Chapter 2

Second Order Differential Equations

"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)

2.1 Introduction

In the last section we saw how second order differential equations naturally appear in the derivations for simple oscillating systems. In this section we will look at more general second order linear differential equations.

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.1}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then, Equation (2.1) becomes

$$Ly = f. \tag{2.2}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

a. $L(f+g) = Lf + Lg$
b. $L(af) = aLf$.

One typically solves (2.1) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

[1] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

Then, the general solution of (2.1) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{2.3}$$

There are methods for finding a particular solution of a nonhomogeneous differential equation. These methods range from pure guessing, the Method of Undetermined Coefficients, the Method of Variation of Parameters, or Green's functions. We will review these methods later in the chapter.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, many now famous mathematicians and physicists have studied a variety of second order linear equations and they have saved us the trouble of finding solutions to the differential equations that often appear in applications. We will encounter many of these in the following chapters. We will first begin with some simple homogeneous linear differential equations.

Linearity is also useful in producing the general solution of a homogeneous linear differential equation. If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, then the *linear combination* $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of the homogeneous equation. This is easily proven.

Let $Ly_1 = 0$ and $Ly_12 = 0$. We consider $y = c_1 y_1 + c_2 y_2$. Then, since L is a linear operator,

$$\begin{aligned} Ly &= L(c_1 y_1 + c_2 y_2) \\ &= c_1 Ly_1 + c_2 Ly_2 \\ &= 0. \end{aligned} \tag{2.4}$$

Therefore, y is a solution.

In fact, if $y_1(x)$ and $y_2(x)$ are *linearly independent*, then $y = c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous problem. A set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1 y_1(x) + \ldots + c_n y_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$. Otherwise, they are said to be linearly dependent. Note that for $n = 2$, the general form is $c_1 y_1(x) + c_2 y_2(x) = 0$. If y_1 and y_2 are linearly dependent, then the coefficients are not zero and $y_2(x) = -\frac{c_1}{c_2} y_1(x)$ and is a multiple of $y_1(x)$. We see this in the next example.

Example 2.1. Show that $y_1(x) = x$ and $y_2(x) = 4x$ are linearly dependent.

We set $c_1 y_1(x) + c_2 y_2(x) = 0$ and show that there are nonzero constants, c_1 and c_2 satisfying this equation. Namely, let

$$c_1 x + c_2(4x) = 0.$$

Then, for $c_1 = -4c_2$, this is true for any nonzero c_2. Let $c_2 = 1$ and we have $c1 = -4$.

Next we consider two functions that are not constant multiples of each other.

Example 2.2. Show that $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent.

We set $c_1 y_1(x) + c_2 y_2(x) = 0$ and show that it can only be true if $c_1 = 0$ and $c_2 = 0$. Let

$$c_1 x + c_2 x^2 = 0,$$

for all x. Differentiating, we have two sets of equations that must be true for all x :

$$\begin{aligned} c_1 x + c_2 x^2 &= 0, \\ c_1 + 2c_2 x &= 0. \end{aligned} \tag{2.5}$$

Setting $x = 0$, we get $c_1 = 0$. Setting $x = 1$, then $c_1 + c_2 = 0$. Thus, $c_2 = 0$.

Another approach would be to solve for the constants. Multiplying the second equation by x and subtracting yields $c_2 = 0$. Substituting this result into the second equation, we find $c_1 = 0$.

For second order differential equations we seek two linearly independent functions, $y_1(x)$ and $y_2(x)$. As in the last example, we set $c_1 y_1(x) + c_2 y_2(x) = 0$ and show that it can only be true if $c_1 = 0$ and $c_2 = 0$. Differentiating, we have

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) &= 0, \\ c_1 y_1'(x) + c_2 y_2'(x) &= 0. \end{aligned} \tag{2.6}$$

These must hold for all x in the domain of the solutions.

Now we solve for the constants. Multiplying the first equation by $y_1'(x)$ and the second equation by $y_2(x)$, we have

$$\begin{aligned} c_1 y_1(x) y_2'(x) + c_2 y_2(x) y_2'(x) &= 0, \\ c_1 y_1'(x) y_2(x) + c_2 y_2'(x) y_2(x) &= 0. \end{aligned} \tag{2.7}$$

Subtracting gives

$$\left[y_1(x) y_2'(x) - y_1'(x) y_2(x) \right] c_1 = 0.$$

Therefore, either $c_1 = 0$ or $y_1(x) y_2'(x) - y_1'(x) y_2(x) = 0$. So, if the latter is true, then $c_1 = 0$ and therefore, $c_2 = 0$. This gives a condition for which $y_1(x)$ and $y_2(x)$ are linearly independent:

$$y_1(x) y_2'(x) - y_1'(x) y_2(x) = 0. \tag{2.8}$$

We define this quantity as the Wronskian of $y_1(x)$ and $y_2(x)$.

Linear independence of the solutions of a differential equation can be established by looking at the Wronskian of the solutions. For a second order differential equation the Wronskian is defined as

$$W(y_1, y_2) = y_1(x) y_2'(x) - y_1'(x) y_2(x).$$

The Wronskian can be written as a determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) y_2'(x) - y_1'(x) y_2(x).$$

Thus, the definition of a Wronskian can be generalized to a set of n functions $\{y_i(x)\}_{i=1}^n$ using an $n \times n$ determinant.

Example 2.3. Determine if the set of functions $\{1, x, x^2\}$ are linearly independent.

We compute the Wronskian.

$$\begin{aligned} W(y_1, y_2, y_3) &= \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix} \\ &= \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \\ &= 2. \end{aligned} \tag{2.9}$$

Since, $W(1, x, x^2) = 2 \neq 0$, then the set $\{1, x, x^2\}$ is linearly independent.

2.2 *Constant Coefficient Equations*

The simplest second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{2.10}$$

where a, b, and c are constants.

Solutions to (2.10) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (2.10) leads to the characteristic equation

$$ar^2 + br + c = 0. \tag{2.11}$$

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y(x) = re^{rx}$, and $y(x) = r^2e^{rx}$. Inserting into (2.10), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Two real, distinct roots, r_1 and r_2, give solutions of the form

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

The roots of this equation, r_1, r_2, in turn lead to three types of solutions depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

Example 2.4. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1e^{-2x} + c_2e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1e^{-2x} + 3c_2e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{2.12}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2x)e^{rx}.$$

In the case when there is a repeated real root, one has only one solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second linearly independent solution? Since the solutions should be independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. (This process is called the Method of Reduction of Order. See Section 2.2.1)

For constant coefficient second order equations, we can write the equation as

$$(D - r)^2y = 0,$$

where $D = \frac{d}{dx}$. We now insert $y_2(x) = v(x)e^{rx}$ into this equation. First we compute

For more on the Method of Reduction of Order, see Section 2.2.1.

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$0 = (D - r)^2ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$. The general solution is then

$$y(x) = c_1e^{rx} + c_2xe^{rx}.$$

Example 2.5. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. From the above discussion, we easily find the solution $y(x) = (c_1 + c_2x)e^{-3x}$.

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula[2], which is treated in Section A.36.

$$e^{i\beta x} = \cos\beta x + i \sin\beta x. \tag{2.13}$$

[2] Euler's Formula is found using Maclaurin series expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots.$$

Let $x = i\theta$ and find

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots. \\ &= 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots \\ & \quad i\left[\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right]. \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

Then, the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{2.14}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Example 2.6. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$, and the general solution consists purely of sinusoidal functions, $y(x) = c_1\cos(2x) + c_2\sin(2x)$, since $\alpha = 0$ and $\beta = 2$.

Example 2.7. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]e^{-x}.$$

Example 2.8. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 2.6. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A\sin x$ and determine what A needs to be. Inserting this guess into the differential equation gives $(-A + 4A)\sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x$.

The three cases for constant coefficient linear second order differential equations are summarized below.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$.

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$.

As we will see, one of the most important applications of such equations is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$m\ddot{x} + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are simple sine and cosine functions, leading to simple harmonic motion.

2.2.1 *Reduction of Order*

We have seen the the Method of Reduction of Order was useful in obtaining a second solution of a second order differential equation with constant coefficients when one solution was known. It can also be used to solve other second order differential equations. First, we review the method by example.

Example 2.9. Verify that $y_1(x) = xe^{2x}$ is a solution of $y'' - 4y' + 4y = 0$ and use the Method of Reduction of Order to find a second linearly independent solution.

We note that

$$\begin{aligned} y_1'(x) &= (1+2x)e^{2x}, \\ y_1''(x) &= [2 + 2(1+2x)]e^{2x} = (4+4x)e^{2x}, \end{aligned}$$

Substituting the $y_1(x)$ and its derivatives into the differential equation, we have

$$\begin{aligned} y_1'' - 4y_1' + 4y_1 &= (4+4x)e^{2x} - 4(1+2x)e^{2x} + 4xe^{2x} \\ &= 0. \end{aligned} \quad (2.15)$$

In order to find a second linearly independent solution, $y_2(x)$, we need a solution that is not a constant multiple of $y_1(x)$. So, we guess the form $y_2(x) = v(x)y_1(x)$. For this example, the function and its derivatives are given by

$$\begin{aligned} y_2 &= vy_1. \\ y_2' &= (vy_1)', \\ &= v'y_1 + vy_1'. \\ y_2'' &= (v'y_1 + vy_1')', \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substituting y_2 and its derivatives into the differential equation, we have

$$\begin{aligned} 0 &= y_2'' - 4y_2' + 4y_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') - 4(v'y_1 + vy_1') + 4vy_1 \\ &= v''y_1 + 2v'y_1' - 4v'y_1 + v[y_1'' - 4y_1' + 4y_1] \\ &= v''y_1 + 2v'y_1' - 4v'y_1 \\ &= v''xe^{2x} + 2v'(1+2x)e^{2x} - 4v'xe^{2x} \\ &= [v''x + 2v']e^{2x}. \end{aligned} \quad (2.16)$$

Therefore, $v(x)$ satisfies the equation

$$v''x + 2v' = 0.$$

This is a first order equation for $v'(x)$, which can be seen by introducing $z(x) = v'(x)$, leading to the separable first order equation

$$x\frac{dz}{dx} = -2z.$$

This is readily solved to find $z(x) = \frac{A}{x^2}$. This gives

$$z = \frac{dv}{dx} = \frac{A}{x^2}.$$

Further integration leads to

$$v(x) = -\frac{A}{x} + C.$$

This gives

$$\begin{aligned} y_2(x) &= \left(-\frac{A}{x} + C\right)xe^{2x} \\ &= -Ae^{2x} + Cxe^{2x}. \end{aligned}$$

Note that the second term is the original $y_1(x)$, so we do not need this term and can set $C = 0$. Since the second linearly independent solution can be determined up to a multiplicative constant, we can set $A = -1$ to obtain the answer $y_2(x) = e^{2x}$. Note that this argument for obtaining the simple form is reason enough to ignore the integration constants when employing the Method of Reduction of Order.

For an example without constant coefficients, consider the following example.

Example 2.10. Verify that $y_1(x) = x$ is a solution of $x^2y'' - 4xy' + 4y = 0$ and use the Method of Reduction of Order to find a second linearly independent solution.

Substituting the $y_1(x) = x$ and its derivatives into the differential equation, we have

$$\begin{aligned} x^2y_1'' - 4xy_1' + 4y_1 &= 0 - 4x + 4x \\ &= 0. \end{aligned} \tag{2.17}$$

In order to find a second linearly independent solution, $y_2(x)$, we need a solution that is not a constant multiple of $y_1(x)$. So, we guess the form $y_2(x) = v(x)y_1(x)$. For this example, the function and its derivatives are given by

$$\begin{aligned} y_2 &= xv. \\ y_2' &= (xv)', \\ &= v + xv'. \\ y_2'' &= (v + xv')', \\ &= 2v' + xv''. \end{aligned}$$

Substituting $y_2 = xv(x)$ and its derivatives into the differential equation, we have

$$\begin{aligned} 0 &= x^2y_2'' - 4xy_2' + 4y_2 \\ &= x^2(2v' + xv'') - 4x(v + xv') + 4xv \\ &= x^3v'' - 2x^2v'. \end{aligned} \tag{2.18}$$

Note how the v-terms cancel, leaving

$$xv'' = 2v'.$$

This equation is solved by introducing $z(x) = v'(x)$. Then, the equation becomes

$$x\frac{dz}{dx} = 2z.$$

Using separation of variables, we have

$$z = \frac{dv}{dx} = Ax^2.$$

Integrating, we obtain

$$v = \frac{1}{3}Ax^3 + B.$$

This leads to the second solution in the form

$$y_2(x) = x\left(\frac{1}{3}Ax^3 + B\right) = \frac{1}{3}Ax^4 + Bx.$$

Since the general solution is

$$y(x) = c_1 x + c_2\left(\frac{1}{3}Ax^4 + Bx\right),$$

we see that we can choose $B = 0$ and $A = 3$ to obtian the general solution as

$$y(x) = c_1 x + c_2 x^4.$$

Therefore, we typically do not need the arbitrary constants found in using reduction of order and simply report that $y_2(x) = x^4$.

2.3 *Simple Harmonic Oscillators*

The next physical problem of interest is that of simple harmonic motion. Such motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. This is the beginning of a major thread running throughout this course. You have seen simple harmonic motion in your introductory physics class. We will review SHM (or SHO in some texts) by looking at springs, pendula (the plural of pendulum), and simple circuits.

2.3.1 *Mass-Spring Systems*

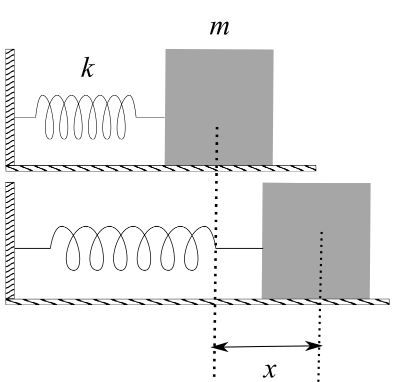

Figure 2.1: Spring-Mass system.

We begin with the case of a single block on a spring as shown in Figure 2.1. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0. \tag{2.19}$$

Dividing by the mass, this equation can be written in the form

$$\ddot{x} + \omega^2 x = 0, \tag{2.20}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

This is the generic differential equation for simple harmonic motion.

We will later derive solutions of such equations in a methodical way. For now we note that two solutions of this equation are given by

$$\begin{aligned} x(t) &= A\cos\omega t, \\ x(t) &= A\sin\omega t, \end{aligned} \tag{2.21}$$

where ω is the angular frequency, measured in rad/s, and A is called the amplitude of the oscillation. .

The angular frequency is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

2.3.2 *The Simple Pendulum*

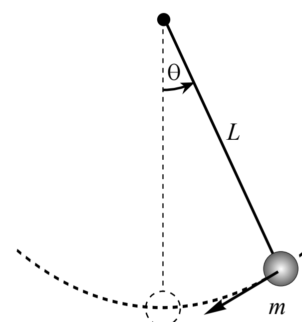

Figure 2.2: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

THE SIMPLE PENDULUM consists of a point mass m hanging on a string of length L from some support. [See Figure 2.2.] One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 2.3 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

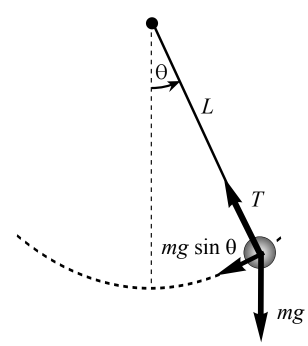

Figure 2.3: There are two forces acting on the mass, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Linear and nonlinear pendulum equation.

Canceling the masses, this then gives us the nonlinear pendulum equation

$$L\ddot{\theta} + g\sin\theta = 0. \tag{2.22}$$

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position ℓ, the torque is given by $\tau = -mg\ell\sin\theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell\sin\theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. For a simple pendulum, we let $\ell = L$ and $I = mL^2$, and obtain $\omega = \sqrt{g/L}$.

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

There are several variations of Equation (2.22) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (2.22) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{2.23}$$

2.3.3 *LRC Circuits*

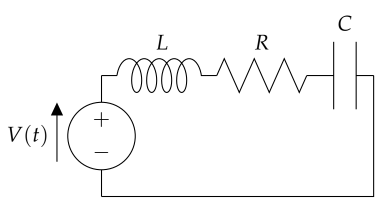

Figure 2.4: Series LRC Circuit.

ANOTHER TYPICAL PROBLEM OFTEN ENCOUNTERED in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.4. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

Furthermore, we need to define the current as $I = \frac{dq}{dt}$. where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.5. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems in the next chapter.

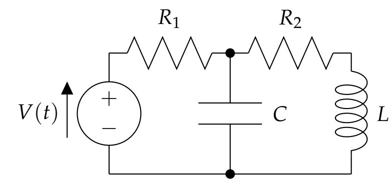

Figure 2.5: Parallel LRC Circuit.

In the following we will look at special cases that arise for the series LRC circuit equation. These include RC circuits, solvable by first order methods and LC circuits, leading to oscillatory behavior.

2.3.4 RC Circuits*

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$) as shown in Figures 2.6 and 2.9.

Charging a capacitor.

For charging a capacitor, we have the initial value problem

$$R\frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \tag{2.24}$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite it and solve it as a separable equation, since V_0 is a constant. We will do the former only as another example of finding the integrating factor.

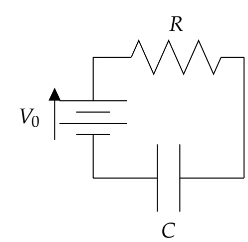

Figure 2.6: RC Circuit for charging.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \tag{2.25}$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt}\left(qe^{t/RC}\right) = \frac{V_0}{R}e^{t/RC}. \tag{2.26}$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R}\int e^{t/RC}\,dt = CV_0e^{t/RC} + K. \tag{2.27}$$

Note that we introduced the integration constant, K. Now divide out the exponential to get the general solution:

$$q = CV_0 + Ke^{-t/RC}. \tag{2.28}$$

(If we had forgotten the K, we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get the particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = CV_0 + K.$$

So, $K = -CV_0$. Inserting this into the solution, we have

$$q(t) = CV_0(1 - e^{-t/RC}). \tag{2.29}$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = CV_0$. This is what we expect, because the current is no longer flowing over R and this just gives the relation between the potential difference across the capacitor plates when a charge of q_0 is established on the plates.

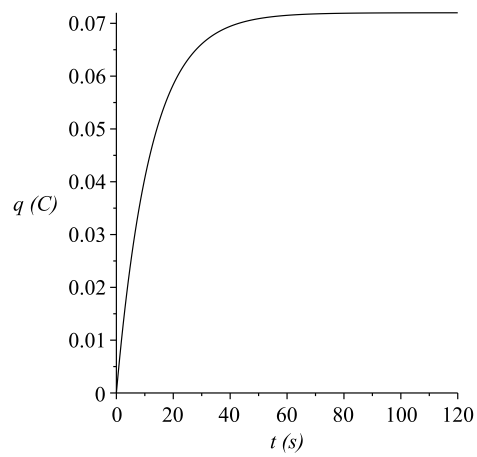

Figure 2.7: The charge as a function of time for a charging capacitor with $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V.

Let's put in some values for the parameters. We let $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V. A plot of the solution is given in Figure 2.7. We see that the charge builds up to the value of $CV_0 = 0.072$ C. If we use a smaller resistance, $R = 200\ \Omega$, we see in Figure 2.8 that the capacitor charges to the same value, but much faster.

Time constant, $\tau = RC$.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays. If we set $t = \tau$, we find that

$$q(\tau) = CV_0(1 - e^{-1}) = (1 - 0.3678794412\ldots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12$s. For the second set, $\tau = 1.2$s.

Discharging a capacitor.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit as shown in Figure 2.9, the charge will then move off the plates, discharging

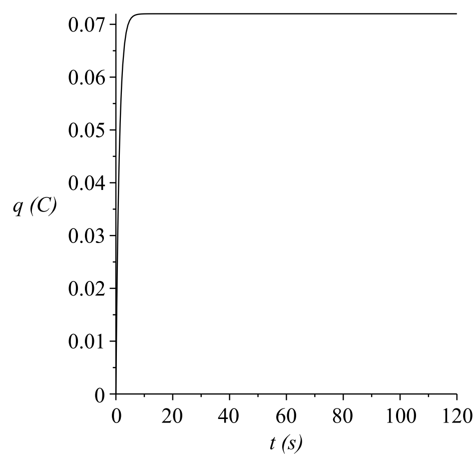

Figure 2.8: The charge as a function of time for a charging capacitor with $R = 200\ \Omega$, $C = 6.00$ mF, and $V_0 = 12$ V.

the capacitor. The relevant form of the initial value problem becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \tag{2.30}$$

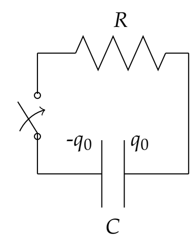

Figure 2.9: RC Circuit for discharging.

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \tag{2.31}$$

This is a simple exponential decay problem, which one can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

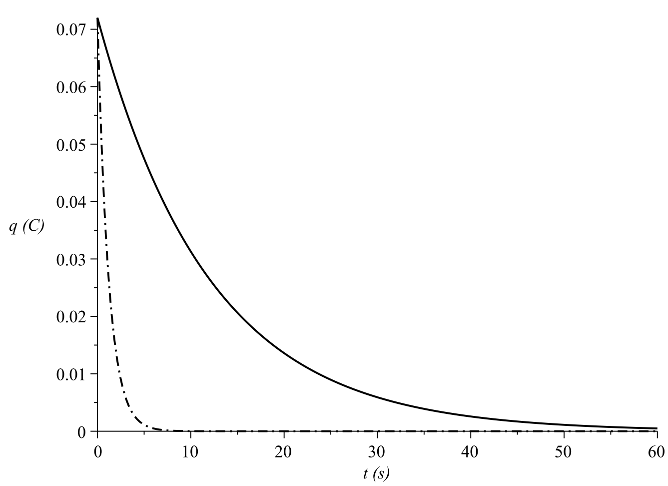

Figure 2.10: The charge as a function of time for a discharging capacitor with $R = 2.00$ kΩ (solid) or $R = 200\ \Omega$ (dashed), and $C = 6.00$ mF, and $q_0 = 0.072$ C.

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.10 we show the discharging of the two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412\ldots)q_0 \approx 0.37q_0.$$

So, at this time the capacitor only has about a third of its original value.

2.3.5 LC Circuits*

LC Oscillators.

Another simple result comes from studying LC circuits. We will now connect a charged capacitor to an inductor as shown in Figure 2.11. In this case, we consider the initial value problem

$$L\ddot{q} + \frac{1}{C}q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \tag{2.32}$$

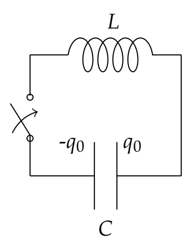

Figure 2.11: An LC circuit.

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC}q = 0. \tag{2.33}$$

This equation is a second order, constant coefficient equation. It is of the same form as the ones for simple harmonic motion of a mass on a spring or the linear pendulum. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm\frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.33) is of the form

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \omega = (LC)^{-1/2}. \tag{2.34}$$

Inserting the initial conditions yields

$$q(t) = q_0 \cos(\omega t). \tag{2.35}$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}. \tag{2.36}$$

This is called the tuning frequency because of its role in tuning circuits.

Example 2.11. Find the resonant frequency for $C = 10\mu$F and $L = 100$mH.

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{(10 \times 10^{-6})(100 \times 10^{-3})}} = 160\text{Hz}.$$

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

2.3.6 Damped Oscillations

AS WE HAVE INDICATED, simple harmonic motion is an ideal situation. In real systems we often have to contend with some energy loss in the system. This leads to the damping of the oscillations. A standard example is a spring-mass-damper system as shown in Figure 2.12 A mass is attached to a spring and a damper is added which can absorb some of the energy of the oscillations. the damping is modeled with a term proportional to the velocity.

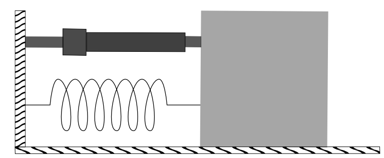

Figure 2.12: A spring-mass-damper system has a damper added which can absorb some of the energy of the oscillations and is modeled with a term proportional to the velocity.

There are other models for oscillations in which energy loss could be in the spring, in the way a pendulum is attached to its support, or in the resistance to the flow of current in an LC circuit. The simplest models of resistance are the addition of a term proportional to first derivative of the dependent variable. Thus, our three main examples with damping added look like:

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{2.37}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.38}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \tag{2.39}$$

These are all examples of the general constant coefficient equation

$$ay''(x) + by'(x) + cy(x) = 0. \tag{2.40}$$

We have seen that solutions are obtained by looking at the characteristic equation $ar^2 + br + c = 0$. This leads to three different behaviors depending on the discriminant in the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.41}$$

We will consider the example of the damped spring. Then we have

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \tag{2.42}$$

For $b > 0$, there are three types of damping.

Damped oscillator cases: Overdamped, critically damped, and underdamped.

I. Overdamped, $b^2 > 4mk$

In this case we obtain two real root. Since this is Case I for constant coefficient equations, we have that

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We note that $b^2 - 4mk < b^2$. Thus, the roots are both negative. So, both terms in the solution exponentially decay. The damping is so strong that there is no oscillation in the system.

II. Critically Damped, $b^2 = 4mk$

In this case we obtain one real root. This is Case II for constant coefficient equations and the solution is given by

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where $r = -b/2m$. Once again, the solution decays exponentially. The damping is just strong enough to hinder any oscillation. If it were any weaker the discriminant would be negative and we would need the third case.

III. Underdamped, $b^2 < 4mk$

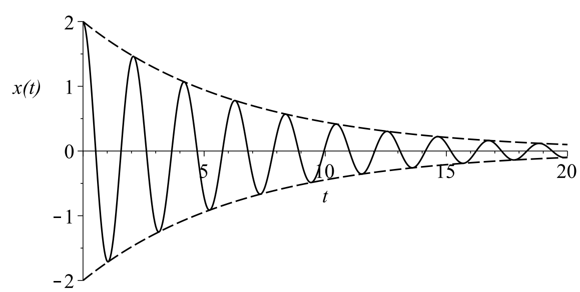

Figure 2.13: A plot of underdamped oscillation given by $x(t) = 2e^{0.15t}\cos 3t$. The dashed lines are given by $x(t) = \pm 2e^{0.15t}$, indicating the bounds on the amplitude of the motion.

In this case we have complex conjugate roots. We can write $\alpha = -b/2m$ and $\beta = \sqrt{4mk - b^2}/2m$. Then the solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

These solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due the overall factor of $e^{\alpha t}$ when $\alpha < 0$. Consider the case that the initial conditions give $c_1 = A$ and $c_2 = 0$. (When is this?) Then, the solution, $x(t) = Ae^{\alpha t} \cos \beta t$, looks like the plot in Figure 2.13.

2.4 *Forced Systems*

All of the systems presented at the beginning of the last section exhibit the same general behavior when a damping term is present. An additional term can be added that might cause even more complicated behavior. In the case of LRC circuits, we have seen that the voltage source makes the system nonhomogeneous. It provides what is called a source term. Such terms can also arise in the mass-spring and pendulum systems. One can drive such systems by periodically pushing the mass, or having the entire system moved, or impacted by an outside force. Such systems are called forced, or driven.

Typical systems in physics can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.43}$$

As noted in Section 2.1, one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (2.1) is simply given as $y = y_h + y_p$.

So far, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous term to such equations we will need to find the particular solution to the nonhomogeneous equation.

We could guess a solution, but that is not usually possible without a little bit of experience. So, we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to the Method of Variation of Parameters and we will also introduce the powerful machinery of Green's functions later in this section.

2.4.1 *Method of Undetermined Coefficients*

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 2.12. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{2.44}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{2.45}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (2.44). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (2.44), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (2.44). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example 2.13. What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{2.46}$$

The only thing that would change is the particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (2.46), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (2.46)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{2.47}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{2.48}$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 2.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

$f(x)$	Guess
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Table 2.1: Forms used in the Method of Undetermined Coefficients.

Example 2.14. Solve

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{2.49}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$ and differentiate this guess to obtain the derivatives $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$.

Inserting these derivatives into the differential equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we have

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$. Thus, the solution to the problem is

$$y(x) = \left(2 - \frac{1}{2}x\right)e^{-3x}.$$

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

2.4.2 *Periodically Forced Oscillations*

A SPECIAL TYPE OF FORCING is periodic forcing. Realistic oscillations will dampen and eventually stop if left unattended. For example, mechanical clocks are driven by compound or torsional pendula and electric oscillators are often designed with the need to continue for long periods of time. However, they are not perpetual motion machines and will need a periodic injection of energy. This can be done systematically by adding periodic forcing. Another simple example is the motion of a child on a swing in the park. This simple damped pendulum system will naturally slow down to equilibrium (stopped) if left alone. However, if the child pumps energy into the swing at the right time, or if an adult pushes the child at the right time, then the amplitude of the swing can be increased.

There are other systems, such as airplane wings and long bridge spans, in which external driving forces might cause damage to the system. A well know example is the wind induced collapse of the Tacoma Narrows Bridge due to strong winds. Of course, if one is not careful, the child in the last example might get too much energy pumped into the system causing a similar failure of the desired motion.

The Tacoma Narrows Bridge opened in Washington State (U.S.) in mid 1940. However, in November of the same year the winds excited a transverse mode of vibration, which eventually (in a few hours) lead to large amplitude oscillations and then collapse.

While there are many types of forced systems, and some fairly complicated, we can easily get to the basic characteristics of forced oscillations by modifying the mass-spring system by adding an external, time-dependent, driving force. Such as system satisfies the equation

$$m\ddot{x} + b\dot{(x)} + kx = F(t), \tag{2.50}$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. Since the systems we have considered so far are similar, one could easily apply the following to pendula or circuits.

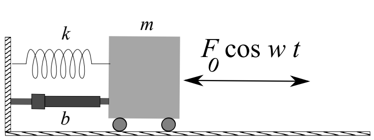

Figure 2.14: An external driving force is added to the spring-mass-damper system.

As the damping term only complicates the solution, we will consider the simpler case of undamped motion and assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$ in order to study periodic forcing. This leads to the simple periodically driven mass on a spring system

$$m\ddot{x} + kx = F_0 \cos \omega t. \tag{2.51}$$

In order to find the general solution, we first obtain the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Next, we seek a particular solution to the nonhomogeneous problem. We will apply the Method of Undetermined Coefficients.

A natural guess for the particular solution would be to use $x_p = A \cos \omega t + B\ sin \omega t$. However, recall that the guess should not be a solution of the homogeneous problem. Comparing x_p with x_h, this would hold if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. So, we have two cases to consider.

Dividing through by the mass, we solve the simple driven system,

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

Example 2.15. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$, for $\omega \neq \omega_0$.

In this case we continue with the guess $x_p = A \cos \omega t + B\ sin \omega t$. Since there is no damping term, one quickly finds that $B = 0$. Inserting $x_p = A \cos \omega t$ into the differential equation, we find that

$$\left(-\omega^2 + \omega_0^2\right) A \cos \omega t = \frac{F_0}{m} \cos \omega t.$$

Solving for A, we obtain

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}.$$

The general solution for this case is thus,

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \tag{2.52}$$

Example 2.16. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t$.

In this case, we need to employ the Modified Method of Undetermined Coefficients. So, we make the guess $x_p = t\left(A \cos \omega_0 t + B\ sin \omega_0 t\right)$. Since there is no damping term, one finds that $A = 0$. Inserting the guess in to the differential equation, we find that

$$B = \frac{F_0}{2m\omega_0},$$

or the general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega} t \sin \omega t. \tag{2.53}$$

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases} \tag{2.54}$$

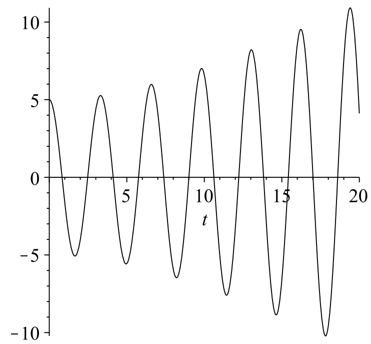

Figure 2.15: Plot of

$$x(t) = 5\cos 2t + \frac{1}{2} t \sin 2t,$$

a solution of $\ddot{x} + 4x = 2\cos 2t$ showing resonance.

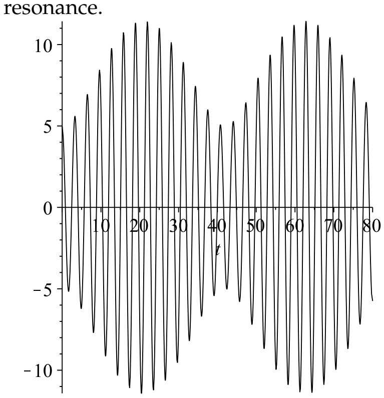

Figure 2.16: Plot of

$$x(t) = \frac{1}{249}\left(2045\cos 2t - 800\cos\frac{43}{20}t\right),$$

a solution of $\ddot{x} + 4x = 2\cos 2.15t$, showing beats.

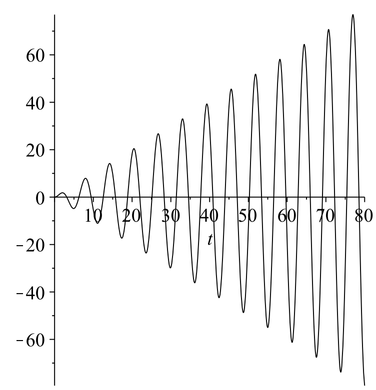

Figure 2.17: Plot of

$$x(t) = t\sin 2t,$$

a solution of $\ddot{x} + x = 2\cos t$.

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks. An example of such an oscillation is shown in Figure 2.15.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions that $c_1 = -F_0/(m(\omega_0^2 - \omega^2))$, $c_2 = 0$. Then one has (see Problem 13)

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \sin\frac{(\omega_0 + \omega)t}{2}. \tag{2.55}$$

For values of ω near ω_0, one finds the solution consists of a rapid oscillation, due to the $\sin\frac{(\omega_0+\omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2-\omega^2)} \sin\frac{(\omega_0-\omega)t}{2}$. The reader can investigate this solution.

This slow variation is called a beat and the beat frequency is given by $f = \frac{|\omega_0-\omega|}{4\pi}$. In Figure 2.16 we see the high frequency oscillations are contained by the lower beat frequency, $f = \frac{0.15}{4\pi}$s. This corresponds to a period of $T = 1/f \approx 83.7$ Hz, which looks about right from the figure.

Example 2.17. Solve $\ddot{x} + x = 2\cos\omega t$, $x(0) = 0$, $\dot{x}(0) = 0$, for $\omega = 1, 1.15$. For each case, we need the solution of the homogeneous problem,

$$x_h(t) = c_1\cos t + c_2 \sin t.$$

The particular solution depends on the value of ω.

For $\omega = 1$, the driving term, $2\cos\omega t$, is a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = At\cos t + Bt\sin t.$$

Inserting this into the differential equation, we find $A = 0$ and $B = 1$. So, the general solution is

$$x(t) = c_1\cos t + c_2\sin t + t\sin t.$$

Imposing the initial conditions, we find

$$x(t) = t\sin t.$$

This solution is shown in Figure 2.17.

For $\omega = 1.15$, the driving term, $2\cos\omega 1.15t$, is not a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = A\cos 1.15t + B\sin 1.15t.$$

Inserting this into the differential equation, we find $A = -\frac{800}{129}$ and $B = 0$. So, the general solution is

$$x(t) = c_1\cos t + c_2\sin t - \frac{800}{129}\cos t.$$

Imposing the initial conditions, we find

$$x(t) = \frac{800}{129}\left(\cos t - \cos 1.15t\right).$$

This solution is shown in Figure 2.18. The beat frequency in this case is the same as with Figure 2.16.

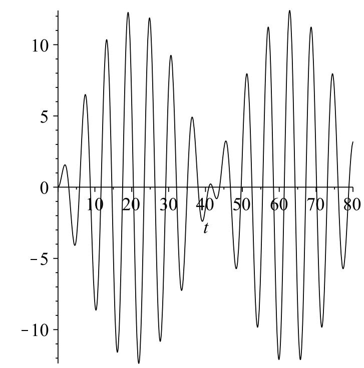

Figure 2.18: Plot of

$$x(t) = \frac{800}{129}\left(\cos t - \cos \frac{23}{20}t\right),$$

a solution of $\ddot{x} + x = 2\cos 1.15t$.

2.4.3 *Reduction of Order for Nonhomogeneous Equations*

The Method of Reduction of Order is also useful for solving nonhomogeneous problems. In this case if we know one solution of the homogeneous problem, then we can use it to obtain a particular solution of the nonhomogeneous problem. For example, consider the nonhomogeneous differential equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.56}$$

Let's assume that $y_1(x)$ satisfies the homogeneous differential equation

$$a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x) = 0. \tag{2.57}$$

Then, we seek a particular solution, $y_p(x) = v(x)y_1(x)$. Its derivatives are given by

$$\begin{aligned} y_p' &= (vy_1)', \\ &= v'y_1 + vy_1'. \\ y_p'' &= (v'y_1 + vy_1')', \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substituting y_p and its derivatives into the differential equation, we have

$$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + cvy_1 \\ &= av''y_1 + 2av'y_1' + bv'y_1 + v[ay_1'' + by_1' + cy_1] \\ &= av''y_1 + 2av'y_1' + bv'y_1 \end{aligned}$$

Therefore, $v(x)$ satisfies the second order equation

$$a(x)y_(x)v''(x) + [2a(x)y_1'(x) + b(x)y_1(x)]v'(x) = f(x).$$

Letting $z = v'$, we see that we have the linear first order equation for $z(x)$:

$$a(x)y_(x)z'(x) + [2a(x)y_1'(x) + b(x)y_1(x)]z(x) = f(x).$$

Example 2.18. Use the Method of Reduction of Order to solve $y'' + y = \sec x$.

Solutions of the homogeneous equation, $y'' + y = 0$ are $\sin x$ and $\cos x$. We can choose either to begin using the Method of Reduction of

Order. Let's take $y_p = v \cos x$. Its derivatives are given by

$$\begin{aligned} y_p' &= (v \cos x)', \\ &= v' \cos x - v \sin x. \\ y_p'' &= (v' \cos x - v \sin x)', \\ &= v'' \cos x - 2v' \sin x - v \cos x. \end{aligned}$$

Substituting into the nonhomogeneous equation, we have

$$\begin{aligned} \sec x &= y_p'' + y_p \\ &= v'' \cos x - 2v' \sin x - v \cos x + v \cos x \\ &= v'' \cos x - 2v' \sin x \end{aligned}$$

Letting $v' = z$, we have the linear first order differential equation

$$(\cos x) z' - (2 \sin x) z = \sec x.$$

Rewriting the equation as,

$$z' - (2 \tan x) z = \sec^2 x.$$

Multiplying by the integrating factor,

$$\begin{aligned} \mu(x) &= -\exp \int^x 2 \tan \xi \, d\xi \\ &= -\exp 2 \ln |\sec x| \\ &= \cos^2 x, \end{aligned}$$

we obtain

$$(z \cos^2 x)' = 1.$$

Integrating,

$$v' = z = x \sec^2 x.$$

This can be integrated using integration by parts (letting $U = x$ and $V = \tan x$):

$$\begin{aligned} v &= \int x \sec^2 x \, dx \\ &= x \tan x - \int \tan x \, dx \\ &= x \tan x - \ln |\sec x|. \end{aligned}$$

We now have enough to write out the solution. The particular solution is given by

$$\begin{aligned} y_p &= v y_1 \\ &= (x \tan x - \ln |\sec x|) \cos x \\ &= x \sin x + \cos x \ln |\cos x|. \end{aligned}$$

The general solution is then

$$y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln |\cos x|.$$

2.4.4 *Method of Variation of Parameters*

A MORE SYSTEMATIC WAY to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.58}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1y_1(x) + c_2y_2(x).$$

Replacing the constants with functions, then we no longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{2.59}$$

If this is to be a solution, then insertion into the differential equation should make the equation hold. To do this we will first need to compute some derivatives.

The first derivative is given by

$$y'_p(x) = c_1(x)y'_1(x) + c_2(x)y'_2(x) + c'_1(x)y_1(x) + c'_2(x)y_2(x). \tag{2.60}$$

Next we will need the second derivative. But, this will yield eight terms. So, we will first make a simplifying assumption. Let's assume that the last two terms add to zero:

$$c'_1(x)y_1(x) + c'_2(x)y_2(x) = 0. \tag{2.61}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

Under the assumption the first derivative simplifies to

$$y'_p(x) = c_1(x)y'_1(x) + c_2(x)y'_2(x). \tag{2.62}$$

The second derivative now only has four terms:

$$y'_p(x) = c_1(x)y''_1(x) + c_2(x)y''_2(x) + c'_1(x)y'_1(x) + c'_2(x)y'_2(x). \tag{2.63}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) &= a(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ &\quad + b(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] \\ &\quad + c(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned} \tag{2.64}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) &= c_1(x)\left[a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)\right] \\ &\quad + c_2(x)\left[a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)\right] \\ &\quad + a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{2.65}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$f(x) = a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right],$$

which can be rearranged as

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{2.66}$$

In order to solve the differential equation $Ly = f$, we assume

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations (2.67).

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

System (2.67) can be solved as

$$c_1'(x) = -\frac{fy_2}{aW(y_1,y_2)},$$

$$c_1'(x) = \frac{fy_1}{aW(y_1,y_2)},$$

where $W(y_1,y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. We use this solution in the next section.

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned} \tag{2.67}$$

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just as easily start from this system and solve the system for each problem encountered.

Example 2.19. Find the general solution of the nonhomogeneous problem: $y'' - y = e^{2x}$.

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (2.67), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \tag{2.68}$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x\,dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \to c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p:

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \tag{2.69}$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1e^x + c_2e^{-x} + \frac{1}{3}e^{2x}.$$

Example 2.20. Now consider the problem: $y'' + 4y = \sin x$.

The solution to the homogeneous problem is

$$y_h(x) = c_1\cos 2x + c_2\sin 2x. \tag{2.70}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x)\cos 2x + c_2(x)\sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (2.67):

$$\begin{aligned} c_1'(x)\cos 2x + c_2'(x)\sin 2x &= 0 \\ -2c_1'(x)\sin 2x + 2c_2'(x)\cos 2x &= \sin x. \end{aligned} \tag{2.71}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2}\sin x\cos 2x = \frac{1}{2}(2\cos^2 x - 1)\sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x)\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2}\int (2\cos^2 x - 1)\sin x\, dx = \frac{1}{2}(\cos x - \frac{2}{3}\cos^3 x).$$

$$c_1(x) = -\int \sin^x \cos x\, dx = -\frac{1}{3}\sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (-\frac{1}{3}\sin^3 x)\cos 2x + (\frac{1}{2}\cos x - \frac{1}{3}\cos^3 x)\sin x \\ &= \frac{1}{3}\sin x. \end{aligned} \tag{2.72}$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x. \tag{2.73}$$

2.4.5 *Initial Value Green's Functions**

In this section we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \tag{2.74}$$

subject to the initial conditions

$$y(0) = y_0 \quad y'(0) = v_0.$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, t.

Equation (2.74) can be written compactly as

$$L[y] = f,$$

where L is the differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(t) = \int G(t,\tau) f(\tau)\, d\tau.$$

The function $G(t,\tau)$ is referred to as the kernel of the integral operator and is called the Green's function.

The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". In this section we will derive the initial value Green's function for ordinary differential equations. Later in the book we will return to boundary value Green's functions and Green's functions for partial differential equations.

George Green (1793-1841), a British mathematical physicist who had little formal education and worked as a miller and a baker, published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in which he not only introduced what is now known as Green's function, but he also introduced potential theory and Green's Theorem in his studies of electricity and magnetism. Recently his paper was posted at arXiv.org, arXiv:0807.0088.

In the last section we solved nonhomogeneous equations like (2.74) using the Method of Variation of Parameters. Letting,

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t), \tag{2.75}$$

we found that we have to solve the system of equations

$$\begin{aligned} c_1'(t)y_1(t) + c_2'(t)y_2(t) &= 0. \\ c_1'(t)y_1'(t) + c_2'(t)y_2'(t) &= \frac{f(t)}{q(t)}. \end{aligned} \tag{2.76}$$

This system is easily solved to give

$$\begin{aligned} c_1'(t) &= -\frac{f(t)y_2(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} \\ c_2'(t) &= \frac{f(t)y_1(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}. \end{aligned} \tag{2.77}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(t) &= -\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ c_2(t) &= \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau, \end{aligned} \tag{2.78}$$

where t_0 and t_1 are arbitrary constants to be determined from the initial conditions.

Therefore, the particular solution of (2.74) can be written as

$$y_p(t) = y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\,d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\,d\tau. \qquad (2.79)$$

We begin with the particular solution (2.79) of the nonhomogeneous differential equation (2.74). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1y_1(t) + c_2y_2(t) + y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\,d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\,d\tau. \qquad (2.80)$$

However, an appropriate choice of t_0 and t_1 can be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(t) + c_2y_2(t)$. However, setting up the solution in this form will allow us to use t_0 and t_1 to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (2.80) can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_0^t G(t,\tau)f(\tau)\,d\tau, \qquad (2.81)$$

where the function $G(t,\tau)$ will be identified as the Green's function.

The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0. \qquad (2.82)$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = v_0. \qquad (2.83)$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t), \quad y_p(0) = 0, \quad y_p'(0) = 0. \qquad (2.84)$$

Since the differential equation is linear, then we know that

$$y(t) = y_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$

$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for t_0 and t_1 in Equation (2.79) which satisfy the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0) \int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(0) \int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{2.85}$$

Here, $y_1(t)$ and $y_2(t)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then, we have

$$y_p(0) = y_2(0) \int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau \tag{2.86}$$

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t) \int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1'(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau, \tag{2.87}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $t = 0$, we have

$$y_p'(0) = -y_1'(0) \int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{2.88}$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(t) \int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t) \int_{0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ &= \int_0^t \left[\frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \right] f(\tau)\, d\tau. \end{aligned} \tag{2.89}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau, \tag{2.90}$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}.$$

We summarize

Solution of IVP Using the Green's Function

The solution of the initial value problem,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

takes the form

$$y(t) = y_h(t) + \int_0^t G(t,\tau)f(\tau)\,d\tau, \tag{2.91}$$

where

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \tag{2.92}$$

is the Green's function and y_1, y_2, y_h are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

Example 2.21. Solve the forced oscillator problem

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, to the homogeneous differential equation satisfying different homogeneous conditions, $y_1(0) = 0$ and $y_2'(0) = 0$. The simplest solutions are $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $a(t) = 1$ in this problem, we compute the Green's function,

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{2.93}$$

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form when carrying out the integration.

We can now determine the particular solution of the nonhomogeneous differential equation. We have

$$\begin{aligned} x_p(t) &= \int_0^t G(t,\tau)f(\tau)\,d\tau \\ &= \int_0^t (\sin t\cos\tau - \sin\tau\cos t)\,(2\cos\tau)\ d\tau \end{aligned}$$

$$\begin{aligned} &= 2\sin t\int_0^t \cos^2\tau d\tau - 2\cos t\int_0^t \sin\tau\cos\tau d\tau \\ &= 2\sin t\left[\frac{\tau}{2}+\frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \\ &= t\sin t. \end{aligned} \tag{2.94}$$

Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution: $x(t) = 4\cos t + t\sin t$.

2.5 *Cauchy-Euler Equations*

ANOTHER CLASS OF SOLVABLE LINEAR DIFFERENTIAL EQUATIONS that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{2.95}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (2.95), we have

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1) + br + c = 0. \tag{2.96}$$

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1) + br + c = 0$.

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

Example 2.22. Find the general solution: $x^2y'' + 5xy' + 12y = 0$.

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{2.97}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1\cos(2\sqrt{2}\ln|x|) + c_2\sin(2\sqrt{2}\ln|x|)\right]x^{-2}$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1 r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Dividing this equation by a and rewriting, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the Cauchy-Euler equation for this case can be written in the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1 - 1}, \\ y_2''(x) &= (x^2 v'' + 2r_1 xv' + r_1(r_1 - 1)v)x^{r_1 - 2}. \end{aligned} \tag{2.98}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1 + 1}. \end{aligned} \tag{2.99}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln |v'| = -\ln |x| + C,$$

where $A = \pm e^C$ absorbs C and the signs from the absolute values. Exponentiating, we obtain one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln |x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln |x|.$$

Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln |x|)x^r$.

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2 \ln |x|)x^r.$$

Example 2.23. Solve the initial value problem: $t^2 y'' + 3ty' + y = 0$, with the initial conditions $y(1) = 0$, $y'(1) = 1$.

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln |t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln |t| t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln |t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln |t| t^{-1}$.

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha + i\beta}$. The key to obtaining real solutions is to first rewrite x^y :

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha + i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln |x|) + i \sin(\beta \ln |x|),$$

we can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln |x|)$ and $x^\alpha \sin(\beta \ln |x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

Example 2.24. Solve: $x^2y'' - xy' + 5y = 0$.

The characteristic equation takes the form

$$r(r-1) - r + 5 = 0,$$

or

$$r^2 - 2r + 5 = 0.$$

The roots of this equation are complex, $r_{1,2} = 1 \pm 2i$. Therefore, the general solution is $y(x) = x(c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|))$.

The three cases are summarized in the table below.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

Example 2.25. Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1x^{-1} + c_2x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{2.100}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3} x^2.$$

Example 2.26. Find the solution of $x^2 y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2 y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

Example 2.27. Find the solution of $x^2 y'' - xy' - 3y = 2x^3$ using Variation of Parameters.

As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions, $y_1(x) = x^{-1}$ and $y_2(x) = x^3$. Assuming a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, we need to solve the system (2.67):

$$\begin{aligned} c_1'(x)x^{-1} + c_2'(x)x^3 &= 0 \\ -c_1'(x)x^{-2} + 3c_2'(x)x^2 &= \frac{2x^3}{x^2} = 2x. \end{aligned} \tag{2.101}$$

From the first equation of the system we have $c_1'(x) = -x^4 c_2'(x)$. Substituting this into the second equation gives $c_2'(x) = \frac{1}{2x}$. So, $c_2(x) = \frac{1}{2} \ln |x|$ and, therefore, $c_1(x) = \frac{1}{8} x^4$. The particular solution is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8} x^3 + \frac{1}{2} x^3 \ln |x|.$$

Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since $\frac{1}{8} x^3$ is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

Problems

1. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

2. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin(x^2).$

c. $(1 - x^2)y'' - 2xy' + 2y = 0, \quad y_1(x) = x.$ [Note: This is one solution of Legendre's differential equation in Example 4.4.]

d. $(x - 1)y'' - xy' + y = 0, \quad y_1(x) = e^x.$

3. Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

4. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

5. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$, Undetermined Coefficients.

b. $y'' + 2y' + y = 5 + 10\sin 2x$, Undetermined Coefficients.

c. $y'' - 5y' + 6y = 3e^x$, Reduction of Order.

d. $y'' + 5y' - 6y = 3e^x$, Reduction of Order.

e. $y'' + y = \sec^3 x$, Reduction of Order.

f. $y'' + y' = 3x^2$, Variation of Parameters.

g. $y'' - y = e^x + 1$, Variation of Parameters.

6. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x.$

b. $y'' - 4y' + 4y = 6xe^{2x}.$

c. $y'' - 2y' + y = \frac{e^{2x}}{(1+e^x)^2}$.

d. $y'' - 3y' + 2y = \cos(e^x)$.

7. Instead of assuming that $c_1' y_1 + c_2' y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1' y_1 + c_2' y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

8. Find all of the solutions of the second order differential equations for $x > 0$. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2 y'' + 3xy' + 2y = 0$.

b. $x^2 y'' - 3xy' + 3y = 0, \quad y(1) = 1, y'(1) = 0$.

c. $x^2 y'' + 5xy' + 4y = 0$.

d. $x^2 y'' - 2xy' + 3y = 0, \quad y(1) = 3, y'(1) = 0$.

e. $x^2 y'' + 3xy' - 3y = 0$.

9. Another approach to solving Cauchy-Euler equations is by transforming the equation to one with constant coefficients.

a. Consider the equation

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

Make the change of variables $x = e^t$ and $y(x) = v(t)$. Show that

$$\frac{dy}{dx} = \frac{1}{x}\frac{dv}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2v}{dt^2} - \frac{dv}{dt}\right)$$

b. Use the above transformation to solve the following equations:

i. $x^2 y'' + 3xy' - 3y = 0$.

ii. $2x^2 y'' + 5xy' + y = 0$.

iii. $4x^2 y'' + y = 0$.

iii. $x^3 y''' + xy' - y = 0$.

10. Solve the following nonhomogenous Cauchy-Euler equations for $x > 0$.

a. $x^2 y'' + 3xy' - 3y = 3x^2$.

b. $2x^2 y'' + 5xy' + y = x^2 + x$.

c. $x^2 y'' + 5xy' + 4y = 2x^3$.

d. $x^2 y'' - 2xy' + 3y = 5x^2, \quad y(1) = 3, y'(1) = 0$.

11. A spring fixed at its upper end is stretched six inches by a 10-pound weight attached at its lower end. The spring-mass system is suspended in a viscous medium so that the system is subjected to a damping force of $5\frac{dx}{dt}$ lbs. Describe the motion of the system if the weight is drawn down an additional 4 inches and released. What would happen if you changed the coefficient "5" to "4"? [You may need to consult your introductory physics text. For example, the weight and mass are related by $W = mg$, where the mass is in slugs and $g = 32$ ft/s^2.]

12. Consider an LRC circuit with $L = 1.00$ H, $R = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ f, and $V = 1.00 \times 10^3$ V. Suppose that no charge is present and no current is flowing at time $t = 0$ when a battery of voltage V is inserted. Find the current and the charge on the capacitor as functions of time. Describe how the system behaves over time.

13. Consider the problem of forced oscillations as described in section 2.4.2.

b. Plot the solutions in Equation (2.77) for the following cases: Let $c_1 = 0.5$, $c_2 = 0$, $F_0 = 1.0$ N, and $m = 1.0$ kg for $t \in [0, 100]$.
   i. $\omega_0 = 2.0$ rad/s, $\omega = 0.1$ rad/s.
   ii. $\omega_0 = 2.0$ rad/s, $\omega = 0.5$ rad/s.
   iii. $\omega_0 = 2.0$ rad/s, $\omega = 1.5$ rad/s.
   iv. $\omega_0 = 2.0$ rad/s, $\omega = 2.2$ rad/s.
   v. $\omega_0 = 1.0$ rad/s, $\omega = 1.2$ rad/s.
   vi. $\omega_0 = 1.5$ rad/s, $\omega = 1.5$ rad/s.

d. Confirm that the solution in Equation (2.78) is the same as the solution in Equation (2.77) for $F_0 = 2.0$ N, $m = 10.0$ kg, $\omega_0 = 1.5$ rad/s, and $\omega = 1.25$ rad/s, by plotting both solutions for $t \in [0, 100]$.

14. A certain model of the motion light plastic ball tossed into the air is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, g=9.8 m/s^2 is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 5\ \text{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$ versus the time, using computer software.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

15. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

16. Use the initial value Green's function for $x'' + x = f(t)$, $x(0) = 4$, $x'(0) = 0$, to solve the following problems.

a. $x'' + x = 5t^2.$

b. $x'' + x = 2\tan t.$

17. For the problem $y'' - k^2y = f(x)$, $y(0) = 0$, $y'(0) = 1$,

a. Find the initial value Green's function.

b. Use the Green's function to solve $y'' - y = e^{-x}$.

c. Use the Green's function to solve $y'' - 4y = e^{2x}$.

18. Find and use the initial value Green's function to solve

$$x^2y'' + 3xy' - 15y = x^4e^x, \quad y(1) = 1, y'(1) = 0.$$