4

Transfer Functions and State Space Blocks

4.1 State Space Formulation

There are other more elegant approaches to solving a differential equation in Simullink. Take for example the differential equation for a forced, damped harmonic oscillator,

$$mx'' + bx' + kx = u(t). \tag{4.1}$$

Note that we changed the driving force to $u(t)$.

Defining $x_1 = x'$ and $x_2 = x'$, this second order differential equation can be written as a system of two first order differential equations,

$$\begin{aligned} x_1' &= -\frac{b}{m}x_1 - \frac{k}{m}x_2 + \frac{1}{m}u(t) \\ x_2' &= x_1 \end{aligned} \tag{4.2}$$

Note that $x_2'' = x_1'$ gives the second order equation with $x = x_2$. Also, this is not the typical order of equations usually encountered when studying systems of differential equations. This order is chosen to be consistent with the **State Space Block** which we will use later.

This system can be written in matrix form: $\mathbf{x}' = A\mathbf{x} + B\mathbf{u}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$A = \begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}.$$

We now think of $\mathbf{x}' = A\mathbf{x} + B\mathbf{u}$, as a system whose input is given by the forcing term $u(t)$ and we need to integrate the right hand side for a given input function. The output of this system is the solution vector, $\mathbf{x}$. Also, we might want to output a plot of the forcing function. Thus, the complete output from the system can be written as

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u},$$

where C is a row vector and $D = 0$ or 1. In particular, we might only want to output the solution component x_2. So, we let $C = [0, 1]$ and $D = 0$. The block diagram for this process is shown in Figure 4.1.

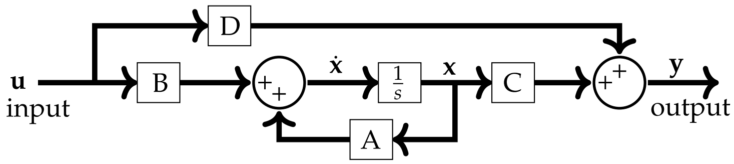

Figure 4.1: State space representation of the system $\mathbf{x}' = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$,

The whole process is captured in the **State Space Block**. This block is found in the Continuous group. The implementation of this system with a sinusoidal forcing term is depicted in Figure 4.2. This shows the pair of equations

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}. \end{aligned} \tag{4.3}$$

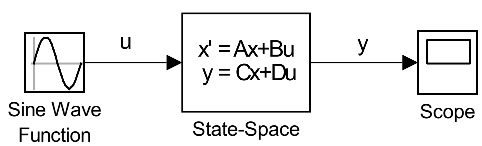

Figure 4.2: The use of the**State Space Block** displaying a **Sine Wave** input and output to a **Scope**.

As an example, we consider the case where $m = 2$ kg, $b = 0.2$ kg/s, and $k = 1.0$ N/m. We also let $u(t) = \sin t$. Then, we have

$$A = \begin{bmatrix} -0.1 & -0.5 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

$C = [0, 1]$, and $D = 0$. These values are put into the block by going into the **Function Block Parameters** dialog box for the **State Space** block as shown in Figure 4.3. Note that there is a place to enter the initial condition, such as $[x_1(0), x_2(0)]^T = [0, 1]^T$. In this case one would type **[0; 1]**. A comparison of outputs from using this initial condition to zero initial conditions is shown in Figure 4.4. Figure 4.5 shows the system needed to produce this plot.

4.2 Transfer Functions

ANOTHER METHOD FOR SOLVING THE DIFFERENTIAL EQUATION compactly is to use the **Transfer Fcn** block. This is shown in Figure 4.6. One needs to enter the transfer function numerator and denominator in the **Function Block Parameter** box, shown in Figure 4.7. Essentially, this is

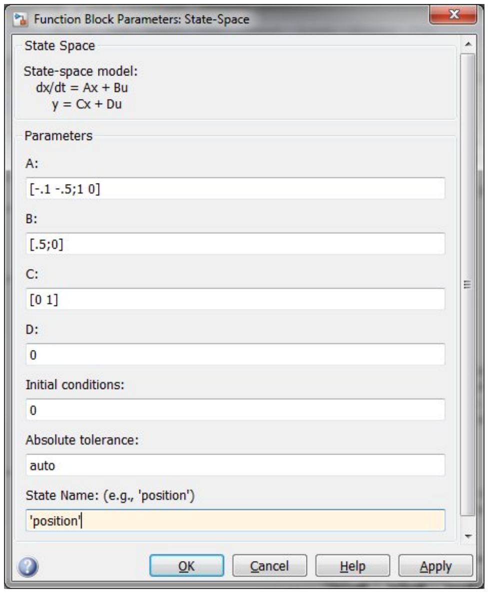

Figure 4.3: State space block parameters.

recognized as the Laplace transform of the differential equation with zero initial conditions.

Recall, the Laplace transform of a function $f(t)$ is defined as

$$X(s) = \mathcal{L}[x](s) = \int_0^\infty x(t)e^{-st}\,dt, \quad s > 0. \tag{4.4}$$

Also, we have the properties

$$\begin{aligned} \mathcal{L}\left[\frac{dx}{dt}\right] &= sX(s) - x(0) \\ \mathcal{L}\left[\frac{d^2x}{dt^2}\right] &= s^2X(s) - sx(0) - x'(0). \end{aligned} \tag{4.5}$$

Taking the Lplace transform of the differential equation, $2x'' + .2x' + x = \sin t$, with $x(0) = x'(0) = 0$,

$$(2s^2 + .2s + 1)X(s) = \frac{1}{s^2+1}.$$

Solving for $X(s)$,

$$X(s) = \frac{1}{2s^2 + .2s + 1}\mathcal{L}[u(t)].$$

Therefore, the transfer function comes from the factor multiplying $\mathcal{L}[u(t)]$. In this case, one enters the coefficients of the second order differential equation into the denominator as **[2 .2 1]**. Unfortunately, one can only solve problems with zero initial conditions.

If one knows the transfer function, then one can use it to create an equivalent **State Space block**. This is done using the MATLAB function **tf2ss(1,[2,.2,1])**. We note that this produces the parameters A, B, C, D, but what it gives is

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

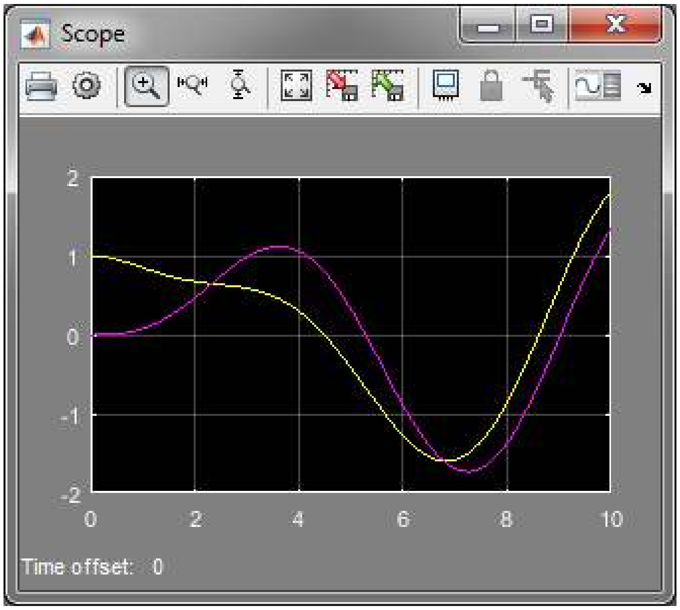

Figure 4.4: Solution to the forced, damped harmonic oscillator problem with initial conditions set to 0 or [0;1].

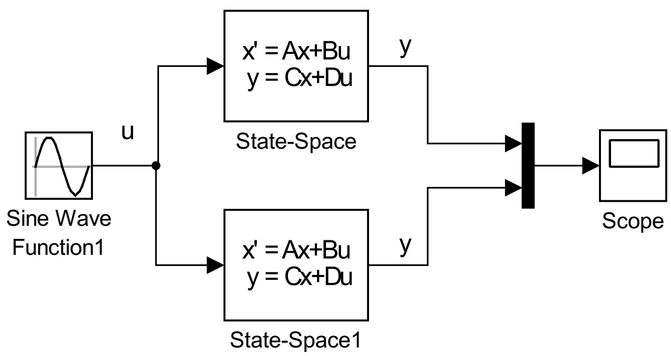

Figure 4.5: The use of the**State Space Block** dispaying a Sine Wave input and output to a **Scope**. The **Mux** block (from Signal Routing) is used to feed solutions from two systems using different initial conditions.

and $C = [0, 0.5]$, This differs from what we derived above. The difference lies in the fact that we can multiply B by any constant and divide C by that constant and not affect the solution of the problem. In this case, the constant in question is the mass, $m = 2.0$.

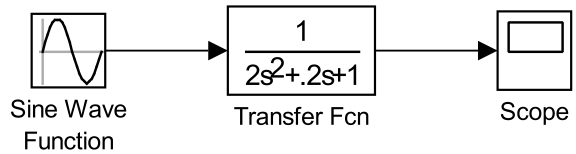

Figure 4.6: The use of the **Transfer Fcn** Block with a **Sine Wave** input and output to a **Scope**.

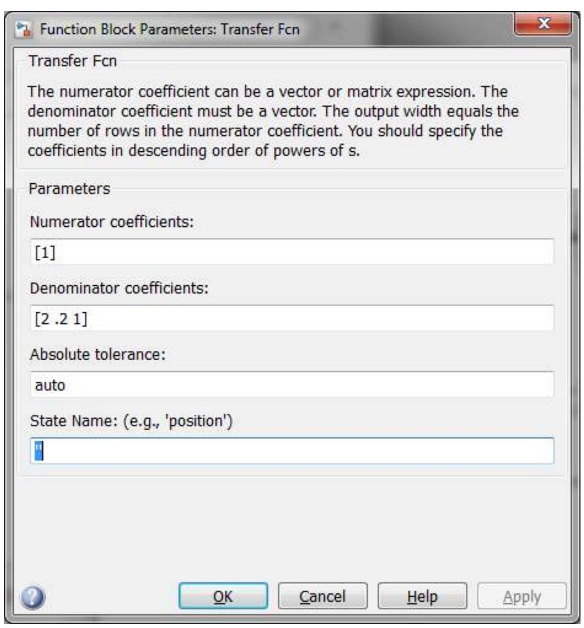

Figure 4.7: Block parameter display for the Transfer Fcn block.