Differential Equations Review II

I. Nonhomogeneous Equations: $a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x)$

a. Get Homogeneous Solution

i. Constant Coefficient Equations: $ay''+by'+cy=0$

Solutions - $y(x)=e^{rx}, ar^2+br+c=0.$

1. Two, real distinct solutions $y=c_1e^{r_1x}+c_2e^{r_2x}$
2. One real solution $y=(c_1+c_2x)e^{rx}$
3. Two complex conjugate solutions
   $y=(c_1\cos\beta x+c_2\sin\beta x)e^{\alpha x}$

ii. Cauchy-Euler Equations: $ax^2y''+bxy'+cy=0$

Solutions - $y(x)=x^r, ar(r-1)+br+c=0.$

1. Two, real distinct solutions $y=c_1x^{r_1}+c_2x^{r_2}$
2. One real solution $y=(c_1+c_2\ln|x|)x^r$
3. Two complex conjugate solutions
   $y=[c_1\cos(\beta\ln|x|)+c_2\sin(\beta\ln|x|)]x^{\alpha}$

b. Get Particular Solution

i. Method of Undetermined Coefficients

$f(x)$	$y_p(x)$
$P_n(x)=a_nx^n+\cdots+a_1x+a_0$	$Ax^n+\cdots+Bx+C$
$P_n(x)e^{ax}$	$(Ax^n+\cdots+Bx+C)e^{ax}$
$(P_n(x)\cos bx+Q_n(x)\sin bx)e^{ax}$	$[(Ax^n+\ldots)\cos bx+(Bx^n+\ldots)\sin bx]e^{ax}$

ii. Modified Method of Undetermined Coefficients

If any term of guess is a solution of the homogeneous equation, then multiply the guess by x repeatedly until no resulting term is a solution of the homogeneous equation.

iii. Reduction of Order

Get one linearly independent solution of the homogeneous problem, $y_1(x)$, and form $y(x)=v(x)y_1(x)$. The function $z(x)=v'(x)$ will satisfy a linear first order differential equation.

iv. Method of Variation of Parameters

1. Determine Two Linearly Independent Solutions of Homogeneous Equation, $y_1(x), y_2(x)$
2. Solve System for c's and integrate

   a. $c_1'y_1+c_2'y_2=0,\ c_1'y_1'+c_2'y_2'=\dfrac{f(x)}{a(x)}$

b. or, $c_1 = -\int \frac{fy_2}{aW}\,dx,\ c_2 = \int \frac{fy_1}{aW}\,dx,\ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

c. Guessing method also applies to First Order! $a(x)y' + b(x)y = f(x)$

II. Oscillations - $x = x(t)$

a. Simple Harmonic Motion $x'' + \omega^2 x = 0,\ \omega = \sqrt{\frac{k}{m}}$

b. Damped Harmonic Motion $mx'' + bx' + kx = 0.$

Recognize solution behavior
underdamped, critically damped, overdamped.

c. Forced, Damped, Harmonic Motion $mx'' + bx' + kx = F_0 \sin(\omega t + \phi).$

III. Euler's Method – Numerical Approximation

Solve $\frac{dy}{dx} = f(x, y),\ y(a) = y_0,\ a \le x \le b,$ using

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}),\ n = 1, \ldots, N$

$x_n = a + nh,\ y_n \approx f(x_n),\ h = \Delta x = \frac{b-a}{N}$

IV. Direction fields – Sketch fields and solutions for $y' = f(x, y).$

V. Power Series

a. Classification of Points

i. Ordinary Point
ii. Regular Singular Point
iii. Irregular Singular Point

b. Taylor Series Approximation from IVP

Use ODE and IC to get coefficients in $y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$

c. Power Series Method - $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$

i. Re-indexing
ii. Recursion relation

d. Frobenius Method - $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$

i. Indicial Equation