Chapter 1

First Order Differential Equations

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

1.1 Free Fall

In this chapter we will study some common differential equations that appear in physics. We will begin with the simplest types of equations and standard techniques for solving them We will end this part of the discussion by returning to the problem of free fall with air resistance. We will then turn to the study of oscillations, which are modeled by second order differential equations.

Free fall example.

Let us begin with a simple example from introductory physics. Recall that free fall is the vertical motion of an object solely under the force of gravity. It has been experimentally determined that an object near the surface of the Earth falls at a constant acceleration in the absence of other forces, such as air resistance. This constant acceleration is denoted by $-g$, where g is called the acceleration due to gravity. The negative sign is an indication that we have chosen a coordinate system in which up is positive.

We are interested in determining the position, $y(t)$, of the falling body as a function of time. From the definition of free fall, we have

$$\ddot{y}(t) = -g. \tag{1.1}$$

Differentiation with respect to time is often denoted by dots instead of primes.

Note that we will occasionally use a dot to indicate time differentiation. This notation is standard in physics and we will begin to introduce you to this notation, though at times we might use the more familiar prime notation to indicate spatial differentiation, or general differentiation.

In Equation (1.1) we know g. It is a constant. Near the Earth's surface it is about 9.81 m/s^2 or 32.2 ft/s^2. What we do not know is $y(t)$. This is our first differential equation. In fact it is natural to see differential equations appear in physics often through Newton's Second Law, $F = ma$, as it plays an important role in classical physics. We will return to this point later.

So, how does one solve the differential equation in (1.1)? We do so by using what we know about calculus. It might be easier to see when we put in a particular number instead of g. You might still be getting used to the fact that some letters are used to represent constants. We will come back to the more general form after we see how to solve the differential equation.

Consider

$$\ddot{y}(t) = 5. \tag{1.2}$$

Recalling that the second derivative is just the derivative of a derivative, we can rewrite this equation as

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 5. \tag{1.3}$$

This tells us that the derivative of dy/dt is 5. Can you think of a function whose derivative is 5? (Do not forget that the independent variable is t.) Yes, the derivative of $5t$ with respect to t is 5. Is this the only function whose derivative is 5? No! You can also differentiate $5t+1$, $5t+\pi$, $5t-6$, etc. In general, the derivative of $5t+C$ is 5, where C is an arbitrary integration constant.

So, Equation (1.2) can be reduced to

$$\frac{dy}{dt} = 5t + C. \tag{1.4}$$

Now we ask if you know a function whose derivative is $5t+C$. Well, you might be able to do this one in your head, but we just need to recall the Fundamental Theorem of Calculus, which relates integrals and derivatives. Thus, we have

$$y(t) = \frac{5}{2}t^2 + Ct + D, \tag{1.5}$$

where D is a second integration constant.

Equation (1.5) gives the solution to the original differential equation. That means that when the solution is placed into the differential equation, both sides of the differential equation give the same expression. You can always check your answer to a differential equation by showing that your solution satisfies the equation. In this case we have

$$\ddot{y}(t) = \frac{d^2}{dt^2}\left(\frac{5}{2}t^2 + Ct + D\right) = \frac{d}{dt}(5t + C) = 5.$$

Therefore, Equation (1.5) gives the general solution of the differential equation.

We also see that there are two arbitrary constants, C and D. Picking any values for these gives a whole family of solutions. As we will see, the equation $\ddot{y}(t) = 5$ is a linear second order ordinary differential equation. The general solution of such an equation always has two arbitrary constants.

Let's return to the free fall problem. We solve it the same way. The only difference is that we can replace the constant 5 with the constant $-g$. So, we find that

$$\frac{dy}{dt} = -gt + C, \tag{1.6}$$

and

$$y(t) = -\frac{1}{2}gt^2 + Ct + D. \tag{1.7}$$

Once you get down the process, it only takes a line or two to solve.

There seems to be a problem. Imagine dropping a ball that then undergoes free fall. We just determined that there are an infinite number of solutions for the position of the ball at any time! Well, that is not possible. Experience tells us that if you drop a ball you expect it to behave the same way every time. Or does it? Actually, you could drop the ball from anywhere. You could also toss it up or throw it down. So, there are many ways you can release the ball before it is in free fall producing many different paths, $y(t)$. That is where the constants come in. They have physical meanings.

If you set $t = 0$ in the equation, then you have that $y(0) = D$. Thus, D gives the initial position of the ball. Typically, we denote initial values with a subscript. So, we will write $y(0) = y_0$. Thus, $D = y_0$.

That leaves us to determine C. It appears at first in Equation (1.6). Recall that $\frac{dy}{dt}$, the derivative of the position, is the vertical velocity, $v(t)$. It is positive when the ball moves upward. We will denote the initial velocity $v(0) = v_0$. Inserting $t = 0$ in Equation (1.6), we find that $\dot{y}(0) = C$. This implies that $C = v(0) = v_0$.

Putting this all together, we have the physical form of the solution for free fall as

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0. \tag{1.8}$$

Doesn't this equation look familiar? Now we see that the infinite family of solutions consists of free fall resulting from initially dropping a ball at position y_0 with initial velocity v_0. The conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are called the initial conditions. A solution of a differential equation satisfying a set of initial conditions is often called a particular solution. Specifying the initial conditions results in a unique solution.

So, we have solved the free fall equation. Along the way we have begun to see some of the features that will appear in the solutions of other problems that are modeled with differential equation. Throughout the book we will see several applications of differential equations. We will extend our analysis to higher dimensions, in which we case will be faced with so-called partial differential equations, which involve the partial derivatives of functions of more that one variable.

But are we done with free fall? Not at all! We can relax some of the conditions that we have imposed. We can add air resistance. We will visit this problem later in this chapter after introducing some more techniques. We can also provide a horizontal component of motion, leading to projectile motion.

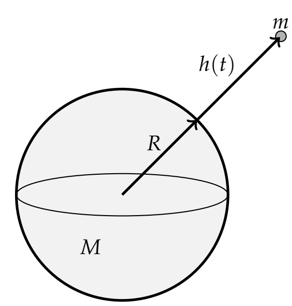

Figure 1.1: Free fall far from the Earth from a height $h(t)$ from the surface.

Finally, we should also note that free fall at constant g only takes place near the surface of the Earth. What if a tile falls off the shuttle far from the surface of the Earth? It will also fall towards the Earth. Actually, the tile also has a velocity component in the direction of the motion of the shuttle. So, it would not necessarily take radial path downwards. For now, let's ignore that component.

To look at this problem in more detail, we need to go to the origins of the acceleration due to gravity. This comes out of Newton's Law of Gravitation. Consider a mass m at some distance $h(t)$ from the surface of the (spherical) Earth. Letting M and R be the Earth's mass and radius, respectively, Newton's Law of Gravitation states that

$$\begin{aligned} ma &= F \\ m\frac{d^2h(t)}{dt^2} &= -G\frac{mM}{(R+h(t))^2}. \end{aligned} \tag{1.9}$$

Here $G = 6.6730 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is the Universal Gravitational Constant, $M = 5.9736 \times 10^{24}$ kg and $R = 6371$ km are the Earth's mass and mean radius, respectively. For $h << R$, $GM/R^2 \approx g$.

Thus, we arrive at a differential equation

$$\frac{d^2h(t)}{dt^2} = -\frac{GM}{(R+h(t))^2}. \tag{1.10}$$

This equation is not as easy to solve. We will leave it as a homework exercise for the reader.

1.2 First Order Differential Equations

n-th order ordinary differential equation

Before moving on, we first define an n-th order ordinary differential equation. It is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.11}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

Initial value problem.

An initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.12}$$

Linear nth order differential equation

A linear nth order differential equation takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{1.13}$$

If $f(x) \equiv 0$, then the equation is said to be homogeneous, otherwise it is called nonhomogeneous.

Homogeneous and nonhomogeneous equations.

First order differential equation

Typically, the first differential equations encountered are first order equations. A first order differential equation takes the form

$$F(y', y, x) = 0. \tag{1.14}$$

There are two common first order differential equations for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

1.2.1 Separable Equations

A FIRST ORDER EQUATION IS SEPARABLE if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1.15}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

The general solution to equation (1.15) is obtained in terms of two integrals:

Separable equations.

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx + C,} \tag{1.16}$$

where C is an integration constant. This yields a 1-parameter family of solutions to the differential equation corresponding to different values of C. If one can solve (1.16) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a particular solution.

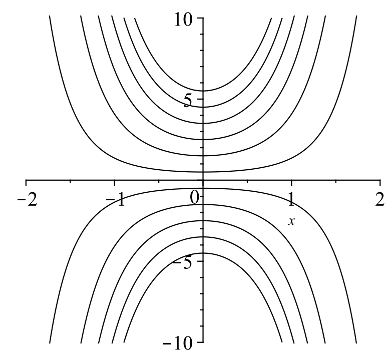

Figure 1.2: Plots of solutions from the 1-parameter family of solutions of Example 1.1 for several initial conditions.

Example 1.1. $y' = 2xy$, $y(0) = 2$.

Applying (1.16), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 1.2.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.

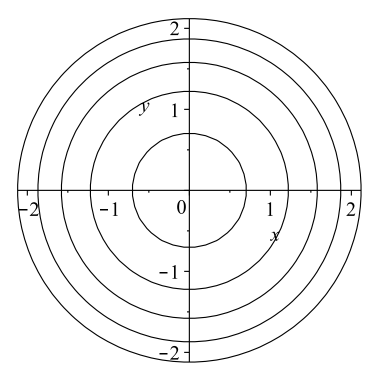

Figure 1.3: Plots of solutions of Example 1.2 for several initial conditions.

Example 1.2. $yy' = -x$. Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 1.3.

1.2.2 *Linear First Order Equations*

THE SECOND TYPE OF FIRST ORDER EQUATION encountered is the linear first order differential equation in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{1.17}$$

In this case one seeks an integrating factor, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x). \tag{1.18}$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\,d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (1.18),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{1.19}$$

and comparing this equation to the one obtained from multiplying (1.17) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{1.20}$$

Note that these last two equations would be the same if the second terms were the same. Thus, we will require that

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

Integrating factor.

This is a separable first order equation for $\mu(x)$ whose solution is the integrating factor:

$$\boxed{\mu(x) = \exp\left(\int^x p(\xi)\,d\xi\right).} \tag{1.21}$$

Equation (1.18) is now easily integrated to obtain the general solution to the linear first order differential equation:

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right].} \tag{1.22}$$

Example 1.3. $xy' + y = x, \quad x > 0,\ y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the equation is not separable. Furthermore, it is not in the standard form (1.17). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{1.23}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (1.23) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}(x - \frac{1}{x}).$$

We can verify that this is the solution. Since $y' = \frac{1}{2} + \frac{1}{2x^2}$, we have

$$xy' + y = \frac{1}{2}x + \frac{1}{2x} + \frac{1}{2}\left(x - \frac{1}{x}\right) = x.$$

Also, $y(1) = \frac{1}{2}(1 - 1) = 0$.

Example 1.4. $(\sin x)y' + (\cos x)y = x^2$.

Actually, this problem is easy if you realize that the left hand side is a perfect derivative. Namely,

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, we rewrite the original differential equation in standard form. We divide the equation by $\sin x$ to obtain

$$y' + (\cot x)y = x^2 \csc x.$$

Then, we compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \xi \, d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the standard form equation becomes

$$\frac{d}{dx}((\sin x)y) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y(x) = \left(\frac{1}{3}x^3 + C\right) \csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields or numerical methods. We will return to a discussion of the qualitative behavior of differential equations later and numerical solutions of ordinary differential equations later in the book.

1.2.3 Exact Differential Equations

Some first order differential equations can be solved easily if they are what are called exact differential equations. These equations are typically written using differentials. For example, the differential equation

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0 \tag{1.24}$$

can be written in the form

$$M(x,y)dx + N(x,y)dy = 0.$$

This is seen by multiplying Equation (1.24) by dx and noting from calculus that for a function $y = y(x)$, the relation between the differentials dx and dy is

$$dy = \frac{dy}{dx}\,dx.$$

Differential one-forms.

The expression $M(x,y)dx + N(x,y)dy$ is called a differential one-form. Such a one-form is called exact if there is a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du.$$

Exact one-form.

However, from calculus we know that for any function $u(x,y)$,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

If $du = M(x,y)dx + N(x,y)dy$, then we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= M(x,y) \\ \frac{\partial u}{\partial y} &= N(x,y). \end{aligned} \tag{1.25}$$

Since

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

when these second derivatives are continuous, by Clairaut's Theorem, then we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

must hold if $M(x,y)dx + N(x,y)dy$ is to be an exact one-form.

In summary, we have found that

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is exact in the domain D of the xy-plane for M, N, M_y, and N_x continuous functions in D if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds in the domain.

Condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact.

Furthermore, if $du = M(x,y)dx + N(x,y)dy = 0$, then $u(x,y) = C$, for C an arbitrary constant. Thus, an implicit solution can be found as

$$\int_{x_0}^{x} M(x,y)\,dx + \int_{y_0}^{y} N(x,y)\,dy = C.$$

We show this in the following example.

Example 1.5. Show that $(x^3 + xy^2)\,dx + (x^2y + y^3)\,dy = 0$ is an exact differential equation and obtain the corresponding implicit solution

We first note that

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = 2xy.$$

Since these partial derivatives are the same, the differential equation is exact. So, we need to find the function $u(x,y)$ such that $du = (x^3 + xy^2)\,dx + (x^2y + y^3)\,dy$.

First, we note that $x^3 = d\left(\frac{x^4}{4}\right)$ and $y^3 = d\left(\frac{y^4}{4}\right)$. The remaining terms can be combined to find that

$$\begin{aligned} xy^2\,dx + x^2y\,dy &= xy(y\,dx + x\,dy) \\ &= xy\,d(xy) \\ &= d\left(\frac{(xy)^2}{2}\right). \end{aligned} \tag{1.26}$$

Combining these results, we have

$$u = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4} = C.$$

What if the one-form is not exact?

So ,what if $M(x,y)dx + N(x,y)dy$ is not exact? We can multiply the one-form by an integrating factor, $\mu(x)$, and try to make he resulting form exact. We let

$$du = \mu M dx + \mu N dy.$$

For the new form to be exact, we have to require that

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

Carrying out the differentiation, we have

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

Thus, the integrating factor satisfies a partial differential equation. If the integrating factor is a function of only x or y, then this equation reduces to ordinary differential equations for μ.

As an example, if $\mu = \mu(x)$, then the integrating factor satisfies

$$N\frac{d\mu}{dx} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right),$$

or

$$N\frac{d\ln\mu}{dx} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

If

$$\frac{\mu}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is only a function of x, then $\mu = \mu(x)$.

If

$$\frac{\mu}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

is only a function of y, then $\mu = \mu(y)$.

Example 1.6. Find the general solution to the differential equation $(1 + y^2)\,dx + xy\,dy = 0$.

First, we note that this is not exact. We have $M(x, y) = 1 + y^2$ and $N(x, y) = xy$. Then,

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = y.$$

Therefore, the differential equation is not exact.

Next, we seek the integrating factor. We let

$$du = \mu(1 + y^2)\,dx + \mu xy\,dy.$$

For the new form to be exact, we have to require that

$$xy\frac{\partial \mu}{\partial x} - (1 + y^2)\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial(1 + y^2)}{\partial y} - \frac{\partial xy}{\partial x}\right) = \mu y.$$

If $\mu = \mu(x)$, then

$$x\frac{d\mu}{dx} = \mu.$$

This is easily solved as a separable first order equation. We find that $\mu(x) = x$.

Multiplying the original equation by $\mu = x$, we obtain

$$0 = x(1 + y^2)\,dx + x^2 y\,dy = d\left(\frac{x^2}{2} + \frac{x^2y^2}{2}\right).$$

Thus,

$$\frac{x^2}{2} + \frac{x^2y^2}{2} = C$$

gives the solution.

1.3 Applications

In this section we will look at some simple applications which are modeled with first order differential equations. We will begin with simple exponential models of growth and decay.

1.3.1 Growth and Decay

Some of the simplest models are those involving growth or decay. For example, a population model can be obtained under simple assumptions. Let $P(t)$ be the population at time t. We want to find an expression for the rate of change of the population, $\frac{dP}{dt}$. Assuming that there is no migration of population, the only way the population can change is by adding or subtracting individuals in the population. The equation would take the form

$$\frac{dP}{dt} = \text{Rate In} - \text{Rate Out}.$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. The simplest forms for these rates would be given by

$$\text{Rate In } = bP \text{ and the } \text{Rate Out } = mP.$$

Here we have denoted the birth rate as b and the mortality rate as m. This gives the total rate of change of population as

$$\frac{dP}{dt} = bP - mP \equiv kP. \tag{1.27}$$

Equation (1.27) is a separable equation. The separation follows as we have seen earlier in the chapter. Rearranging the equation, its differential form is

$$\frac{dP}{P} = k\,dt.$$

Integrating, we have

$$\begin{aligned} \int \frac{dP}{P} &= \int k\,dt \\ \ln|P| &= kt + C. \end{aligned} \tag{1.28}$$

Next, we solve for $P(t)$ through exponentiation, Integrating, we have

More generally, the initial value problem $dP/dt = kP$, $P(t_0) = P_0$ has the solution

$$P(t) = P_0 e^{k(t-t_0)}.$$

$$\begin{aligned} |P(t)| &= e^{kt+C} \\ P(t) &= \pm e^{kt+C} \\ &= \pm e^C e^{kt} \\ &= Ae^{kt}. \end{aligned} \tag{1.29}$$

Here we renamed the arbitrary constant, $\pm e^C$, as A.

If the population at $t = 0$ is P_0, i.e., $P(0) = P_0$, then the solution gives $P(0) = Ae^0 = A = P_0$. So, the solution of the initial value problem is

$$P(t) = P_0 e^{kt}.$$

Equation (1.27) the familiar exponential model of population growth:

Malthusian population growth.

$$\frac{dP}{dt} = kP.$$

This is easily solved and one obtains exponential growth ($k > 0$) or decay ($k < 0$). This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

Example 1.7. Consider a bacteria population of weight 20 g. If the population doubles every 20 minutes, then what is the population after 30 minutes? [Note: It is easier to weigh this population than to count it.]

One looks at the given information before trying to answer the question. First, we have the initial condition $P_0 = 20$ g. Since the population doubles every 20 minutes, then $P(20) = 2P_0 = 40$. Here we have take the time units as minutes. We are then asked to find $P(30)$.

We do not need to solve the differential equation. We will assume a simple growth model. Using the general solution, $P(t) = 20e^{kt}$, we have

$$P(20) = 20e^{20k} = 40,$$

or

$$e^{20k} = 2.$$

We can solve this for k,

$$20k = \ln 2, \quad \Rightarrow k = \frac{\ln 2}{20} \approx 0.035.$$

This gives an approximate solution, $P(t) \approx 20e^{.035t}$. Now we can answer the original question. Namely, $P(30) \approx 57$.

Of course, we could get an exact solution. With some simple manipulations, we have

$$\begin{aligned} P(t) &= 20e^{kt} \\ &= 20e^{\left(\frac{\ln 2}{20}\right)t} \\ &= 20\left(e^{\ln 2}\right)^{\frac{t}{20}} \\ &= 20\left(2^{\frac{t}{20}}\right). \end{aligned} \tag{1.30}$$

This answer takes the general form for population doubling, $P(t) = P_0 2^{\frac{t}{\tau}}$, where τ is the doubling rate.

Radioactive decay problems.

Another standard growth-decay problem is radioactive decay. Certain isotopes are unstable and the nucleus breaks apart, leading to nuclear decay. The products of the decay may also be unstable and undergo further nuclear decay. As an example, Uranium-238 (U-238) decays into Thorium-234 (Th-234). Thorium-234 is unstable and decays into Protactinium (Pa-234). This in turn decays in many steps until lead (Pb-206) is produced as shown in Table 1.1. This lead isotope is stable and the decay process stops. While this is one form of radioactive decay, there are other types. For example, Radon

Isotope	Half-life
U^{238}	$4.468x10^9$ years
Th^{234}	24.1 days
Pa^{234m}	1.17 minutes
U^{234}	$2.47x10^5$ years
Th^{230}	$8.0x10^4$ years
Ra^{226}	1602 years
Rn^{222}	3.823 days
Po^{218}	3.05 minutes
Pb^{214}	26.8 minutes
Bi^{214}	19.7 minutes
Po^{214}	164 microsec
Pb^{210}	21 years
Bi^{210}	5.01 days
Po^{210}	138.4 days
Pb^{206}	Stable

Table 1.1: U-238 decay chain.

222 (Rn-222) gives up an alpha particle (helium nucleus) leaving Polonium (Po-218).

Given a certain amount of radioactive material, it does not all decay at one time. A measure of the tendency of a nucleus to decay is called the half-life. This is the time it takes for half of the material to decay. This is similar to the last example and can be understood using a simple example.

Example 1.8. If 150.0 g of Thorium-234 decays to 137.6 g of Thorium-234 in three days, what is its half-life?

This is another simple decay process. If $Q(t)$ represents the quantity of unstable material, then $Q(t)$ satisfies the rate equation

$$\frac{dQ}{dt} = kQ$$

with $k < 0$. The solution of the initial value problem, as we have seen, is $Q(t) = Q_0 e^{kt}$.

Now, let the half-life be given by τ. Then, $Q(\tau) = \frac{1}{2}Q_0$. Inserting this fact into the solution, we have

$$\begin{aligned} Q(\tau) &= Q_0 e^{k\tau} \\ \frac{1}{2}Q_0 &= Q_0 e^{k\tau} \\ \frac{1}{2} &= e^{k\tau}. \end{aligned} \tag{1.31}$$

Noting that $Q(t) = Q_0 \left(e^k\right)^t$, we solve Equation (1.31) for

$$e^k = 2^{-1/\tau}.$$

Then, the solution can be written in the general form

$$Q(t) = Q_0 2^{-\frac{t}{\tau}}.$$

Note that the decay constant is $k = -\frac{\ln 2}{\tau} < 0$.

Returning to the problem, we are given

$$Q(3) = 1502^{-\frac{3}{\tau}} = 137.6.$$

Solving to τ,

$$\begin{aligned} 2^{-\frac{3}{\tau}} &= \frac{136.7}{150} \\ -3\ln 2 &= \ln .9173\tau \\ \tau &= -\frac{3\ln 2}{\ln .9173} = 24.09. \end{aligned} \tag{1.32}$$

Therefore, the half-life is about 24.1 days.

1.3.2 Newton's Law of Cooling

If you take your hot cup of tea, and let it sit in a cold room, the tea will cool off and reach room temperature after a period of time. The law of cooling is attributed to Isaac Newton (1642-1727) who was probably the first to state results on how bodies cool.[1] The main idea is that a body at temperature $T(t)$ is initially at temperature $T(0) = T_0$. It is placed in an environment at an ambient temperature of T_a. A simple model is given that the rate of change of the temperature of the body is proportional to the difference between the body temperature and its surroundings. Thus, we have

$$\frac{dT}{dt} \propto T - T_a.$$

The proportionality is removed by introducing a cooling constant,

$$\frac{dT}{dt} = -k(T - T_a), \tag{1.33}$$

where $k > 0$.

This differential equation can be solved by noting that the equation can be written in the form

$$\frac{d}{dt}(T - T_a) = -k(T - T_a).$$

This is now of the form of exponential decay of the function $T(t) - T_a$. The solution is easily found as

$$T(t) - T_a = (T_0 - T_a)e^{-kt},$$

or

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

Example 1.9. A cup of tea at 90^oC cools to 85^oC in ten minutes. If the room temperature is 22^oC, what is its temperature after 30 minutes?

Using the general solution with $T_0 = 90^o$C,

$$T(t) = 22 + (90 - 22)e^{-k} = 22 + 68e^{-kt},$$

[1] Newton's 1701 Law of Cooling is an approximation to how bodies cool for small temperature differences ($T - T_a \ll T$) and does not take into account all of the cooling processes. One account is given by C. T. O'Sullivan, Am. J. Phys (1990) p 956-960.

we then find k using the given information, $T(10) = 85^o$C. We have

$$\begin{aligned} 85 &= T(10) \\ &= 22 + 68e^{-10k} \\ 63 &= 68e^{-10k} \\ e^{-10k} &= \frac{63}{68} \approx 0.926 \\ -10k &= \ln 0.926 \\ k &= -\frac{\ln 0.926}{10} = 0.00764. \end{aligned} \tag{1.34}$$

This gives the equation for this model as

$$T(t) = 22 + 68e^{-0.00764t}.$$

Now we can answer the question. What is $T(30)$?

$$T(30) = 22 + 68e^{-0.00764(30)} = 76^o\text{C}.$$

1.3.3 Terminal Velocity

Now let's return to free fall. What if there is air resistance? We first need to model the air resistance. As an object falls faster and faster, the drag force becomes greater. So, this resistive force is a function of the velocity. There are a couple of standard models that people use to test this. The idea is to write $F = ma$ in the form

$$m\ddot{y} = -mg + f(v), \tag{1.35}$$

where $f(v)$ gives the resistive force and mg is the weight. Recall that this applies to free fall near the Earth's surface. Also, for it to be resistive, $f(v)$ should oppose the motion. If the body is falling, then $f(v)$ should be positive. If it is rising, then $f(v)$ would have to be negative to indicate the opposition to the motion.

One common determination derives from the drag force on an object moving through a fluid. This force is given by

$$f(v) = \frac{1}{2}CA\rho v^2, \tag{1.36}$$

where C is the drag coefficient, A is the cross sectional area and ρ is the fluid density. For laminar flow the drag coefficient is constant.

Unless you are into aerodynamics, you do not need to get into the details of the constants. So, it is best to absorb all of the constants into one to simplify the computation. So, we will write $f(v) = bv^2$. The differential equation including drag can then be rewritten as

$$\dot{v} = kv^2 - g, \tag{1.37}$$

where $k = b/m$. Note that this is a first order equation for $v(t)$. It is separable too!

Formally, we can separate the variables and integrate over time to obtain

$$t + K = \int^{v} \frac{dz}{kz^2 - g}. \tag{1.38}$$

This is the first use of Partial Fraction Decomposition. We will explore this method further in the section on Laplace Transforms.

(Note: We used an integration constant of K since C is the drag coefficient in this problem.) If we can do the integral, then we have a solution for v. In fact, we can do this integral. You need to recall another common method of integration, which we have not reviewed yet. Do you remember Partial Fraction Decomposition? It involves factoring the denominator in the integral. In the simplest case there are two linear factors in the denominator and the integral is rewritten:

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{b-a} \int \left[\frac{1}{x-a} - \frac{1}{x-b} \right] dx \tag{1.39}$$

The new integral now has two terms which can be readily integrated.

In order to factor the denominator in the current problem, we first have to rewrite the constants. We let $\alpha^2 = g/k$ and write the integrand as

$$\frac{1}{kz^2 - g} = \frac{1}{k} \frac{1}{z^2 - \alpha^2}. \tag{1.40}$$

Now we use a partial fraction decomposition to obtain

$$\frac{1}{kz^2 - g} = \frac{1}{2\alpha k} \left[\frac{1}{z-\alpha} - \frac{1}{z+\alpha} \right]. \tag{1.41}$$

Now, the integrand can be easily integrated giving

$$t + K = \frac{1}{2\alpha k} \ln \left| \frac{v-\alpha}{v+\alpha} \right|. \tag{1.42}$$

Solving for v, we have

$$v(t) = \frac{1 - Be^{2\alpha k t}}{1 + Be^{2\alpha k t}} \alpha, \tag{1.43}$$

where $B \equiv e^K$. B can be determined using the initial velocity.

There are other forms for the solution in terms of a tanh function, which the reader can determine as an exercise. One important conclusion is that for large times, the ratio in the solution approaches -1. Thus, $v \to -\alpha = -\sqrt{\frac{g}{k}}$ as $t \to \infty$. This means that the falling object will reach a constant terminal velocity.

As a simple computation, we can determine the terminal velocity. We will take an 80 kg skydiver with a cross sectional area of about 0.093 m^2. (The skydiver is falling head first.) Assume that the air density is a constant 1.2 $\mathrm{kg/m}^3$ and the drag coefficient is $C = 2.0$. We first note that

$$v_{\text{terminal}} = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{2mg}{CA\rho}}.$$

So,

$$v_{\text{terminal}} = -\sqrt{\frac{2(70)(9.8)}{(2.0)(0.093)(1.2)}} = -78 m/s.$$

This is about 175 mph, which is slightly higher than the actual terminal velocity of a sky diver with arms and feet fully extended. One would need a more accurate determination of C and A for a more realistic answer. Also, the air density varies along the way.

1.3.4 Mixture Problems

MIXTURE PROBLEMS OFTEN OCCUR IN A FIRST COURSE on differential equations as examples of first order differential equations. In such problems we consider a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake. The goal is to prdict the amount of salt, or pollutant, at some later time.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\boxed{\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out.}}$$

The rates are not often given. One is generally given information about the concentration and flow rates in and out of the system. If one pays attention to the dimentsion and sketches the situation, then one can write out this rate equation as a first order differential equation. We consider a simple example.

Example 1.10. Single Tank Problem

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 6.23.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.

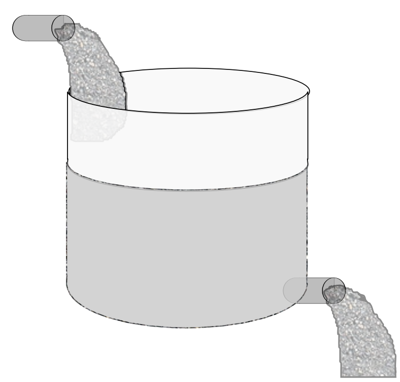

Figure 1.4: A typical mixing problem.

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = 10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x \text{ pounds}}{50 \text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = \frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is solved using the methods for linear first order equations. The integrating factor is $\mu = e^{x/10}$, leading to the general solution

$$x(t) = 100 + Ae^{-t/10}.$$

Using the initial condition, one finds the particular solution

$$x(t) = 100\left(1 - e^{-t/10}\right).$$

Often one is interested in the long time behavior of a system. In this case we have that $\lim_{t\to\infty} x(t) = 100$ lb. This makes sense because 2 pounds per galloon enter during this time to eventually leave the entire 50 gallons with this concentration. Thus,

$$50 \text{ gal} \times 2\frac{\text{lb}}{50 \text{ gal}} = 100 \text{ lb}.$$

1.3.5 *Orthogonal Trajectories of Curves*

THERE ARE MANY PROBLEMS FROM GEOMETRY which have lead to the study of differential equations. One such problem is the construction of orthogonal trajectories. Give a a family of curves, $y_1(x;a)$, we seek another family of curves $y_2(x;c)$ such that the second family of curves are perpendicular the to given family. This means that the tangents of two intersecting curves at the point of intersection are perpendicular to each other. The slopes of the tangent lines are given by the derivatives $y_1'(x)$ and $y_2'(x)$. We recall from elementary geometry that the slopes of two perpendicular lines are related by

$$y_2'(x) = -\frac{1}{y_1'(x)}.$$

Example 1.11. Find a family of orthogonal trajectories to the family of parabolae $y_1(x;a) = ax^2$.

We note that the new collection of curves has to satisfy the equation

$$y_2'(x) = -\frac{1}{y_1'(x)} = -\frac{1}{2ax}.$$

Before solving for $y_2(x)$, we need to eliminate the parameter a. From the give function, we have that $a = \frac{y}{x^2}$. Inserting this into the equation for y_2', we have

$$y'(x) = -\frac{1}{2ax} = -\frac{x}{2y}.$$

Thus, to find $y_2(x)$, we have to solve the differential equation

$$2yy' + x = 0.$$

Noting that $(y^2)' = 2yy'$ and $(\frac{1}{2}x^2)' = x,$, this (exact) equation can be written as

$$\frac{d}{dx}\left(y^2 + \frac{1}{2}x^2\right) = 0.$$

Integrating, we find the family of solutions,

$$y^2 + \frac{1}{2}x^2 = k.$$

In Figure 1.5 we plot both families of orthogonal curves.

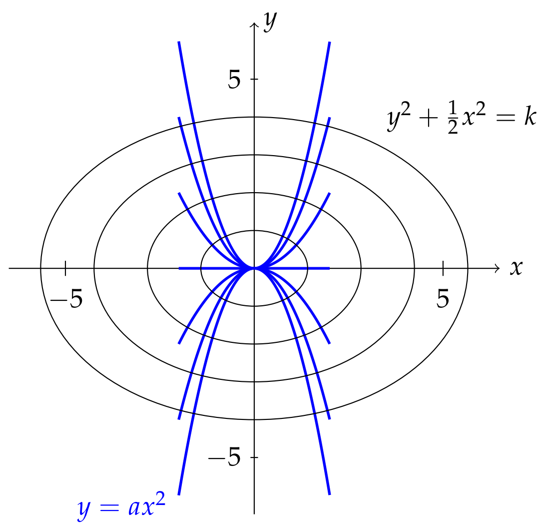

Figure 1.5: Plot of orthogonal families of curves, $y = ax^2$ and $y^2 + \frac{1}{2}x^2 = k$.

1.3.6 *Pursuit Curves**

ANOTHER APPLICATION THAT IS INTERESTING IS TO FIND the path that a body traces out as it moves towards a fixed point or another moving body. Such curses are know as pursuit curves. These could model aircraft or submarines following targets, or predators following prey. We demonstrate this with an example.

Example 1.12. A hawk at point (x, y) sees a sparrow traveling at speed v along a straight line. The hawk flies towards the sparrow at constant speed w but always in a direction along line of sight between their positions. If the hawk starts out at the point $(a, 0)$ at $t = 0$, when the sparrow is at $(0, 0)$, then what is the path the hawk needs to follow? Will the hawk catch the sparrow? The situation is shown in Figure 1.6. We pick the path of the sparrow to be along the $y-$axis. Therefore, the sparrow is at position $(0, vt)$.

First we need the equation of the line of sight between the points (x, y) and $(0, vt)$. Considering that the slope of the line is the same as the slope of the tangent to the path, $y = y(x)$, we have

$$y' = \frac{y - vt}{x}.$$

The hawk is moving at a constant speed, w. Since the speed is related to the time through the distance the hawk travels. we need to

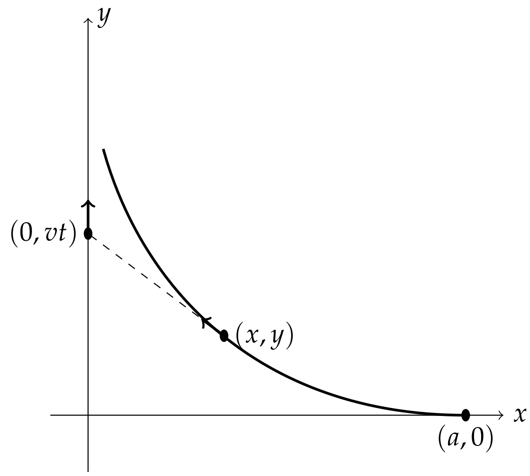

Figure 1.6: A hawk at point (x, y) sees a sparrow at point $(0, vt)$ and always follows the straight line between these points.

find the arclength of the path between $(a, 0)$ and (x, y). This is given by

$$L = \int ds = \int_x^a \sqrt{1 + [y'(x)]^2}\, dx.$$

The distance is related to the speed, w, and the time, t, by $L = wt$.

Eliminating the time using $y' = \frac{y - vt}{x}$, we have

$$\int_x^a \sqrt{1 + [y'(x)]^2}\, dx = \frac{w}{v}(y - xy').$$

Furthermore, we can differentiate this result with respect to x to get rid of the integral,

$$\sqrt{1 + [y'(x)]^2} = \frac{w}{v} xy''.$$

Even though this is a second order differential equation for $y(x)$, it is a first order separable equation in the speed function $z(x) = y'(x)$. Namely,

$$\frac{w}{v} xz' = \sqrt{1 + z^2}.$$

Separating variables, we find

$$\frac{w}{v} \int \frac{dz}{\sqrt{1 + z^2}} = \int \frac{dx}{x}.$$

The integrals can be computed using standard methods from calculus. We can easily integrate the right hand side,

$$\int \frac{dx}{x} = \ln |x| + c_1.$$

The left hand side takes a little extra work, or looking the value up in Tables or using a CAS package. Recall a trigonometric substitution is in order. [See the Appendix.] We let $z = \tan\theta$. Then $dz = \sec^2\theta\, d\theta$. The methods proceeds as follows:

$$\int \frac{dz}{\sqrt{1 + z^2}} = \int \frac{\sec^2\theta}{\sqrt{1 + \tan^2\theta}}\, d\theta$$

$$\begin{aligned} &= \int \sec\theta \, d\theta \\ &= \ln(\tan\theta + \sec\theta) + c_2 \\ &= \ln(z + \sqrt{1+z^2}) + c_2. \end{aligned} \tag{1.44}$$

Putting these together, we have for $x > 0$,

$$\ln(z + \sqrt{1+z^2}) = \frac{v}{w}\ln x + C.$$

Using the initial condition $z = y' = 0$ and $x = a$ at $t = 0$,

$$0 = \frac{v}{w}\ln a + C,$$

or $C = -\frac{v}{w}\ln a$.

Using this value for c, we find

$$\begin{aligned} \ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln x - \frac{v}{w}\ln a \\ \ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln\frac{x}{a} \\ \ln(z + \sqrt{1+z^2}) &= \ln\left(\frac{x}{a}\right)^{\frac{v}{w}} \\ z + \sqrt{1+z^2} &= \left(\frac{x}{a}\right)^{\frac{v}{w}}. \end{aligned} \tag{1.45}$$

We can solve for $z = y'$, to find

$$y' = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{v}{w}} - \left(\frac{x}{a}\right)^{-\frac{v}{w}}\right]$$

Integrating,

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + k.$$

The integration constant, k, can be found knowing $y(a) = 0$. This gives

$$\begin{aligned} 0 &= \frac{a}{2}\left[\frac{1}{1+\frac{v}{w}} - \frac{1}{1-\frac{v}{w}}\right] + k \\ k &= \frac{a}{2}\left[\frac{1}{1-\frac{v}{w}} - \frac{1}{1+\frac{v}{w}}\right] \\ &= \frac{avw}{w^2 - v^2}. \end{aligned} \tag{1.46}$$

The full solution for the path is given by

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + \frac{avw}{w^2 - v^2}.$$

Can the hawk catch the sparrow? This would happen if there is a time when $y(0) = vt$. Inserting $x = 0$ into the solution, we have $y(0) = \frac{avw}{w^2 - v^2} = vt$. This is possible if $w > v$.

1.4 Other First Order Equations*

There are several nonlinear first order equations whose solution can be obtained using special techniques. We conclude this chapter by looking at a few of these equations named after famous mathematicians of the 17-18th century inspired by various applications.

1.4.1 Bernoulli Equation*

We begin with the Bernoulli equation, named after Jacob Bernoulli (1655-1705). The Bernoulli equation is of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 0, 1.$$

Note that when $n = 0, 1$ the equation is linear and can be solved using an integrating factor. The key to solving this equation is using the transformation $z(x) = \frac{1}{y^{n-1}(x)}$ to make the equation for $z(x)$ linear. We demonstrate the procedure using an example.

The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler (1707-1783) and Daniel Bernoulli, who is known for his work in hydrodynamics.

Example 1.13. Solve the Bernoulli equation $xy' + y = y^2 \ln x$ for $x > 0$.

In this example $p(x) = 1$, $q(x) = \ln x$, and $n = 2$. Therefore, we let $z = \frac{1}{y}$. Then,

$$z' = -\frac{1}{y^2}y' = z^2 y'.$$

Inserting $z = y^{-1}$ and $z' = z^2 y'$ into the differential equation, we have

$$\begin{aligned} xy' + y &= y^2 \ln x \\ -x\frac{z'}{z^2} + \frac{1}{z} &= \frac{\ln x}{z^2} \\ -xz' + z &= \ln x \\ z' - \frac{1}{x}z = -\frac{\ln x}{x}. \end{aligned} \tag{1.47}$$

Thus, the resulting equation is a linear first order differential equation. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \frac{1}{x}.$$

Multiplying the differential equation by the integrating factor, we have

$$\left(\frac{z}{x}\right)' = \frac{\ln x}{x^2}.$$

Integrating, we obtain

$$\begin{aligned} \frac{z}{x} &= -\int \frac{\ln x}{x^2} + C \\ &= \frac{\ln x}{x} + \int \frac{dx}{x^2} + C \\ &= \frac{\ln x}{x} + \frac{1}{x} + C. \end{aligned} \tag{1.48}$$

Multiplying by x, we have $z = \ln x + 1 + Cx$. Since $z = y^{-1}$, the general solution to the problem is

$$y = \frac{1}{\ln x + 1 + Cx}.$$

1.4.2 *Lagrange and Clairaut Equations**

ALEXIS CLAUDE CLAIRAUT (1713-1765) SOLVED the differential equation

$$y = xy' + g(y').$$

This is a special case of the family of Lagrange equations,

$$y = xf(y') + g(y'),$$

named after Joseph Louis Lagrange (1736-1813). These equations also have solutions called singular solutions. Singular solution are solutions for which there is a failure of uniqueness to the initial value problem at every point on the curve. A singular solution is often one that is tangent to every solution in a family of solutions.

Lagrange equations, $y = xf(y') + g(y')$.

First, we consider solving the more general Lagrange equation. Let $p = y'$ in the Lagrange equation, giving

$$y = xf(p) + g(p). \tag{1.49}$$

Next, we differentiate with respect to x to find

$$y' = p = f(p) + xf'(p)p' + g'(p)p'.$$

Here we used the Chain Rule. For example,

$$\frac{dg(p)}{dx} = \frac{dg}{dp}\frac{dp}{dx}.$$

Solving for p', we have

$$\frac{dp}{dx} = \frac{p - f(p)}{xf'(p) + g'(p)}. \tag{1.50}$$

We have introduced $p = p(x)$, viewed as a function of x. Let's assume that we can invert this function to find $x = x(p)$. Then, from introductory calculus, we know that the derivatives of a function and its inverse are related,

$$\frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}.$$

Applying this to Equation (1.50), we have

$$\begin{aligned} \frac{dx}{dp} &= \frac{xf'(p) + g'(p)}{p - f(p)} \\ x' - \frac{f'(p)}{p - f(p)}x &= \frac{g'(p)}{p - f(p)}, \end{aligned} \tag{1.51}$$

assuming that $p - f(p) \neq 0$.

As can be seen, we have transformed the Lagrange equation into a first order linear differential equation (1.51) for $x(p)$. Using methods from earlier in the chapter, we can in principle obtain a family of solutions

$$x = F(p, C),$$

where C is an arbitrary integration constant. Using Equation (1.49), one might be able to eliminate p in Equation (1.51) to obtain a family of solutions of the Lagrange equation in the form

$$\varphi(x, y, C) = 0.$$

If it is not possible to eliminate p from Equations (1.49) and (1.51), then one could report the family of solutions as a parametric family of solutions with p the parameter. So, the parametric solutions would take the form

$$\begin{aligned} x &= F(p, C), \\ y &= F(p, C) f(p) + g(p). \end{aligned} \tag{1.52}$$

Singular solutions are possible for Lagrange equations.

We had also assumed the $p - f(p) \neq 0$. However, there might also be solutions of Lagrange's equation for which $p - f(p) = 0$. Such solutions are called singular solutions.

Example 1.14. Solve the Lagrange equation $y = 2xy' - y'^2$.

We will start with Equation (1.51). Noting that $f(p) = 2p$, $g(p) = -p^2$, we have

$$\begin{aligned} x' - \frac{f'(p)}{p - f(p)} x &= \frac{g'(p)}{p - f(p)} \\ x' - \frac{2}{p - 2p} x &= \frac{-2p}{p - 2p} \\ x' + \frac{2}{p} x &= 2. \end{aligned} \tag{1.53}$$

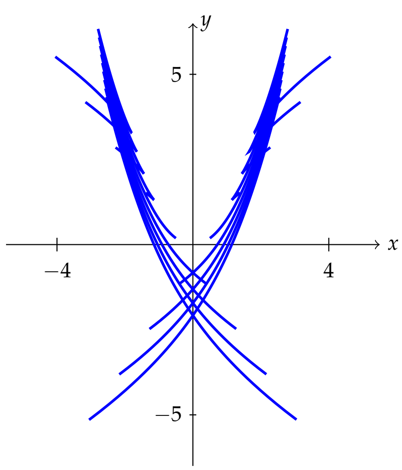

Figure 1.7: Family of solutions of the Lagrange equation $y = 2xy' - y'^2$.

This first order linear differential equation can be solved using an integrating factor. Namely,

$$\mu(p) = \exp\left(\int \frac{2}{p}\, dp\right) = e^{2 \ln p} = p^2.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dp}\left(xp^2\right) = 2p^2.$$

Integrating,

$$xp^2 = \frac{2}{3}p^3 + C.$$

This gives the general solution

$$x(p) = \frac{2}{3}p + \frac{C}{p^2}.$$

Replacing $y' = p$ in the original differential equation, we have $y = 2xp - p^2$. The family of solutions is then given by the parametric equations

$$\begin{aligned} x &= \frac{2}{3}p + \frac{C}{p^2}, \\ y &= 2\left(\frac{2}{3}p + \frac{C}{p^2}\right)p - p^2 \\ &= \frac{1}{3}p^2 + \frac{2C}{p}. \end{aligned} \tag{1.54}$$

The plots of these solutions is shown in Figure 1.7.

We also need to check for a singular solution. We solve the equation $p - f(p) = 0$, or $p = 0$. This gives the solution $y(x) = (2xp - p^2)_{p=0} = 0$.

The Clairaut differential equation is given by

Clairaut equations, $y = xy' + g(y')$.

$$y = xy' + g(y').$$

Letting $p = y'$, we have

$$y = xp + g(p).$$

This is the Lagrange equation with $f(p) = p$. Differentiating with respect to x,

$$p = p + xp' + g'(p)p'.$$

Rearranging, we find

$$x = -g'(p)$$

So, we have the parametric solution

$$\begin{aligned} x &= -g'(p), \\ y &= -pg'(p) + g(p). \end{aligned} \tag{1.55}$$

For the case that $y' = C$, it can be seen that $y = Cx + g(C)$ is a general solution solution.

Example 1.15. Find the solutions of $y = xy' - y'^2$.

As noted, there is a family of straight line solutions $y = Cx - C^2$, since $g(p) = -p^2$. There might also by a parametric solution not contained n this family. It would be given by the set of equations

$$\begin{aligned} x &= -g'(p) = 2p, \\ y &= -pg'(p) + g(p) = 2p^2 - p^2 = p^2. \end{aligned} \tag{1.56}$$

Eliminating p, we have the parabolic curve $y = x^2/4$.

In Figure 1.8 we plot these solutions. The family of straight line solutions are shown in blue. The limiting curve traced out, much like string figures one might create, is the parametric curve.

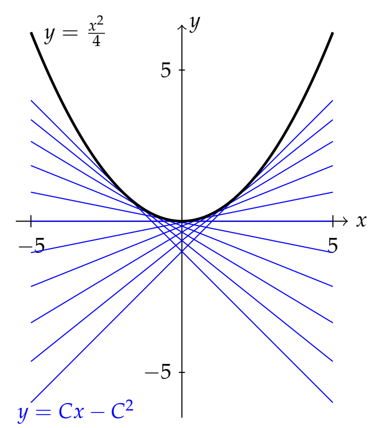

Figure 1.8: Plot of solutions to the Clairaut equation $y = xy' - y'^2$. The straight line solutions are a family of curves whose limit is the parametric slution.

1.4.3 *Riccati Equation**

Jacopo Francesco Riccati (1676-1754) studied curves with some specified curvature. He proposed an equation of the form

$$y' + a(x)y^2 + b(x)y + c(x) = 0$$

around 1720. He communicated this to the Bernoulli's. It was Daniel Bernoulli who had actually solved this equation. As noted by Ranjan Roy (2011), Riccati had published his equation in 1722 with a note that D. Bernoulli giving the solution in terms of an anagram. Furthermore, when $a \equiv 0$, the Riccati equation reduces to a Bernoulli equation.

In Section 7.2.1, we will show that the Ricatti equation can be transformed into a second order linear differential equation. However, there are special cases in which we can get our hands on the solutions. For example, if a, b, and c are constants, then the differential equation can be integrated directly. We have

$$\frac{dy}{dx} = -(ay^2 + by + c).$$

This equation is separable and we obtain

$$x - C = -\int \frac{dy}{ay^2 + by + c}.$$

When a differential equation is left in this form, it is said to be solved by quadrature when the resulting integral in principle can be computed in terms of elementary functions.[2]

[2] By elementary functions we mean well known functions like polynomials, trigonometric, hyperbolic, and some not so well know to undergraduates, such as Jacobi or Weierstrass elliptic functions.

If a particular solution is known, then one can obtain a solution to the Riccati equation. Let the known solution be $y_1(x)$ and assume that the general solution takes the form $y(x) = y_1(x) + z(x)$ for some unknown function $z(x)$. Substituting this form into the differential equation, we can show that $v(x) = 1/z(x)$ satisfies a first order linear differential equation.

Inserting $y = y_1 + z$ into the general Riccati equation, we have

$$\begin{aligned}
0 &= \frac{dy}{dx} + a(x)y^2 + b(x)y + c \\
&= \frac{dz}{dx} + az^2 + 2azy_1 + bz + \\
&\quad + \frac{dy_1}{dx} + ay_1^2 + by_1 + c \\
&= \frac{dz}{dx} + a(x)[2y_1 z + z^2] + b(x)z \\
-a(x)z^2 &= \frac{dz}{dx} + [2a(x)y_1 + b(x)]z. \qquad (1.57)
\end{aligned}$$

The last equation is a Bernoulli equation with $n = 2$. So, we can make it a linear equation with the substitution $z = \frac{1}{v}$, $z' = -\frac{z'}{v^2}$. Then, we obtain a differential equation for $v(x)$. It is given by

$$v' - (2a(x)y_1(x) + b(x))v = a(x).$$

Example 1.16. Find the general solution of the Riccati equation, $y' - y^2 + 2e^x y - e^{2x} - e^x = 0$, using the particular solution $y_1(x) = e^x$.

We let the sought solution take the form $y(x) = z(x) + e^x$. Then, the equation for $z(x)$ is found as

$$\frac{dz}{dx} = z^2.$$

This equation is simple enough to integrate directly to obtain $z = \frac{1}{C-x}$. Then, the solution to the problem becomes

$$y(x) = \frac{1}{C-x} + e^x.$$

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1$.

h. $x' - 2x = te^{2t}$.

i. $\dfrac{dy}{dx} + y = \sin x$, $y(0) = 0$.

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3$, $y(1) = 4$.

2. For the following determine if the differential equation is exact. If it is not exact, find the integrating factor. Integrate the equations to obtain solutions.

a. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$.

b. $(x + y^2)\,dx - 2xy\,dy = 0$.

c. $(\sin xy + xy\cos xy)\,dx + x^2\cos xy\,dy = 0$.

d. $(x^2 + y)\,dx - x\,dy = 0$.

e. $(2xy^2 - 3y^3)\,dx + (7 - 3xy^2)\,dy = 0$.

3. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

4. A ball is thrown upward with an initial velocity of 49 m/s from 539 m high. How high does the ball get and how long does in take before it hits the ground? [Use results from the simple free fall problem, $y'' = -g$.]

5. Consider the case of free fall with a damping force proportional to the velocity, $f_D = \pm kv$ with $k = 0.1$ kg/s.

a. Using the correct sign, consider a 50 kg mass falling from rest at a height of 100m. Find the velocity as a function of time. Does the mass reach terminal velocity?

b. Let the mass be thrown upward from the ground with an initial speed of 50 m/s. Find the velocity as a function of time as it travels upward and then falls to the ground. How high does the mass get? What is its speed when it returns to the ground?

6. An piece of a satellite falls to the ground from a height of 10,000 m. Ignoring air resistance, find the height as a function of time. [Hint: For free fall from large distances,

$$\ddot{h} = -\frac{GM}{(R+h)^2}.$$

Multiplying both sides by $\dot{h}$, show that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{h}^2\right) = \frac{d}{dt}\left(\frac{GM}{R+h}\right).$$

Integrate and solve for $\dot{h}$. Further integrating gives $h(t)$.]

7. The problem of growth and decay is stated as follows: The rate of change of a quantity is proportional to the quantity. The differential equation for such a problem is

$$\frac{dy}{dt} = \pm ky.$$

The solution of this growth and decay problem is $y(t) = y_0 e^{\pm kt}$. Use this solution to answer the following questions if forty percent of a radioactive substance disappears in 100 years.

a. What is the half-life of the substance?

b. After how many years will 90% be gone?

8. Uranium 237 has a half-life of 6.78 days. If there are 10.0 grams of U-237 now, then how much will be left after two weeks?

9. The cells of a particular bacteria culture divide every three and a half hours. If there are initially 250 cells, how many will there be after ten hours?

10. The population of a city has doubled in 25 years. How many years will it take for the population to triple?

11. Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}$.

b. $y = 2xy' + \ln y'$.

c. $y' + 2xy = 2xy^2$.

d. $y' + 2xy = y^2 e^{x^2}$.

12. Find the general solution of the Riccati equation given the particular solution.

a. $xy' - y^2 + (2x+1)y = x^2 + 2x$, $y_1(x) = x$.

b. $y'e^{-x} + y^2 - 2ye^x = 1 - e^{2x}$, $y_1(x) = e^x$.

A function $F(x, y)$ is said to be homogeneous of degree k if $F(tx, ty) = t^k F(x, y)$.

13. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in this chapter. The function on the right-hand side is a homogeneous function of degree zero. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

14. If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, then M/N can be written as a function of y/x. This suggests that a substitution of $y(x) = xz(x)$ into $M(x, y)\,dx + N(x, y)\,dy$ might simplify the equation. For the following problems use this method to find the family of solutions.

a. $(x^2 - xy + y^2)\,dx - xy\,dy = 0$.

b. $xy\,dx - (x^2 + y^2)\,dy = 0$.

c. $(x^2 + 2xy - 4y^2)\,dx - (x^2 - 8xy - 4y^2)\,dy = 0$.

15. Find the family of orthogonal curves to the given family of curves.

a. $y = ax$

b. $y = ax^2$.

c. $x^2 + y^2 = 2ax$.

16. The temperature inside your house is 70^{o}F and it is 30^{o}F outside. At 1:00 A.M. the furnace breaks down. At 3:00 A.M. the temperature in the house has dropped to 50^{o}F. Assuming the outside temperature is constant and that Newton's Law of Cooling applies, determine when the temperature inside your house reaches 40^{o}F.

17. A body is discovered during a murder investigation at 8:00 P.M. and the temperature of the body is 70^oF. Two hours later the body temperature has dropped to 60^oF in a room that is at 50^oF. Assuming that Newton's Law of Cooling applies and the body temperature of the person was 98.6^oF at the time of death, determine when the murder occurred.

18. Newton's Law of Cooling states that the rate of heat loss of an object is proportional to the temperature gradient, or

$$\frac{dQ}{dt} = hA\Delta T,$$

where Q is the thermal energy, h is the heat transfer coefficient, A is the surface area of the body, and $\Delta T = T - T_a$. If $Q = CT$, where C is the heat capacity, then we recover Equation (1.33) with $k = hA/C$.

However, there are modifications which include convection or radiation. Solve the following models and compare the solution behaviors.

a. Newton $T' = -k(T - T_a)$

b. Dulong-Petit $T' = -k(T - T_a)^{5/4}$

c. Newton-Stefan $T' = -k(T - T_a) - \epsilon\sigma(T^4 - T_a^4) \approx -k(T - T_a) - b(T - T_a)^2$.

19. Initially a 200 gallon tank is filled with pure water. At time $t = 0$ a salt concentration with 3 pounds of salt per gallon is added to the container at the rate of 4 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

d. Assuming that the tank holds much more than 200 gallons, and everything is the same except that the mixture is drained at 3 gallons per minute, what would the answers to parts a and b become?

20. You make two gallons of chili for a party. The recipe calls for two teaspoons of hot sauce per gallon, but you had accidentally put in two tablespoons per gallon. You decide to feed your guests the chili anyway. Assume that the guests take 1 cup/min of chili and you replace what was taken with beans and tomatoes without any hot sauce. [1 gal = 16 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of hot sauce as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the chili back to the recipe's suggested concentration?