The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus. Let M be a bounded, twice continuously differentiable, oriented $k+1$-dimensional manifold in $\mathbf{R}^n$, $n \geq k+1$, with a k-dimensional boundary ∂M, and let ω be a continuously differentiable k-form on $\mathbf{R}^n$, then $d\omega$ is a $k+1$-form and $\int_{\partial M} \omega = \int_M d\omega.$

Special Cases

1. ω is a 0-form (Scalar Field) and M a curve from **a** to **b**. $\int_M d\omega = \omega(\mathbf{b}) - \omega(\mathbf{a}).$ For M an interval in **R**, this is the Calculus I version of the ***Fundamental Theorem***:
$$\text{If } \frac{dF(x)}{dx} = f(x), \text{ then } \int_a^b f(x)\,dx = F(b) - F(a).$$
2. $\omega = f dx + g dy$ is a 1-form in $\mathbf{R}^2$ and M a region in $\mathbf{R}^2$ bounded by a closed curve. Then $\int_{\partial M} f\,dx + g\,dy = \int_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy.$ This is ***Green's Theorem in the Plane.*** It is a two dimensional version of Stokes' Theorem below.
3. $\omega = f\,dydz + g\,dzdx + h\,dxdy$ is a 2-form in $\mathbf{R}^3$ and M a solid region in $\mathbf{R}^3$. Then, $\int_{\partial M} f\,dydz + g\,dzdx + h\,dxdy = \int_M \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dxdydz.$
This is ***Gauss' Divergence Theorem***, which can be written in vector form:
Let $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$ be a vector field in $\mathbf{R}^3$ and define the ***divergence,***
$\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$ Then $\int_{\partial M} \mathbf{F} \cdot \mathbf{n}\,d\sigma = \int_M \nabla \cdot \mathbf{F}\,dV.$
It says that we can find the ***flux*** of the vector field, $\int_{\partial M} \mathbf{F} \cdot \mathbf{n}\,d\sigma,$ across the boundary by computing the volume integral of the divergence of the vector field.
4. $\omega = f dx + g dy + h\,dz$ is a 1-form in $\mathbf{R}^3$ and M a region in $\mathbf{R}^3$ bounded by a closed curve. Then,
$$\int_{\partial M} f\,dx + g\,dy + h\,dz = \int_M \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy.$$
This is ***Stokes' Theorem***, which can be written in vector form:
Let $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$ be a vector field in $\mathbf{R}^3$ and define the ***curl*** by
$\nabla \times \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\mathbf{k}.$ Then $\int_{\partial M} \mathbf{F} \cdot d\mathbf{r} = \int_M \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma.$
5. The path integral is called the ***circulation integral*** when **F** describes a flow. If $\nabla \times \mathbf{F} = 0,$ then the flow is called ***irrotational***. $\int_{\partial M} \mathbf{F} \cdot d\mathbf{r}$ is the ***work integral*** when **F** is a force field. If $\nabla \times \mathbf{F} = 0,$ then the field is called a ***conservative*** field. In both cases one says that the path integral over any curve is ***path independent***.

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Poincaré's Lemma. If ω is a differential form that is differentiable and <u>closed</u> in a simply connected region, *R*, then ω is <u>exact</u> in *R*.

Examples:

1. ω is a continuously differentiable 1-form in $\mathbf{R}^2$ and *M* a bounded, simply connected region in $\mathbf{R}^2$, bounded by a closed curve. Let $\omega = f\,dx + g\,dy = dF$ be ***exact***. Then $\oint_{\partial M} f\,dx + g\,dy = \oint_{\partial M} \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0,$ since the starting and ending points are the same around a closed curve.
   Note: for $\frac{\partial F}{\partial x} = f$ and $\frac{\partial F}{\partial y} = g,$ then $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right).$ Thus, $\boxed{\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}}$, or, $\oint_{\partial M} \omega = 0,$ and for any curve *C* in *M* the value of $\int_C \omega$ is ***path independent***.
2. ω is a continuously differentiable 2-form in $\mathbf{R}^3$ and *M* a simply connected, bounded region in $\mathbf{R}^3$. Let $\omega = f\,dydz + g\,dzdx + h\,dxdy$ and $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}.$ If $\nabla\cdot\mathbf{F} = 0,$ then **F** is called *incompressible*. In this case the net flow out of *M* equals the net flow into *M*. If the field is a magnetic field, it is called ***solenoidal***. Poincaré's Lemma then tells us that $\oint_{\partial M} \omega = 0,$ and for surfaces S with a common boundary such that the orientation on the boundary is the same, one obtains the same value for the surface integral $\int_S \omega$.
3. If $\omega = f dx + g dy + h\,dz$ is a continuously differentiable 1-form in $\mathbf{R}^3$ that is ***closed*** in a simply ***loop connected*** region, then ω is ***exact*** in that region. This tells us that for the continuously differentiable, ***irrotational*** ($\nabla\times\mathbf{F} = 0$), vector field $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\mathbf{k}$, there exists a scalar ***potential field,*** ϕ, such that $\mathbf{F} = \nabla\phi$ in that region. Thus, $\oint_{\partial M} \omega = 0,$ and for any curve *C* in *M* the value of $\int_C \omega$ ***is path independent***.
   In fact, we have the **Fundamental Theorem for Line Integrals**:
   $$\boxed{\int_C \nabla\phi\cdot d\mathbf{r} = \int_C \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int_C d\phi = \phi(\mathbf{b}) - \phi(\mathbf{a}).}$$
   Thus, the work done by a conservative force along a path is just the difference in potential energy between the endpoints of the paths.
4. **Operator results:**
   $\nabla\times(\nabla f) = 0 \leftrightarrow d(d\omega) = 0$ for 0-forms
   $\nabla\cdot(\nabla\times\mathbf{G}) = 0 \leftrightarrow d(d\omega) = 0$ for 1-forms
5. **Poincaré's Lemma – Special Cases:** In a simply connected region,
   $\nabla\times\mathbf{F} = 0 \Rightarrow \exists\phi \ni \mathbf{F} = \nabla\phi,$ or $d\omega = 0 \Rightarrow \exists\phi \ni \omega = d\phi$, ω a 1-form
   $\nabla\cdot\mathbf{F} = 0 \Rightarrow \exists\mathbf{G} \ni \mathbf{F} = \nabla\times\mathbf{G},$ or $d\omega = 0 \Rightarrow \exists\nu \ni \omega = d\nu$, ω a 2-form