

## 2.8 The Binomial Expansion

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression  $(a + b)^p$  in powers of  $a$  and  $b$ . We will investigate this expansion first for nonnegative integer powers  $p$  and then derive the expansion for other values of  $p$ . While the binomial expansion can be obtained using Taylor series in the last section, we will provide a more interesting derivation here to show that

$$(a + b)^p = \sum_{r=0}^{\infty} C_r^p a^{n-r} b^r, \quad (2.15)$$

where the  $C_r^p$  are called the binomial coefficients.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned} (a + b)^0 &= 1 \\ (a + b)^1 &= a + b \\ (a + b)^2 &= a^2 + 2ab + b^2 \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &\dots \end{aligned} \quad (2.16)$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of  $a$  and a power of  $b$ . The powers of  $a$  are decreasing from  $n$  to 0 in the expansion of  $(a + b)^n$ .

Similarly, the powers of  $b$  increase from 0 to  $n$ . The sums of the exponents in each term is  $n$ . So, we can write the  $k + 1$ st term in the expansion as  $a^{n-k}b^k$ . For example, in the expansion of  $(a + b)^{51}$  the 6th term is  $a^{51-5}b^5 = a^{46}b^5$ . However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{array}{rcccccc} n = 0 : & & & & & & 1 \\ n = 1 : & & & & & 1 & 1 \\ n = 2 : & & & 1 & 2 & 1 & \\ n = 3 : & & 1 & 3 & 3 & 1 & \\ n = 4 : & 1 & 4 & 6 & 4 & 1 & \end{array} \quad (2.17)$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. The second term and next to last term each have coefficients of  $n$ . Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows  $n = 2$  and  $n = 3$  that  $1 + 2 = 3$  and  $2 + 1 = 3$ :

$$\begin{array}{ccccccc}
 n = 2 : & & 1 & & 2 & & 1 \\
 & & & \searrow & \swarrow & \searrow & \swarrow \\
 n = 3 : & 1 & & 3 & & 3 & & 1
 \end{array} \tag{2.18}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{ccccccc}
 n = 3 : & & 1 & & 3 & & 3 & & 1 \\
 n = 4 : & & 1 & & 4 & & 6 & & 4 & & 1 \\
 n = 5 : & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 n = 6 : & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
 \end{array} \tag{2.19}$$

So, for rows  $n = 4$  and  $n = 5$ , we see that  $1 + 4 = 5$ ,  $4 + 6 = 10$ , etc.

Of course, it would take a while to compute each row up to the desired  $n$ . We need a simple expression for computing a specific coefficient. Consider the  $k$ th term in the expansion of  $(a + b)^n$ . Let  $r = k - 1$ . Then this term is of the form  $C_r^n a^{n-r} b^r$ . We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \binom{n}{r}.$$

This is nothing other than the combinatoric symbol for determining how to choose  $n$  things  $r$  at a time. In our case, this makes sense. We have to count the number of ways that we can arrange the  $r$  products of  $b$  with  $n - r$  products of  $a$ . There are  $n$  slots to place the  $b$ 's. For example, the  $r = 2$  case for  $n = 4$  involves the six products:  $aabb$ ,  $abab$ ,  $abba$ ,  $baab$ ,  $baba$ , and  $bbaa$ . Thus, it is natural to use this notation. The original problem that concerned Pascal was in gambling. However, we will not go into that here.

So, we have found that

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r. \quad (2.20)$$

What if  $a \gg b$ ? Can we use this to get an approximation to  $(a + b)^n$ ? If we neglect  $b$  then  $(a + b)^n \simeq a^n$ . How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big  $O$  notation. In order to do this we first divide out  $a$  as

$$(a + b)^n = a^n \left(1 + \frac{b}{a}\right)^n.$$

Now we have a small parameter,  $\frac{b}{a}$ . According to what we have seen above, we can use the binomial expansion to write

$$\left(1 + \frac{b}{a}\right)^n = \sum_{r=0}^n \binom{n}{r} \left(\frac{b}{a}\right)^r. \quad (2.21)$$

Thus, we have a finite sum of terms involving powers of  $\frac{b}{a}$ . Since  $a \gg b$ , most of these terms can be neglected. So, we can write

$$\left(1 + \frac{b}{a}\right)^n = 1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

note that we have used the observation that the second coefficient in the  $n$ th row of Pascal's triangle is  $n$ .

Summarizing, this then gives

$$\begin{aligned} (a + b)^n &= a^n \left(1 + \frac{b}{a}\right)^n \\ &= a^n \left(1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^n + na^n\frac{b}{a} + a^n O\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \quad (2.22)$$

Therefore, we can approximate  $(a + b)^n \simeq a^n + nba^{n-1}$ , with an error on the order of  $ba^{n-2}$ . Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation

that  $(a + b)^n \simeq O(a^n)$ , but it is not typically good enough in applications because the error in this case is of the order  $ba^{n-1}$ .

Now consider the geometric series  $1 + x + x^2 + \dots$ . We have seen that such a series converges for  $|x| < 1$ , giving

$$1 + x + x^2 + \dots = \frac{1}{1 - x}.$$

But,  $\frac{1}{1-x} = (1-x)^{-1}$ . This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (2.20).

This example suggests that our sum may no longer be finite. So, for  $p$  a real number, we write

$$(1 + x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r. \quad (2.23)$$

However, we quickly run into problems with this form. Consider the coefficient for  $r = 1$  in an expansion of  $(1 + x)^{-1}$ . This is given by

$$\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is  $(-1)!$ ? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

Here we have used the fact that  $n! = n(n-1)!$ .

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \binom{p}{r} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \quad (2.24)$$

With this in mind we now state the theorem:

**General Binomial Expansion** The general binomial expansion for  $(1+x)^p$  is a simple generalization of Equation (2.20). For  $p$  real, we have that

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r. \quad (2.25)$$

Often we need the first few terms for the case that  $x \ll 1$ :

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \quad (2.26)$$