van Roomen’s Problem

Adriaan van Roomen (1561-1615) challenged “mathematicians all over the world” to find the solution of a 45th degree polynomial in *Ideae mathematicae* (1593). As the story goes, a Dutch ambassador told King Henry IV of France that there was no mathematician in France who could solve it. When François Viète (1540-1603) saw the problem, he solved it within minutes. This was because of his intimate knowledge of trigonometric identities. Before solving the 45th degree polynomial, we will first consider cubic equations.

![Figure 1: Viète, Henry IV, and van Roomen.](image)

## 1 Solution of a Depressed Cubic Equation

The solution of a depressed cubic equation,

\[ z^3 - pz - q = 0, \quad p, q > 0, \]  

(1)

can be found using methods attributed to François Viète. Consider the sine function identity \(^1\)

\[ 4 \sin^3 \theta = 3 \sin \theta - \sin 3 \theta. \]

\(^1\)We can also use hyperbolic identities to solve \( z^3 + pz + q = 0, \ p, q > 0. \) In this case we begin with a hyperbolic sine function identity, \( \sinh 3 \theta = 3 \sinh \theta + 4 \sinh^3 \theta. \) Letting \( z = A \sinh \theta \) and \( C = \sinh 3 \theta, \) then

\[ 4 \left( \frac{z}{A} \right)^3 + 3 \frac{z}{A} = C. \]

Rewriting, we have a cubic equation,

\[ z^3 + \frac{3}{4} A^2 z = \frac{CA^3}{4}. \]

Define \( A = 2 \sqrt{\frac{q}{p}} \) and \( C = -\frac{q}{2p}. \) Then, the cubic becomes \( z^3 + pz + q = 0. \)

Since \( C = \sinh 3 \theta, \) we find

\[ \theta = \frac{1}{3} \text{Arcsinh} \left( \frac{q}{2p} \right) + \frac{2k\pi i}{3}, \quad k = 0, 1, 2. \]
Let $z = A \sin \theta$ and $C = \sin 3\theta$. Then,

$$4 \left( \frac{z}{A} \right)^3 = 3 \left( \frac{z}{A} \right) - C.$$ 

Rewriting this equation, we have

$$z^3 - \frac{3}{4} A^2 z + \frac{CA^3}{4} = 0.$$ 

For the depressed cubic in Equation (1) we identify $A = 2\sqrt{\frac{P}{3}}$ and $C = -\frac{4q}{A^3} = -\frac{q}{2\gamma}$.

Since $C = \sin 3\theta$, we find three distinct solutions,

$$\theta = \frac{1}{3} \sin^{-1} \left( -\frac{q}{2\gamma} \right) + \frac{2k\pi}{3}, \quad k = 0, 1, 2.$$ 

From $z = A \sin \theta$, we obtain the solution to the depressed cubic as

$$z = 2 \sqrt{\frac{p}{3}} \sin \left( \frac{1}{3} \sin^{-1} \left( -\frac{q}{2\gamma} \right) + \frac{2k\pi}{3} \right), \quad k = 0, 1, 2. \quad (2)$$

2 Adriaan van Roomen’s Problem

The 45th degree polynomial in van Roomen’s Problem was given by [See Figure 2.]

$$P_{45}(x) = x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33}$$
$$-14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23}$$
$$+ 483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13}$$
$$-34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \quad (3)$$

Then, the problem can be stated as: Solve

$$P_{45}(x) = 2 \sin \left( 45 \sin^{-1} \frac{x}{2} \right),$$

Viète showed that $P_{45}(x) = 2 \sin \left( 45 \sin^{-1} \frac{x}{2} \right)$, which can be solved for $x$:

$$2 \sin \left( 45 \sin^{-1} \frac{x}{2} \right) \approx 0.4158234$$

Using $z = A \sinh \theta$, we obtain the solutions

$$z = 2 \sqrt{\frac{p}{3}} \sinh \left( \frac{1}{3} \sinh^{-1} \left( -\frac{q}{2\gamma} \right) + \frac{2k\pi}{3} \right), \quad k = 0, 1, 2.$$ 

where $\gamma^2 = \frac{p^3}{27}$. 

2
45 \sin^{-1} \frac{x}{2} \approx \sin^{-1}(0.4158234/2)

x \approx 2 \sin \left( \frac{\sin^{-1}(0.4158234/2)}{45} \right)

2.1 Viète’s Form for the $P_{45}(x)$

Consider the equations

\begin{align*}
    c &= 2 \sin 45\theta, \\
    y &= 2 \sin 15\theta, \\
    z &= 2 \sin 5\theta, \\
    x &= 2 \sin \theta.
\end{align*}

The problem of finding $x$, given $c$, is equivalent to the van Roomen Problem.
This is accomplished using two identities,

\[
\sin 3\alpha = 3\sin \alpha - 4\sin^3 \alpha \quad (5)
\]

\[
\sin 5\alpha = 5\sin \alpha - 20\sin^3 \alpha + 16\sin^5 \alpha \quad (6)
\]

We set \(\alpha = 15\theta\) in identity (5) and find

\[
c = 2\sin 45\theta
\]

\[
= 6\sin 15\theta - 8\sin^3 15\theta
\]

\[
= 3y - y^3. \quad (7)
\]

For \(\alpha = 5\theta\), identity (5) becomes

\[
y = 2\sin 15\theta
\]

\[
= 6\sin 5\theta - 8\sin^3 5\theta
\]

\[
= 3z - z^3. \quad (8)
\]

We can use identity (6) to rewrite \(z\)

\[
z = 2\sin 5\theta
\]

\[
= 10\sin \theta - 40\sin^3 \theta + 32\sin^5 \theta
\]

\[
= 5x - 5x^3 + x^5, \quad (9)
\]

since \(x = 2\sin \theta\).

We can use these results to write \(c\) in terms of \(x\):

\[
y = 3z - z^3
\]

\[
= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3
\]

\[
= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \quad (10)
\]

\[
c = 3y - y^3
\]

\[
= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]
\]

\[
-[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3
\]

\[
= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33}
\]

\[
-14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23}
\]

\[
+483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13}
\]

\[
-34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x. \quad (11)
\]

We have established that \(P_{45}(x) = c\).

### 2.2 Complex Form

We will use the binomial expansion of \((\cos(t) + i\sin(t))^{45}\) to compute

\[
P_{45}(x) = 2\sin\left(45\sin^{-1}\frac{x}{2}\right)
\]
Let $x = 2 \sin \frac{\alpha}{45}$ to find

$$t = \sin^{-1} \frac{x}{2} = \frac{\alpha}{45}.$$  

Since $\sin \alpha = \text{Im} \left( e^{i\alpha} \right)$, we have

\[
\sin(45t) = \text{Im} \left( e^{45it} \right) = \text{Im} \left( (e^{it})^{45} \right) = \text{Im} \left( (\cos(t) + i \sin(t))^{45} \right) = \text{Im} \left( \sum_{k=0}^{45} \binom{45}{k} (i \sin(t))^k \cos^{45-k}(t) \right) = \frac{1}{i} \sum_{k=0}^{22} \binom{45}{2k+1} (i \sin(t))^{2k+1} \cos^{44-2k}(t) = \sum_{k=0}^{22} \binom{45}{2k+1} (-1)^k \sin^{2k+1}(t) \cos^{44-2k}(t) = \sum_{k=0}^{22} \binom{45}{2k+1} (-1)^k \sin^{2k+1}(t) \sum_{j=0}^{22-k} \binom{22-k}{j} (-1)^j \sin^{2j}(t) = \sum_{k=0}^{22} \sum_{j=0}^{22-k} \binom{45}{2k+1} \binom{22-k}{j} (-1)^{j+k} \sin^{2k+2j+1}(t) = \sum_{k=0}^{22} \sum_{j=0}^{22-k} \binom{45}{2k+1} \binom{22-k}{j-k} (-1)^j \sin^{2j+1}(t) = \sum_{j=0}^{22} \sum_{k=0}^{j} \binom{45}{2k+1} \binom{22-k}{j-k} (-1)^j \sin^{2j+1}(t) = \sum_{j=0}^{22} (-1)^j \left[ \sum_{k=0}^{j} \binom{45}{2k+1} \binom{22-k}{j-k} \right] \sin^{2j+1}(t) = \sum_{j=0}^{22} (-1)^j \left[ \sum_{k=0}^{j} \binom{45}{2k+1} \binom{22-k}{j-k} \right] \left( \frac{x}{2} \right)^{2j+1} = \sum_{j=0}^{22} \left( -\frac{1}{4} \right)^j \left[ \sum_{k=0}^{j} \binom{45}{2k+1} \binom{22-k}{j-k} \right] x^{2j+1}
\]

\[
x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} - 14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} + 483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} - 34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x = P_{45}(x). \]  

(12)