, dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))]\,(\frac{dx}{dt} + i\frac{dy}{dt})dt. \tag{6.1}$$

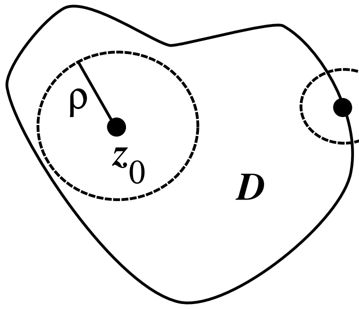

Fig. 6.3. Locations of open disks inside and on the boundary of a region.

Note that we have used

$$dz = dx + idy = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt.$$

This definition gives us a prescription for computing path integrals. Let's see how this works with a couple of examples.

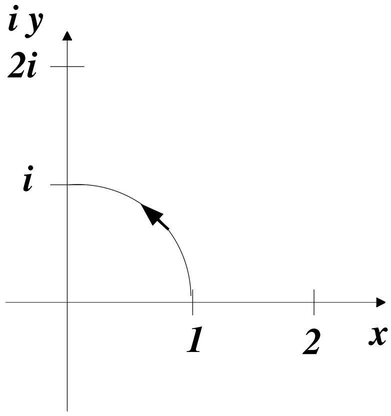

Fig. 6.4. Contour for Example 6.5.

Example 6.5. $\int_C z^2\, dz$, $C =$ the arc of the unit circle in the first quadrant as shown in Figure 6.4.

We first specify the parametrization. There are two ways we could do this. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le 2\pi.$$

For a quarter circle, $0 \le \theta \le \frac{\pi}{2}$, we let $z = \cos\theta + i\sin\theta$. Therefore, $dz = (-\sin\theta + i\cos\theta)\, d\theta$. Then the path integral becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)\,d\theta.$$

We can multiply this out and integrate, having to perform some trigonometric integrations:

$$\int_0^{\frac{\pi}{2}} [\sin^3\theta - 3\cos^2\theta\sin\theta + i(\cos^3\theta - 3\cos\theta\sin^2\theta)]\,d\theta.$$

While this is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ on C. So, $dz = ie^{i\theta}d\theta$. The integration then becomes

$$\begin{aligned}\int_C z^2\,dz &= \int_0^{\frac{\pi}{2}} (e^{i\theta})^2 ie^{i\theta}\,d\theta \\ &= i\int_0^{\frac{\pi}{2}} e^{3i\theta}\,d\theta \\ &= \frac{ie^{3i\theta}}{3i}\Big|_0^{\pi/2} \\ &= -\frac{1+i}{3}. \end{aligned} \tag{6.2}$$

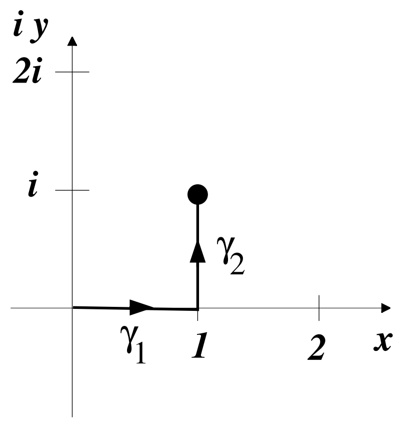

Fig. 6.5. Contour for Example 6.6 with $\Gamma = \gamma_1 \cup \gamma_2$.

Example 6.6. $\int_\Gamma z\,dz$, $\Gamma = \gamma_1 \cup \gamma_2$ is the path shown in Figure 6.5.

In this problem we have path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding up the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 6.5.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0, 1]$. It is natural to take x as the parameter. So, $dz = dx$ and we have

$$\int_{\gamma_1} z\,dz = \int_0^1 x\,dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1+iy$ for $y \in [0,1]$. Inserting z and $dz = i\,dy$, the integral becomes

$$\int_{\gamma_2} z\,dz = \int_0^1 (1+iy)\,idy = i - \frac{1}{2}.$$

Combining these results, we have $\int_\Gamma z\,dz = \frac{1}{2} + (i - \frac{1}{2}) = i$.

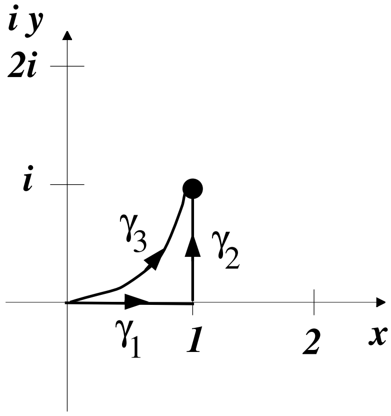

Fig. 6.6. Contour for Example 6.7.

Example 6.7. $\int_{\gamma_3} z\,dz$, γ_3 is the path shown in Figure 6.6.

In this case we take a path from $z = 0$ to $z = 1+i$ along a different path. Let $\gamma_3 = \{(x,y)|y = x^2, x \in [0,1]\} = \{z|z = x + ix^2, x \in [0,1]\}$. Then, $dz = (1+2ix)\,dx$.

The integral becomes

$$\begin{aligned}\int_{\gamma_3} z\,dz &= \int_0^1 (x+ix^2)(1+2ix)\,dx \\ &= \int_0^1 (x+3ix^2-2x^3)\,dx = \\ &= \left[\frac{1}{2}x^2 + ix^3 - \frac{1}{2}x^4\right]_0^1 = i. \end{aligned} \tag{6.3}$$

In the last case we found the same answer as in Example 6.6. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. We will now look into this notion of path independence.

Definition 6.8. *The integral* $\int f(z)\,dz$ *is path independent if*

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all paths from z_1 *to* z_2.

If $\int f(z)\,dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)\,dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)\,dz$. But first we have to define what we mean by a closed loop.

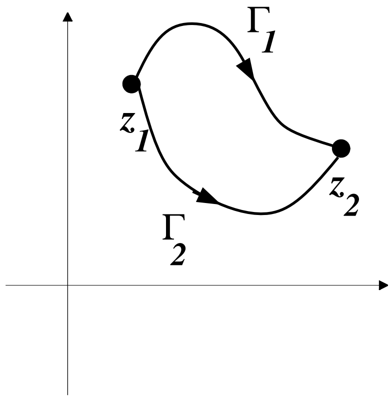

Fig. 6.7. $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.

Definition 6.9. *A simple closed contour is a path satisfying*

a The end point is the same as the beginning point. (This makes the loop closed.)
b The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over the closed loop C shown in Figure 6.8. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite direction. Then,

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2^-} f(z)\,dz. \end{aligned} \tag{6.4}$$

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)\,dz = 0$.

Example 6.10. Consider the integral $\oint_C z\,dz$ for C the closed contour shown in Figure 6.6 starting at $z = 0$ following path γ_1, then γ_2 and returning to $z = 0$. Based on the earlier examples and the fact that going backwards on γ_3 introduces a negative sign, we have

$$\oint_C z\,dz = \int_{\gamma_1} z\,dz + \int_{\gamma_2} z\,dz - \int_{\gamma_3} z\,dz = \frac{1}{2} + \left(i - \frac{1}{2}\right) - i = 0.$$

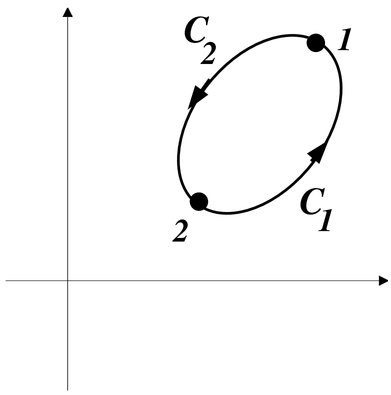

Fig. 6.8. The integral $\oint_C f(z)\,dz$ around C is zero if the integral $\int_\Gamma f(z)\,dz$ is path independent.

6.2 Cauchy's Theorem

Next we want to investigate if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour.

Definition 6.11. *A curve with parametrization* $(x(t), y(t))$ *has a normal* $(n_x, n_y) = (-\frac{dx}{dt}, \frac{dy}{dt})$.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points towards the interior of a closed curve. We will define a positively oriented contour as one that is traversed with the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

We now consider $\oint_C (u + iv)\, dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u+iv)\,dz &= \int_C (u+iv)(dx + i\,dy)\\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \qquad (6.5)\end{aligned}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Green's Theorem in the Plane.

Theorem 6.12. *Let $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions on and inside the simple closed curve C. Denoting the enclosed region S, we have*

$$\int_C P\,dx + Q\,dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \qquad (6.6)$$

Using Green's Theorem to rewrite the first integral in (6.5), we have

$$\int_C u\,dx - v\,dy = \int\int_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy.$$

If u and v satisfy the Cauchy-Riemann equations, then the integrand in the double integral vanishes. Therefore,

$$\int_C u\,dx - v\,dy = 0.$$

In a similar fashion, one can show that

$$\int_C v\,dx + u\,dy = 0.$$

We have thus proven the following theorem:

Cauchy's Theorem

Theorem 6.13. *If u and v satisfy the Cauchy-Riemann equations (5.13) inside and on the simple closed contour C, then*

$$\oint (u+iv)\,dz = 0. \qquad (6.7)$$

Corollary $\oint_C f(z)\,dz = 0$ *when f is differentiable in domain D with $C \subset D$.*

Either one of these is referred to as **Cauchy's Theorem**.

Example 6.14. Consider $\oint_{|z-1|=3} z^4\,dz$. Since $f(z) = z^4$ is differentiable inside the circle $|z-1| = 3$, this integral vanishes.

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

Theorem 6.15. *If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$.*

We consider the two curves as shown in Figure 6.9. Connecting the two contours with contours Γ_1 and Γ_2 (as shown in the figure) C is seen to split into contours C_1 and C_2 and C' into contours C_1' and C_2'. Note that $f(z)$ is differentiable inside the newly formed regions between the curves. Also, the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that integrals of $f(z)$ over these regions are zero.

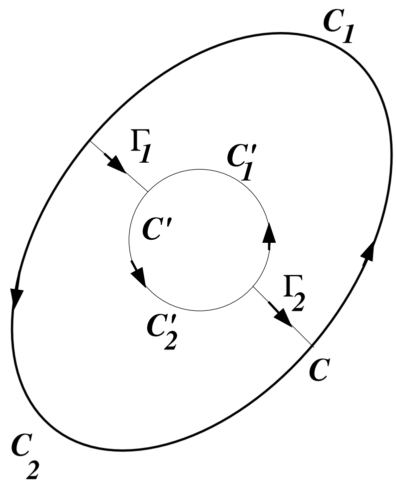

Fig. 6.9. The contours needed to prove that $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$. when $f(z)$ is holomorphic between the contours C and C'.

Noting that integrations over contours opposite to the positive orientation are the negative of integrals in the opposite directions, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)\,dz + \int_{\Gamma_1} f(z)\,dz - \int_{C_1'} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

and

$$\int_{C_2} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{C_2'} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0.$$

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards and Γ_2. The second integral denote the integration over the lower region, but going backwards over all contours except for C_2.

Combining these results, we have

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_1'} f(z)\,dz - \int_{C_2'} f(z)\,dz = 0.$$

Noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$, as was to be proven.

Example 6.16. Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2, 2] \times [-2i, 2i]$.

We can compute this integral by looking at four separate integrals over the sides of the rectangle in the complex plane. One simply parametrizes each line segment, perform the integration and sum the four separate results. From the last theorem, we can instead integrate over a simpler contour by deforming the rectangle into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the rectangle and the circle. So, using the unit circle, as shown in Figure 6.10, the integration might be easier to perform.

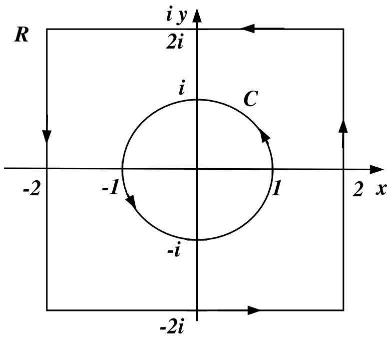

Fig. 6.10. The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

More specifically, the last theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

The latter integral can be computed using the parametrization $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned} \oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned} \tag{6.8}$$

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ by deforming the original simple closed contour.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 6.11. The lower segment, γ_4 of the square can be simple parametrized by noting that along this segment $z = x - 2i$ for $x \in [-2, 2]$. Then, we have

$$\begin{aligned}\oint_{\gamma_4} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\ &= \ln|x-2i|_{-2}^{2} \\ &= (\ln(2\sqrt{2}) - \frac{\pi i}{4}) - (\ln(2\sqrt{2}) - \frac{3\pi i}{4}) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.9}$$

We note that the arguments of the logarithms are determined from the angles make by the diagonals provided in Figure 6.11.

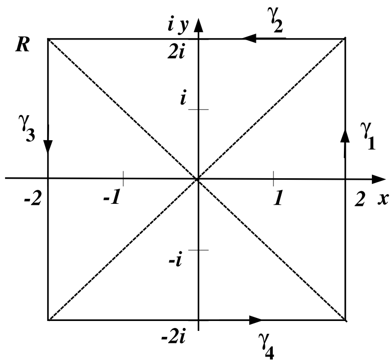

Fig. 6.11. The contours used to compute $\oint_R \frac{dz}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

Similarly, the integral along the top segment is computed as

$$\begin{aligned}\oint_{\gamma_2} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\ &= \ln|x+2i|_{2}^{-2} \\ &= (\ln(2\sqrt{2}) + \frac{3\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{\pi i}{4}) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.10}$$

The integral over the right side is

$$\begin{aligned}\oint_{\gamma_1} \frac{dz}{z} &= \int_{-2}^{2} \frac{idy}{2+iy}\\ &= \ln|2+iy|_{-2}^{2}\\ &= (\ln(2\sqrt{2}) + \frac{\pi i}{4}) - (\ln(2\sqrt{2}) - \frac{\pi i}{4})\\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.11}$$

Finally, the integral over the left side is

$$\begin{aligned}\oint_{\gamma_3} \frac{dz}{z} &= \int_{2}^{-2} \frac{idy}{-2+iy}\\ &= \ln|-2+iy|_{-2}^{2}\\ &= (\ln(2\sqrt{2}) + \frac{5\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{3\pi i}{4})\\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.12}$$

Therefore, we have that

$$\begin{aligned}\oint_{R} \frac{dz}{z} &= \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} + \int_{\gamma_4} \frac{dz}{z}\\ &= \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2}\\ &= 4(\frac{\pi i}{2}) = 2\pi i. \end{aligned} \tag{6.13}$$

This ives the same answer we had found using a simple contour deformation.

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)\,dz = 0$ does not imply that $f(z)$ is differentiable. What we do have is **Morera's Theorem**:

Theorem 6.17. *Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. Then f is differentiable in D.*

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

6.3 Analytic Functions and Cauchy's Integral Formula

In the previous section we saw that Cauchy's Theorem was useful for computing certain integrals without having to parametrize the contours, or to deform contours to simpler ones. The integrand needs to possess certain differentiability properties. In this section, we will generalize our integrand slightly so that we can integrate a larger family of complex functions. This will take the form of what is called *Cauchy's Integral Formula*, which extends Cauchy's Theorem to functions analytic in an annulus. However, first we need to explore the concept of analytic functions.

Definition 6.18. *$f(z)$ is analytic in D if for every open disk $|z - z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

This series converges uniformly and absolutely inside the circle of convergence, $|z - z_0| < R$, with radius of convergence R.

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C (z - z_0)^n \, dz$. As we will see in the homework exercises, these integrals evaluate to zero for most n. Thus, we can show that for $f(z)$ analytic in D and any C lying in D, $\oint_C f(z) \, dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

Let's recall some manipulations from our study of series of real functions.

Example 6.19. $f(z) = \frac{1}{1+z}$ for $z_0 = 0$.

This case is simple. $f(z)$ is the sum of a geometric series for $|z| < 1$. We have

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Thus, this series expansion converges inside the unit circle in the complex plane.

Example 6.20. $f(z) = \frac{1}{1+z}$ for $z_0 = \frac{1}{2}$. We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor' formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in powers of $z - \frac{1}{2}$. So, we rewrite the function in a form that has this term. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. [Note: This is just like what we had done in Chapter 2 wit functions of real variables.] We can get the denominator into such a form by factoring out the $\frac{3}{2}$. Then we would have

$$f(z) = \frac{2}{3}\frac{1}{1+\frac{2}{3}(z-\frac{1}{2})}.$$

The second factor now has the form $\frac{1}{1-r}$, which would be the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{2}{3}(z - \frac{1}{2})$ provided that —r—¡1. Therefore, we have found that

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty}\left[-\frac{2}{3}(z-\frac{1}{2})\right]^n$$

for

$$|-\frac{2}{3}(z-\frac{1}{2})| < 1.$$

This convergence interval can be rewritten as

$$|z-\frac{1}{2}| < \frac{3}{2}.$$

This is a circle centered at $z = \frac{1}{2}$ with radius $\frac{3}{2}$.

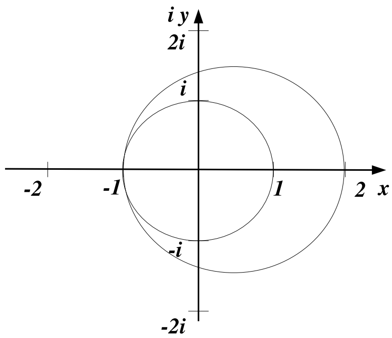

Fig. 6.12. Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

In Figure 6.12 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a region even further outside to the region $|z-\frac{1}{2}| < \frac{3}{2}$. We will see later that there are expansions outside of these regions, though some are expansions involving negative powers of $z-z_0$.

We now present the main theorem of this section:

Cauchy Integral Formula

Theorem 6.21. *Let $f(z)$ be analytic in $|z - z_0| < \rho$ and let C be the boundary (circle) of this disk. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{6.14}$$

In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned} \frac{f(z)}{z-z_0} &= \frac{1}{z-z_0}\left[\sum_{n=0}^{\infty} c_n (z-z_0)^n\right] \\ &= \frac{1}{z-z_0}\left[c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots\right] \\ &= \frac{c_0}{z-z_0} + \underbrace{c_1 + c_2(z-z_0) + \ldots}_{\text{analytic function}}. \end{aligned} \tag{6.15}$$

As noted the integrand can be written as

$$\frac{f(z)}{z-z_0} = \frac{c_0}{z-z_0} + h(z),$$

where $h(z)$ is an analytic function, since $h(z)$ is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)\, dz = 0$. Noting also that $c_0 = f(z_0)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint_C \left[\frac{c_0}{z-z_0} + h(z)\right] dz = f(z_0) \oint_C \frac{1}{z-z_0}\, dz.$$

We need only compute the integral $\oint_C \frac{1}{z-z_0}\, dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $|z - z_0| = \rho$, as shown in Figure 6.13. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

(Note that this has the right complex modulus since $|e^{i\theta}| = 1$. Then $dz = i\rho e^{i\theta} d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i.$$

Therefore,

$$\oint_C \frac{f(z)}{z-z_0}\, dz = f(z_0) \oint_C \frac{1}{z-z_0}\, dz = 2\pi f(z_0),$$

as was to be shown.

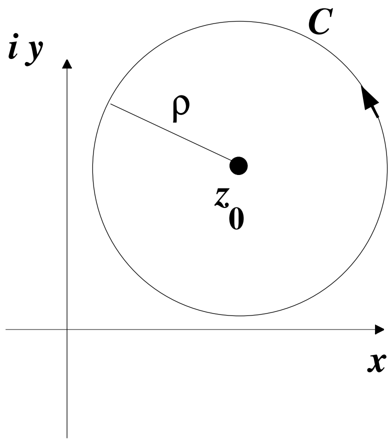

Fig. 6.13. Circular contour used in proving the Cauchy Integral Formula.

Example 6.22. Compute $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\, dz$.

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2 - 6z + 5 = (z-1)(z-5)$. We next locate the locations of the zeros of the denominator. In Figure 6.14 we see the contour and the points $z = 1$ and $z = 5$. The only point inside the region bounded by the contour is $z = 1$. Therefore, we can apply the Cauchy Integral Formula for $f(z) = \frac{\cos z}{z-5}$ to the integral

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\, dz = \int_{|z|=4} \frac{f(z)}{(z-1)}\, dz = 2\pi i f(1).$$

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\, dz = -\frac{\pi i \cos(1)}{2}.$$

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z - z_0| < \rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz. \tag{6.16}$$

This can be proven following a derivation similar to that for the Cauchy Integral Formula. One needs to recall the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

We also need the lemma

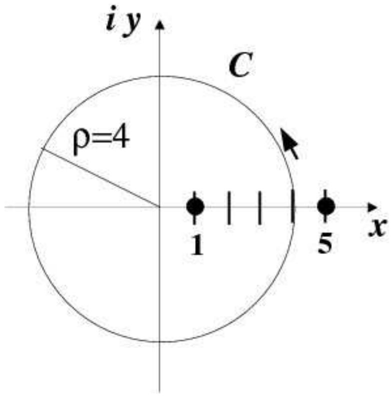

Fig. 6.14. Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\, dz$.

Lemma

$$\oint_C \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 0, & n \neq 0 \\ 2\pi i, & n = 0. \end{cases} \tag{6.17}$$

This is a homework problem. The integrals are similar to the $n = 0$ case above.

6.4 Laurent Series

Until this point we have only talked about series whose terms have nonnegative powers of $z-z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series was shown in Figure 7.6. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found..

Example 6.23. $f(z) = \frac{1}{1+z}$ for $|z| > 1$.

As before, we make use of the geometric series. Since $|z| > 1$, we instead rewrite our function as

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{1}{z}$. We note that $|z| > 1$ implies that $|r| < 1$. Thus, we have the geometric series

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}(-\frac{1}{z})^n.$$

This can be re-indexed as

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{-n-1} = \sum_{j=1}^{\infty} (-1)^{j-1} z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$, has negative powers of z.

Example 6.24. $f(z) = \frac{1}{1+z}$ for $|z - \frac{1}{2}| > \frac{3}{2}$.

As before, we express this in a form in which we use geometric series:

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

Instead of factoring out the $\frac{3}{2}$ we factor out the $(z - \frac{1}{2})$ term. Then, we obtain

$$f(z) = \frac{1}{1+z} = \frac{1}{(z - \frac{1}{2})} \frac{1}{(1 + \frac{3}{2}(z - \frac{1}{2})^{-1})}.$$

Again, we identify $a = 1$ and $r = -\frac{3}{2}(z - \frac{1}{2})^{-1}$. This leads to the series

$$f(z) = \frac{1}{z - \frac{1}{2}} \sum_{n=0}^{\infty} (-\frac{3}{2}(z - \frac{1}{2})^{-1})^n.$$

This converges for $|z - \frac{1}{2}| > \frac{3}{2}$ and can also be re-indexed to verify that this series involves negative powers of $z - \frac{1}{2}$.

This leads to the following theorem:

Theorem 6.25. *Let $f(z)$ be analytic in an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 6.15. Then,*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} b_j (z - z_0)^{-j},$$

with

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{j+1}}\, dz,$$

and

$$b_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-j+1}}\, dz.$$

The above series can be written in the more compact form

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j.$$

Such a series expansion is called a *Laurent series* expansion.

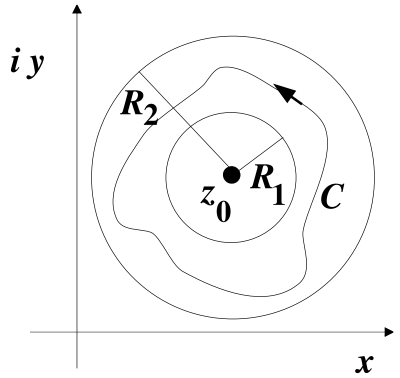

Fig. 6.15. This figure shows an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

Example 6.26. Expand $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$.

Using partial fractions, we can write this as

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z| > 1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z| < 2$. This is done as follows. First, we write

$$\frac{1}{2+z} = \frac{1}{2[1-(-\frac{z}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{z}{2})^n.$$

Then we write

$$\frac{1}{1-z} = -\frac{1}{z[1-\frac{1}{z}]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\begin{aligned}\frac{1}{(1-z)(2+z)} &= \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}(-\frac{z}{2})^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6(2^n)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}. \qquad (6.18)\end{aligned}$$

6.5 Singularities and The Residue Theorem

In the last section we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$, where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a *singularity* of $g(z)$, as $g(z)$ is not defined there. As we saw from the proof of the Cauchy Integral Formula, $g(z)$ has a Laurent series expansion about $z = z_0$,

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2 + \dots.$$

We will first classify singularities.

Definition 6.27. *A singularity of $f(z)$ is a point at which $f(z)$ fails to be analytic.*

Typically these are isolated singularities. In order to classify the singularities of $f(z)$, we look at the *principal part* of the Laurent series of $f(z)$ about $z = z_0$: $\sum_{j-1}^{\infty} b_j(z-z_0)^{-j}$.

1. If $f(z)$ is bounded near z_0, then z_0 is a **removable singularity**.
2. If there are a finite number of terms in the principal part, then one has **poles of order** n.
3. If there are an infinite number of terms in the principal part, then one has an **essential singularity**.

Example 6.28. **Removable singularity:** $f(z) = \frac{\sin z}{z}$.

At first it looks like there is a possible singularity at $z = 0$. However, we know from the first semester of calculus that $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Furthermore, we can expand $\sin z$ about $z = 0$ and see that

$$\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{6!} + \dots) = 1 - \frac{z^2}{6!} + \dots.$$

Thus, there are only nonnegative powers in the series expansion. So, this is an example of a removable singularity.

Example 6.29. **Poles** $f(z) = \frac{e^z}{(z-1)^n}$.

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$ called a simple pole. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1}e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \dots.$$

Note that the principal part of the Laurent series expansion about $z = 1$ only has one term.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2} e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \dots.$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. This is a pole of order 2.

Example 6.30. **Essential Singularity** $f(z) = e^{\frac{1}{z}}$.

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two. In general, we can define poles of order k.

Definition 6.31. *$f(z)$ has a pole of order k at z_0 if and only if $(z-z_0)^k f(z)$ has a removable singularity at z_0, but $(z-z_0)^{k-1} f(z)$ for $k > 0$ does not.*

Let $\phi(z) = (z-z_0)^k f(z)$ be analytic. Then it has a Taylor series expansion about z_0. As we had seen in the last section, we can write the integral representation

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C \frac{\phi(z)}{(z-z_0)^k}\, dz.$$

Inserting the definition of $\phi(z)$ we then have

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C f(z)\, dz.$$

Solving for the integral, we have the result

$$\begin{aligned} \oint_C f(z)\, dz &= \frac{2\pi i}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right]_{z=z_0} \\ &\equiv 2\pi i \ \mathrm{Res}[f(z); z_0] \end{aligned} \tag{6.19}$$

Here we have defined the **residue of** $f(z)$ **at** $z = z_0$**.** For a pole of order k it is given by

Residues

$$\mathrm{Res}[f(z); z_0] = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right] \tag{6.20}$$

Example 6.32. $\oint_{|z|=1} \frac{dz}{\sin z}$.

We begin by looking for the singularities of the integrand, which is when $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \dots,$ are the singularities. However, only $z = 0$

lies inside the contour, as shown in Figure 6.16. We note further that $z = 0$ is a simple pole, since

$$\lim_{z \to 0} (z - 0) \frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

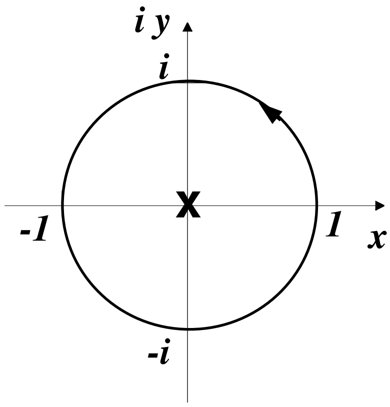

Fig. 6.16. Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2 - 1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 6.18. One could do a partial fraction decomposition and have two integrals with one pole each. However, in cases in which we have many poles, we can use the following theorem, known as the Residue Theorem.

The Residue Theorem

Theorem 6.33. *Let $f(z)$ be a function which has poles z_j, $j = 1, \ldots, N$ inside a simple closed contour C and no other singularities in this region. Then,*

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} Res[f(z); z_j], \tag{6.21}$$

where the residues are computed using Equation (6.20).

The proof of this theorem is based upon the contours shown in Figure 6.17. One constructs a new contour C' by encircling each pole, as show in the figure. Then one connects a path from C to each circle. In the figure two paths are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)\,dz = \oint_{C} f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have be cut out. Solving for $\oint_C f(z)\,dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)\,dz$ is $2\pi i$ times the sum of the residues.

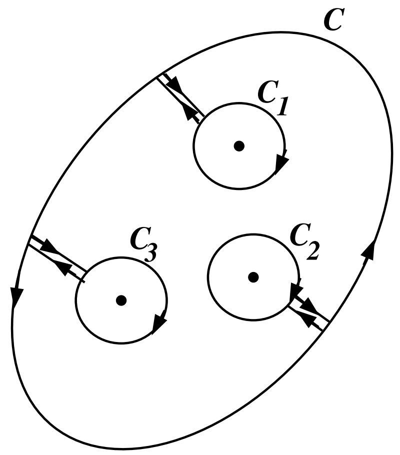

Fig. 6.17. A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

Example 6.34. $\oint_{|z|=2} \frac{dz}{z^2-1}$.

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

In Figure 6.18 we plot the contour and the two poles, denoted by an "x". Since both poles are inside the contour, we need to compute the residues for each one. They are both simple poles, so we have

$$\begin{aligned}\text{Res}\left[\frac{1}{z^2-1}; z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1}\\ &= \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2},\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}\text{Res}\left[\frac{1}{z^2-1}; z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1}\\ &= \lim_{z\to -1}\frac{1}{z-1} = -\frac{1}{2}.\end{aligned} \tag{6.23}$$

Then,

$$\oint_{|z|=2}\frac{dz}{z^2-1} = 2\pi i(\frac{1}{2}-\frac{1}{2}) = 0.$$

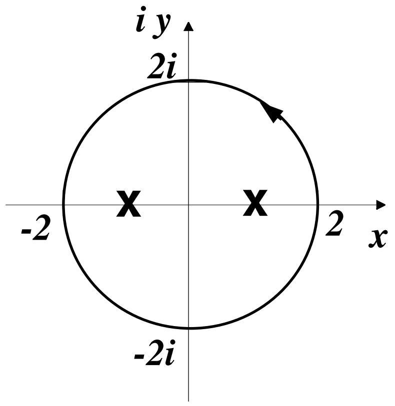

Fig. 6.18. Contour for computing $\oint_{|z|=2}\frac{dz}{z^2-1}$.

Example 6.35. $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply methods from our calculus class to do this integral, attempting to write $1+\cos\theta = 2\cos^2\frac{\theta}{2}$. However, we do not get very far.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta, \sin\theta)$, is to transform the integration to the complex plane through the transformation $z = e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i}{2}\left(z - \frac{1}{z}\right).$$

Under this transformation, $z = e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, we have

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= -i\oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}\left(z^2+1\right)} \\ &= -2i\oint_{|z|=1} \frac{dz}{z^2+4z+1}. \end{aligned} \tag{6.24}$$

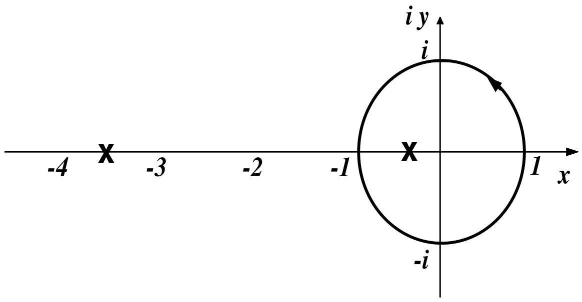

Fig. 6.19. Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

We can apply the Residue Theorem to the resulting integral. The singularities occur for $z^2 + 4z + 1 = 0$. Using the quadratic formula, we have the roots $z = -2 \pm \sqrt{3}$. The location of these poles are shown in Figure 6.19. Only $z = -2 + \sqrt{3}$ lies inside the integration contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$\begin{aligned}\operatorname{Res}[f(z); z = -2+\sqrt{3}] &= \lim_{z\to -2+\sqrt{3}} (z - (-2+\sqrt{3}))\frac{-2i}{z^2+4z+1} \\ &= -2i \lim_{z\to -2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))} \\ &= -2i \lim_{z\to -2+\sqrt{3}} \frac{1}{z-(-2-\sqrt{3})} \\ &= \frac{-2i}{-2+\sqrt{3}-(-2-\sqrt{3})} \\ &= \frac{-i}{\sqrt{3}} \\ &= \frac{-i\sqrt{3}}{3}. \end{aligned} \tag{6.25}$$

Therefore, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i \left(\frac{-i\sqrt{3}}{3}\right) = \frac{2\pi\sqrt{3}}{3}. \quad (6.26)$$

Example 6.36. $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz.$

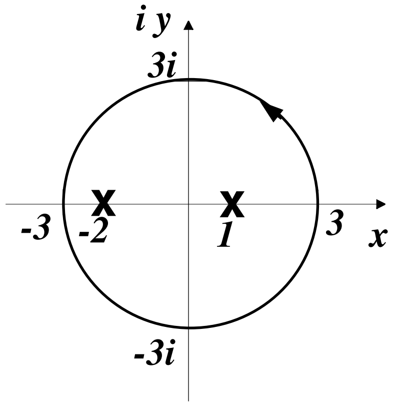

Fig. 6.20. Contour for computing $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz$.

In this example there are two poles $z = 1, -2$ inside the contour. $z = 1$ is a second order pole and $z = -2$ is a simple pole. [See Figure 6.20]. Therefore, we need the residues at each pole of $f(z) = \frac{z^2+1}{(z-1)^2(z+2)}$:

$$\begin{aligned} \operatorname{Res}[f(z); z=1] &= \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2 \frac{z^2+1}{(z-1)^2(z+2)}\right] \\ &= \lim_{z\to 1}\left(\frac{z^2+4z-1}{(z+2)^2}\right) \\ &= \frac{4}{9}. \end{aligned} \quad (6.27)$$

$$\begin{aligned} \operatorname{Res}[f(z); z=1] &= \lim_{z\to -2} (z+2)\frac{z^2+1}{(z-1)^2(z+2)} \\ &= \lim_{z\to -2} \frac{z^2+1}{(z-1)^2} \\ &= \frac{5}{9}. \end{aligned} \quad (6.28)$$

The evaluation of the integral is found by computing $2\pi i$ times the sum of the residues:

$$\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz = 2\pi i \left(\frac{4}{9} + \frac{5}{9}\right) = 2\pi i.$$

6.6 Infinite Integrals

As our final application of complex integration techniques, we will turn to the evaluation of infinite integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$. These types of integrals will appear later in the text and will help to tie in what seems to be a digression in our study of Fourier Analysis. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall that such integrals are improper integrals and you had seen them in your calculus classes. The way that one determines if such integrals exist, or converge, is to compute the integral using a limit:

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx.$$

For example,

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2}\, dx = \lim_{R\to\infty} \left(-\frac{2}{R}\right) = 0.$$

Similarly,

$$\int_{-\infty}^{\infty} x\, dx = \lim_{R\to\infty} \int_{-R}^{R} x\, dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{(-R)^2}{2}\right) = 0.$$

However, The integrals $\int_0^{\infty} x\, dx$ and $\int_{-\infty}^{0} x\, dx$ do not exist. Note that

$$\int_0^{\infty} x\, dx = \lim_{R\to\infty} \int_0^{R} x\, dx = \lim_{R\to\infty} \left(\frac{R^2}{2}\right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx$$

does not exist while $\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the *Cauchy Principal Value Integral* and is denoted with either a P or PV prefix:

$$P \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{6.29}$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

We now proceed to the evaluation of such principal value integrals using complex integration methods. We want to evaluate the integral $\int_{-\infty}^{\infty} f(x)\,dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$). We then consider the integral $\int_{-R}^{R} f(x)\,dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 6.21. Note, a similar construction is sometimes needed extending the integration into the lower half plane ($\text{Im}(z) < 0$)when $f(z)$ is analytic there.

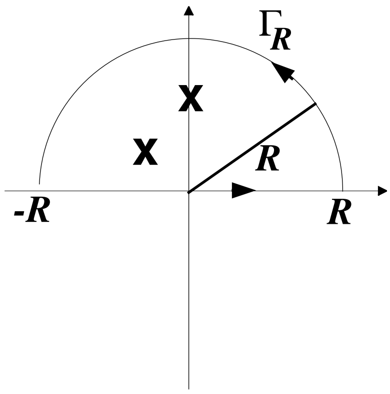

Fig. 6.21. Contours for computing $P\int_{-\infty}^{\infty} f(x)\,dx$.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz. \tag{6.30}$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)\,dx = \oint_{C} f(z)\,dz - \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz, \tag{6.31}$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We can parametrize the contour Γ_R using $z = Re^{i\theta}$. Then, when $|f(z)| < M(R)$,

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \int_{\Gamma_R} f(z)\,dz = \left|\int_0^{2\pi} f(Re^{i\theta})Re^{i\theta}\,d\theta\right|$$

$$\leq R \int_0^{2\pi} \left| f(Re^{i\theta}) \right| \, d\theta$$
$$< RM(R) \int_0^{2\pi} \, d\theta$$
$$= 2\pi RM(R). \tag{6.32}$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz$=0.
We show how this applies some examples.

Example 6.37. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

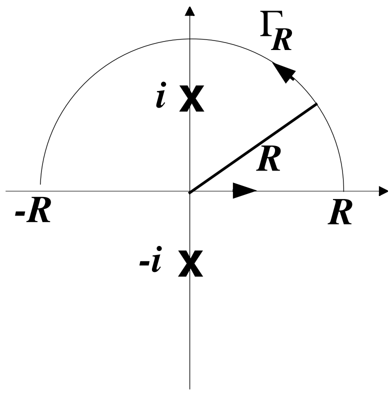

Fig. 6.22. Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We already know how to do this integral from our calculus. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left(2\tan^{-1} R\right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 6.22 and the poles of the integrand are at $z = \pm i$.

We first note that $f(z) = \frac{1}{1+z^2}$ goes to zero fast enough on Γ_R as R gets large.

$$R|f(z)| = \frac{R}{|1 + R^2 e^{2i\theta}|} = \frac{R}{\sqrt{1 + 2R^2\cos\theta + R^4}}.$$

Thus, as $R \to \infty$, $R|f(z)| \to 0$. So,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$\text{Res}[f(z); z = i] = \lim_{z \to i}(z - i)\frac{1}{1 + z^2} = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = 2\pi i \left(\frac{1}{2i}\right) = \pi.$$

Example 6.38. $P \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$.

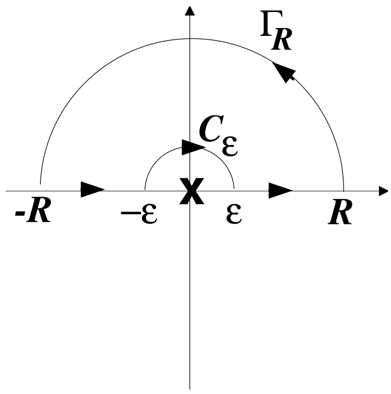

Fig. 6.23. Contour for computing $P \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$.

There are several new techniques that have to be introduced in order to carry out this integration. We need to handle the pole at $z = 0$ in a special way and we need something called Jordan's Lemma to guarantee that integral over the contour Γ_R vanishes.

For this example the integral is unbounded at $z = 0$. Constructing the contours as before we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with our computation by carefully going around the pole with a small semicircle of radius ϵ, as shown in Figure 6.23. Then our principal value integral computation becomes

$$P \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} \frac{\sin x}{x} \, dx + \int_{\epsilon}^{\infty} \frac{\sin x}{x} \, dx \right). \tag{6.33}$$

We will also need to rewrite the sine function in term of exponentials in this integral.

$$P \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{2i} \left(P \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx - P \int_{-\infty}^{\infty} \frac{e^{-ix}}{x} \, dx \right). \tag{6.34}$$

We now employ **Jordan's Lemma**.

Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

A similar result applies for $k < 0$, but one closes the contour in the lower half plane. [See Section 6.6.1 for the proof of Jordan's Lemma.]

We now put these ideas together to compute the given integral. According to Jordan's lemma, we will need to compute the above exponential integrals using two different contours. We first consider $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$. We use the contour in Figure 6.23. Then we have

$$\oint_{C_R} \frac{e^{iz}}{z}\,dz = \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\,dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}\,dz.$$

The integral $\oint_{C_R} \frac{e^{iz}}{z}\,dz$ vanishes since there are no poles enclosed in the contour! The integral over Γ_R will vanish as R gets large according to Jordan's Lemma. The sum of the second and fourth integrals is the integral we seek as $\epsilon \to 0$ and $R \to \infty$.

The remaining integral around the small circle has to be done separately. We have

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{\exp(i\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta = -\int_{0}^{\pi} i\exp(i\epsilon e^{i\theta})\,d\theta.$$

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = -\pi i.$$

So far, we have that

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = -\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \pi i.$$

We can compute $P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx$ in a similar manner, being careful with the sign changes due to the orientations of the contours. In this case, we find the same value

$$P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx = \pi i.$$

Finally, we can compute the original integral as

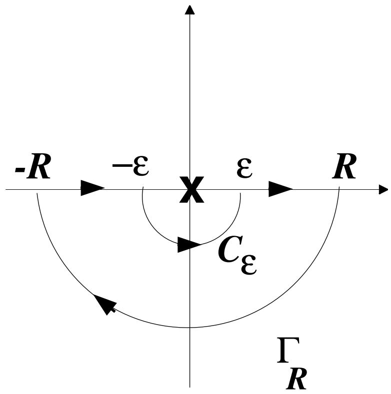

Fig. 6.24. Contour in the lower half plane for computing $P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx$.

$$\begin{aligned} P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx\right) \\ &= \frac{1}{2i}\left(\pi i + \pi i\right) \\ &= \pi. \end{aligned} \tag{6.35}$$

6.6.1 Jordan's Lemma - *optional*

For completeness, we prove Jordan's Lemma.

Theorem 6.39. *If $f(z)$ converges uniformly to zero as $z \to \infty$, then*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

Proof. We consider the integral

$$I_R = \int_{C_R} f(z)e^{ikz}\,dz,$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$ in the complex plane. Let $z = Re^{i\theta}$ be a parametrization of C_R. Then,

$$I_R = \int_0^{\pi} f(Re^{i\theta})e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\,d\theta.$$

Since

$$\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg z \le \pi,$$

then for large $|R|$, $|f(z)| < \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned}|I_R| &= \left|\int_0^\pi f(Re^{i\theta}) e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta\right| \\ &\le \int_0^\pi \left|f(Re^{i\theta})\right| \left|e^{ikR\cos\theta}\right| \left|e^{-aR\sin\theta}\right| \left|iRe^{i\theta}\right| d\theta \\ &\le \epsilon R \int_0^\pi e^{-aR\sin\theta}\, d\theta \\ &= 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \end{aligned} \tag{6.36}$$

The last integral still cannot be computed, but we can get a bound on it over the range $\theta \in [0, \pi/2]$. Note that

$$\sin\theta \ge \frac{\pi}{2}\theta, \quad \theta \in [0, \pi/2].$$

Therefore, we have

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = \frac{2\epsilon R}{2aR/\pi}(1 - e^{-aR}).$$

For large R we have

$$\lim_{R\to\infty} |I_R| \le \frac{\pi\epsilon}{a}.$$

So, as $\epsilon \to 0$, the integral vanishes.

Problems

6.1. Evaluate the following integrals:

a. $\int_C \overline{z}\, dz$, where C is the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$.
b. $\int_C f(z)\, dz$, where $f(z) = z + 2\overline{z}$ and C is the path from $z = 0$ to $z = 1 + 2i$ consisting of two line segments from $z = 0$ to $z = 1$ and then $z = 1$ to $z = 1 + 2i$.
c. $\int_C \frac{1}{z^2+4}\, dz$ for C the positively oriented circle, $|z| = 2$. [Hint: Parametrize the circle as $z = 2e^{i\theta}$, multiply numerator and denominator by $e^{-i\theta}$, and put in trigonometric form.]

6.2. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the counterclockwise direction. Define

$$g(z_0) = \int_C \frac{z^2 + z + 1}{z - z_0}\, dz.$$

Find $g(i)$ and $g(4i)$. [Hint: Sketch the ellipse in the complex plane. Use the Cauchy Integral Theorem with an appropriate $f(z)$.]

6.3. Show that

$$\int_C \frac{dz}{(z-1-i)^{n+1}} = \begin{cases} 0, & n \neq 0, \\ 2\pi i, & n = 0, \end{cases}$$

for C the boundary of the square $0 \leq x \leq 2, 0 \leq y \leq 2$ taken counterclockwise. [Hint: Use the fact that contours can be deformed into simpler shapes (like a circle) as long as the integrand is analytic in the region between them. After picking a simpler contour, integrate using parametrization.]

6.4. For the following determine if the given point is a removable singularity, an essential singularity, or a pole (indicate its order).

a. $\frac{1-\cos z}{z^2}, \quad z = 0.$
b. $\frac{\sin z}{z^2}, \quad z = 0.$
c. $\frac{z^2-1}{(z-1)^2}, \quad z = 1.$
d. $ze^{1/z}, \quad z = 0.$
e. $\cos \frac{\pi}{z-\pi}, \quad z = \pi.$

6.5. Find the Laurent series expansion for $f(z) = \frac{\sinh z}{z^3}$ about $z = 0$. [Hint: You need to first do a MacLaurin series expansion for the hyperbolic sine.]

6.6. Find series representations for all indicated regions.

a. $f(z) = \frac{z}{z-1}$, $|z| < 1$, $|z| > 1$.
b. $f(z) = \frac{1}{(z-i)(z+2)}$, $|z| < 1$, $1 < |z| < 2$, $|z| > 2$. [Hint: Use partial fractions to write this as a sum of two functions first.]

6.7. Find the residues at the given points:

a. $\frac{2z^2+3z}{z-1}$ at $z = 1$.
b. $\frac{\ln(1+2z)}{z}$ at $z = 0$.
c. $\frac{\cos z}{(2z-\pi)^3}$ at $z = \frac{\pi}{2}$.

6.8. Consider the integral $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$.

a. Evaluate this integral by making the substitution $2\cos\theta = z + \frac{1}{z}$, $z = e^{i\theta}$ and using complex integration methods.
b. In the 1800's Weierstrass introduced a method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t.
   i. Show that
   $$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$
   ii. Show that
   $$d\theta = \frac{2dt}{1+t^2}$$
   iii. Use the Weierstrass substitution to compute the above integral.

6.9. Do the following integrals.

a.
$$\oint_{|z-i|=3} \frac{e^z}{z^2+\pi^2}\, dz.$$

b.
$$\oint_{|z-i|=3} \frac{z^2-3z+4}{z^2-4z+3}\, dz.$$

c.
$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4}\, dx.$$

[Hint: This is $\operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx$.]