Prologue

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?." - Albert Einstein (1879-1955)

Introduction

This book is written for an undergraduate course on the introduction to differential equations typically taken by majors in mathematics, the physical sciences, and engineering. In this course we will investigate analytical, graphical, and approximate solutions of differential equations. We will study the theory, methods of solution and applications of ordinary differential equations. This will include common methods of finding solutions, such as using Laplace transform and power series methods.

Students should also be prepared to review their calculus, especially if they have been away from calculus for a while. Some of the key topics are reviewed in the appendix. In particular, students should know how to differentiate and integrate all elementary functions, including hyperbolic functions. They should review the methods of integration as the need arises, including methods of substitution and integration by parts. For the most part, we will just need material from Calculus I and II. Other topics from Calculus II that we will review are infinite series and introductory differential equations and applications.

Most students will have just come out of the calculus sequence knowing all about differentiation and integration. We hope that they have also seen plenty of applications. In this course, we will extend these applications to those connected with differential equations. Differential equations are equations involving an unknown function and its derivatives. If the function is a function of a single variable, then the equations are known as ordinary differential equations, the subject of this book. If the unknown function is a function of several independent variables, then the equation is a partial differential equation, which we will not deal with in this course. Finally, there may be several unknown functions satisfying several coupled differential equations. These systems of differential equations will be treated later in the course and are often the subject of a second course in differential equations.

In all cases we will be interested in specific solutions satisfying a set of initial conditions, or values, of the function and some of its derivatives at a given point of its domain. These are known as initial value problems. When such conditions are given at several points, then one is dealing with boundary value problems. Boundary value problems would be the subject of a second course in differential equations and in partial differential equations.

We will begin the study of differential equations with first order ordinary differential equations. These equations involve only derivatives of first order. Typical examples occur in population modeling and in free fall problems. There are a few standard forms which can be solved quite easily. In the second chapter we move up to second order equations. As the order increases, it becomes harder to solve differential equations analytically. So, we either need to deal with simple equations or turn to other methods of finding approximate solutions.

For second order differential equations there is a theory for linear second order differential equations and the simplest equations are constant coefficient second order linear differential equations. We will spend some time looking at these solutions. Even though constant coefficient equations are relatively simple, there are plenty of applications and the simple harmonic oscillator is one of these. The solutions make physical sense and adding damping and forcing terms leads to interesting solutions and additional methods of solving these equations.

Not all differential equations can be solved in terms of elementary functions. So, we turn to the numerical solution of differential equations using the solvable models as test beds for numerical schemes. This also allows for the introduction of more realistic models. Using Computer Algebra Systems (CAS) or other programming environments, we can explore these examples.

A couple hundred years ago there were no computers. So, mathematicians of the day sought series solutions of differential equations. These series solutions led to the discovery of now famous functions, such as Legendre polynomials and Bessel functions. These functions are quite common in applications and the use of power series solutions is a well known approach to finding approximate solutions by hand.

Another common technique for solving differential equation, both ordinary and partial, are transform methods. One of the simplest of these is the Laplace transform. This integral transform is used to transform the ordinary differentia equation to an algebraic equation. The solution of the algebraic equation is then used to uncover the solution to the differential equation. These techniques are often useful in systems theory or electrical engineering.

In recent decades the inclusion of technology in the classroom has allowed for the introduction of systems of differential equations into the typical course on differential equations. Solutions of linear systems of equations is an important tool in the study of nonlinear differential equations and nonlinear differential equations have been the subject of many research papers over the last several decades. We will look at systems of differential equations at the end of the book and discuss the stability of solutions in dynamical systems.

Technology and Tables

As you progress through the course, you will often have to compute integrals and derivatives by hand. However, many readers know that some of the tedium can be alleviated by using computers, or even looking up what you need in tables. In some cases you might even find applets online that can quickly give you the answers you seek.

You also need to be comfortable in doing many computations by hand. This is necessary, especially in your early studies, for several reasons. For example, you should try to evaluate integrals by hand when asked to do them. This reinforces the techniques, as outlined earlier. It exercises your brain in much the same way that you might jog daily to exercise your body. The more comfortable you are with derivations and evaluations, the easier it is to follow future lectures without getting bogged down by the details, wondering how your professor got from step A to step D. You can always use a computer algebra system, or a Table of Integrals, to check on your work.

Problems can arise when depending purely on the output of computers, or other "black boxes." Once you have a firm grasp on the techniques and a feeling as to what answers should look like, then you can feel comfortable with what the computer gives you. Sometimes, Computer Algebra Systems (CAS) like Maple, can give you strange looking answers and sometimes even wrong answers. Also, these programs cannot do every integral or solve every differential equation that you ask them to do. Even some of the simplest looking expressions can cause computer algebra systems problems. Other times you might even provide wrong input, leading to erroneous results.

Another source of indefinite integrals, derivatives, series expansions, etc, is a Table of Mathematical Formulae. There are several good books that have been printed. Even some of these have typos in them, so you need to be careful. However, it may be worth the investment to have such a book in your personal library. Go to the library, or the bookstore, and look at some of these tables to see how useful they might be.

There are plenty of online resources as well. For example, there is the Wolfram Integrator at http://integrals.wolfram.com/ as well as the recent http://www.wolframalpha.com/. There is also a wealth of information at the following sites: http://www.sosmath.com/,
http://www.math2.org/, http://mathworld.wolfram.com/, and
http://functions.wolfram.com/.

While these resources are useful for problems which have analytical solutions, at some point you will need to realize that most problems in texts, especially those from a few decades ago, are mostly aimed at solutions which either have nice analytical solutions or have solutions that can be approximated using pencil and paper.

More and more you will see problems which need to be solved numerically. While most of this book (97%) stresses the traditional methods used for determining the exact or approximate behavior of systems based upon solid mathematical methods, there are times that an basic understanding of computational methods is useful. Therefore, we will occasionally discuss some numerical methods related to the subject matter in the text. In particular, we will discuss some methods of computational physics such as the numerical solution of differential equations and fitting data to curves. Applications will be discussed which can only be solved using these methods.

There are many programming languages and software packages which can be used to determine numerical solutions to algebraic equations or differential equations. For example, CAS (Computer Algebra Systems) such as Maple and Mathematica are available. Open source packages such as Maxima, which has been around for a while, Mathomatic, and the SAGE Project, do exist as alternatives. One can use built in routines and do some programming. The main features are that they can produce symbolic solutions. Generally, they are slow in generating numerical solutions.

For serious programming, one can use standard programming languages like FORTRAN, C and its derivatives. Recently, Python has become an alternative and much accepted resource as an open source programming language and is useful for doing scientific computing using the right packages.

Also, there is MATLAB. MATLAB was developed in the 1980's as a Matrix Laboratory and for a long time was the standard outside "normal" programming languages to handle non-symbolic solutions in computational science. Similar open source clones have appeared, such as Octave. Octave can run most MATLAB files and some of its own. Other clones of MATLAB are SciLab, Rlab, FreeMat, and PyLab.

In this text there are some snippets provided of Maple and MATLAB routines. Most of the text does not rely on these; however, the MATLAB snippets should be relatively readable to anyone with some knowledge of computer packages, or easy to pass to the open source clones, such as Octave. Maple routines are not so simple, but may be translatable to other packages with a little effort. However, the theory lying behind the use of any of these routines is described in the text and the text can be read without explicit understanding of the particular computer software.

Acknowledgments

MOST, IF NOT ALL, OF THE IDEAS AND EXAMPLES are not my own. These notes are a compendium of topics and examples that I have used in teaching not only differential equations, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken.

I would also like to express my gratitude to the many students who have

found typos, or suggested sections needing more clarity in the core set of notes upon which this book was based. This applies to the set of notes used in my mathematical physics course, applied mathematics course, *An Introduction to Fourier and Complex Analysis with Application to the Spectral Analysis of Signals*, and ordinary differential equations course, *A Second Course in Ordinary Differential Equations: Dynamical Systems and Boundary Value Problems*, all of which have some significant overlap with this book.