Solving Linear Systems of Differential Equations:

You are given a linear system of differential equations:

$$\frac{d}{dt}\mathbf{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{Y} = \mathbf{AY}, \quad \mathbf{Y}(0) = \mathbf{Y}_0.$$

The type of behavior depends upon the eigenvalues of matrix **A**. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If you have an initial condition, you can determine your two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two linearly independent solutions, then the general solution is given as $\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t)$. Then, setting $t = 0$, you get two linear equations for c_1 and c_2 : $c_1\mathbf{Y}_1(0) + c_2\mathbf{Y}_2(0) = \mathbf{Y}_0$.

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that you obtain from solving the characteristic equation, $\det(\mathbf{Y} - \lambda\mathbf{I}) = 0$.

I. Two real, distinct roots.

a. Solve the eigenvalue problem $\mathbf{AV} = \lambda\mathbf{V}$ for each eigenvalue obtaining two eigenvectors $\mathbf{V_1}, \mathbf{V_2}$.

b. Write the general solution as a linear combination $\mathbf{Y}(t) = c_1 e^{\lambda_1 t}\mathbf{V}_1 + c_2 e^{\lambda_2 t}\mathbf{V_2}$

II. Two complex conjugate roots.

a. Solve the eigenvalue problem $\mathbf{AV} = \lambda\mathbf{V}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{V}$. Note that this eigenvector may have complex entries.

b. Write the vector $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{V}$.

c. Construct two linearly independents solutions to the problem by letting $\mathbf{V_1} = \operatorname{Re}(\mathbf{Y}(t))$ and $\mathbf{V}_2 = \operatorname{Im}(\mathbf{Y}(t))$.

Note that since the original system of equations does not have any *i*'s, then we have $\frac{d}{dt}[\operatorname{Re}(\mathbf{Y}(t)) + i\operatorname{Im}(\mathbf{Y}(t))] = \mathbf{A}[\operatorname{Re}(\mathbf{Y}(t)) + i\operatorname{Im}(\mathbf{Y}(t))]$. Differentiating the sum and splitting the real and imaginary parts of the equation, gives $\frac{d}{dt}\operatorname{Re}(\mathbf{Y}(t)) + i\frac{d}{dt}\operatorname{Im}(\mathbf{Y}(t)) = \mathbf{A}[\operatorname{Re}(\mathbf{Y}(t))] + i\mathbf{A}[\operatorname{Im}(\mathbf{Y}(t))]$. Setting the real and imaginary parts equal, we have $\frac{d}{dt}\operatorname{Re}(\mathbf{Y}(t)) = \mathbf{A}[\operatorname{Re}(\mathbf{Y}(t))]$, and $\frac{d}{dt}\operatorname{Im}(\mathbf{Y}(t)) = \mathbf{A}[\operatorname{Im}(\mathbf{Y}(t))]$. Therefore, the real and imaginary parts each are solution of the system!

d. Write the general solution as a linear combination $\mathbf{Y}(t) = c_1\mathbf{V}_1 + c_2\mathbf{V_2}$

III. One Repeated Root

a. Solve the eigenvalue problem $\mathbf{AV} = \lambda \mathbf{V}$ for the one eigenvalue obtaining the first eigenvector $\mathbf{V}_1$.

b. Solve the eigenvalue problem $\mathbf{AV}_2 - \lambda \mathbf{V}_2 = \mathbf{V}_1$ for $\mathbf{V}_2$.

c. The general solution is then given by $\mathbf{Y}(t) = c_1 e^{\lambda t}\mathbf{V}_1 + c_2 e^{\lambda t}(\mathbf{V}_2 + t\mathbf{V}_1)$.

IV. Examples

a. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix}$$
$$0 = (4-\lambda)(3-\lambda) - 6$$
$$0 = \lambda^2 - 7\lambda + 6$$
$$0 = (\lambda - 1)(\lambda - 6)$$

Thus, $\lambda = 1, 6$.

Eigenvectors:

$\lambda = 1$:

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$3v_1 + 2v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

$\lambda = 6$:

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$-2v_1 + 2v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General Solution:

$$\mathbf{Y}(t) = c_1 e^{\lambda_1 t}\mathbf{V}_1 + c_2 e^{\lambda_2 t}\mathbf{V}_2$$
$$\mathbf{Y}(t) = c_1 e^{t}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2c_1 e^{t} + c_2 e^{6t} \\ -3c_1 e^{t} + c_2 e^{6t} \end{pmatrix}.$$

b. $\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix}$$
$$0 = (3-\lambda)(-1-\lambda) + 6$$
$$0 = \lambda^2 - 2\lambda + 2$$
$$\lambda = \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i.$$

Thus, $\lambda = 1+i, 1-i$.

Eigenvectors:

$$\lambda = 1+i:$$
$$\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1+i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$(2-i)v_1 - 5v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

Complex Solution:

$$\begin{aligned} e^{\lambda t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} &= e^{(1+i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^{t}(\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^{t}\begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^{t}\begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^{t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

General Solution:

$$\begin{aligned} \mathbf{Y}(t) &= c_1 e^{t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\ &= e^{t}\begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1\cos t + c_2\sin t \end{pmatrix}. \end{aligned}$$

Note: This can be rewritten as $\mathbf{Y}(t) = e^{t}\cos t\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^{t}\sin t\begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.$

c. $\mathbf{A} = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}.$

Eigenvalues:

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix}$$
$$0 = (7-\lambda)(1-\lambda) + 9$$
$$0 = \lambda^2 - 8\lambda + 16$$
$$0 = (\lambda - 4)^2$$
.

Thus, $\lambda = 4$.

Eigenvectors:

$$\lambda = 4:$$
$$\begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Second Linearly Independent Solution:

Solve $\mathbf{A}\mathbf{V}_2 - \lambda \mathbf{V}_2 = \mathbf{V}_1$

$$\begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$\left.\begin{matrix} 3u_1 - u_2 = 1 \\ 9u_1 - 3u_2 = 3 \end{matrix}\right\} \quad \Rightarrow \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

General Solution:

$$\mathbf{Y}(t) = c_1 e^{\lambda t}\mathbf{V}_1 + c_2 e^{\lambda t}(\mathbf{V_2} + t\mathbf{V}_1)$$
$$\mathbf{Y}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]$$
$$= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}.$$