Babylonian Mathematics

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1 Babylonian Numerals

The Babylonians numerals are shown in Table 1. They were probably created using a stylus against a wet clay tablet.

1	𒐕	26	𒎙𒐚	51	𒐐𒐕	76	𒐕 𒌋𒐚
2	𒐖	27	𒎙𒑂	52	𒐐𒐖	77	𒐕 𒌋𒑂
3	𒐗	28	𒎙𒑄	53	𒐐𒐗	78	𒐕 𒌋𒑄
4	𒐘	29	𒎙𒑆	54	𒐐𒐘	79	𒐕 𒌋𒑆
5	𒐙	30	𒌍	55	𒐐𒐙	80	𒐕 𒎙
6	𒐚	31	𒌍𒐕	56	𒐐𒐚	81	𒐕 𒎙𒐕
7	𒑂	32	𒌍𒐖	57	𒐐𒑂	82	𒐕 𒎙𒐖
8	𒑄	33	𒌍𒐗	58	𒐐𒑄	83	𒐕 𒎙𒐗
9	𒑆	34	𒌍𒐘	59	𒐐𒑆	84	𒐕 𒎙𒐘
10	𒌋	35	𒌍𒐙	60	𒐕	85	𒐕 𒎙𒐙
11	𒌋𒐕	36	𒌍𒐚	61	𒐕 𒐕	86	𒐕 𒎙𒐚
12	𒌋𒐖	37	𒌍𒑂	62	𒐕 𒐖	87	𒐕 𒎙𒑂
13	𒌋𒐗	38	𒌍𒑄	63	𒐕 𒐗	88	𒐕 𒎙𒑄
14	𒌋𒐘	39	𒌍𒑆	64	𒐕 𒐘	89	𒐕 𒎙𒑆
15	𒌋𒐙	40	𒐏	65	𒐕 𒐙	90	𒐕 𒌍
16	𒌋𒐚	41	𒐏𒐕	66	𒐕 𒐚	91	𒐕 𒌍𒐕
17	𒌋𒑂	42	𒐏𒐖	67	𒐕 𒑂	92	𒐕 𒌍𒐖
18	𒌋𒑄	43	𒐏𒐗	68	𒐕 𒑄	93	𒐕 𒌍𒐗
19	𒌋𒑆	44	𒐏𒐘	69	𒐕 𒑆	94	𒐕 𒌍𒐘
20	𒎙	45	𒐏𒐙	70	𒐕 𒌋	95	𒐕 𒌍𒐙
21	𒎙𒐕	46	𒐏𒐚	71	𒐕 𒌋𒐕	96	𒐕 𒌍𒐚
22	𒎙𒐖	47	𒐏𒑂	72	𒐕 𒌋𒐖	97	𒐕 𒌍𒑂
23	𒎙𒐗	48	𒐏𒑄	73	𒐕 𒌋𒐗	98	𒐕 𒌍𒑄
24	𒎙𒐘	49	𒐏𒑆	74	𒐕 𒌋𒐘	99	𒐕 𒌍𒑆
25	𒎙𒐙	50	𒐐	75	𒐕 𒌋𒐙	100	𒐕 𒐏

Table 1: A table of Babylonian numerals from 1 to 100.

The Babylonians used base 60. So, the number $26,008 = 7(60)^2 + 13(60) + 28$. In Babylonian numerals, this would be written as 𒑂 𒌋𒐗 𒎙𒑄 using the numbers 7, 13, 28. For the number $7(60)^2 + 28 = 25228$ 𒑂   𒎙𒑄 there is a space where the 13 used to be.

Problem 1: Show that

1. 424000 is 𒁹 𒐐𒐌 𒐏𒐋 𒐏.
2. 21609 is 𒐋 𒐎 (Note the space).
3. 123456789 is 𒐎 𒌍𒁹 𒌍𒐈 𒌍𒐈 𒐎.
4. In the YBC 7289 tablet the Babylonians approximated $\sqrt{2}$ by 𒁹 ; 𒎙𒐉 𒐐𒁹 𒌋, where we placed the equivalent of a decimal point as a semicolon. How good of an approximation is this?
5. Find the base ten equivalents of the two numbers a) 𒌋𒐌 𒐉 and b) 𒐐𒈫.

We should note that going from the cuneiform numerals to base ten might not be so easy. The Babylonians did not use decimal places. So, the numbers $7(60)^2+13(60)+28 = 26,008$, $7(60)^3+13(60)^2+28(60) = 1,560,480$, and $7+\frac{13}{60}+\frac{28}{60^2} = \frac{3251}{450}$ are represented by 𒐌 𒌋𒐈 𒎙𒐍.

2 Akkadian Tablet (1700 BCE)

In the paper "Sherlock Holmes in Babylon," *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9's.

𒁹	𒐎	𒌋𒐈	𒁹 𒐐𒐌
𒈫	𒌋𒐍	𒌋𒐉	𒈫 𒐋
𒐈	𒎙𒐌	𒌋𒐊	𒈫 𒌋𒐊
𒐉	𒌍𒐋	𒌋𒐋	𒈫 𒎙𒐉
𒐊	𒐏𒐊	𒌋𒐌	𒈫 𒌍𒐈
𒐋	𒐐𒐉	𒌋𒐍	𒈫 𒐏𒈫
𒐌	𒁹 𒐈	𒌋𒐎	𒈫 𒐐𒁹
𒐍	𒁹 𒌋𒈫	𒎙	𒐈
𒐎	𒁹 𒎙𒁹	𒌍	𒐉 𒌍
𒌋	𒁹 𒌍	𒐏	𒐋
𒌋𒁹	𒁹 𒌍𒐎	𒐐	𒐌 𒌍
𒌋𒈫	𒁹 𒐏𒐍		

Table 2: Table of 9's.

As an example, the last entry in the first column is 12 = 𒌋𒈫. The last entry in the second column is $9 \times 12 = 108 =$ 𒁹 𒐏𒐍. Note that in base 60 we have $108 = 1(60) + 48$. This is a one (𒁹) and 48 (𒐏𒐍) separated by a small pace. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

Problem 2

1. Verify that this is true by doing base 60 addition.
2. What is the decimal (base ten) equivalent?

3 Multiplication

Several multiplication tables have been found. Buck [1] also describes the use of a table of reciprocals that could be used for carrying out division. Reciprocals are the sexagesimal numbers with which they multiply to give 1. However,

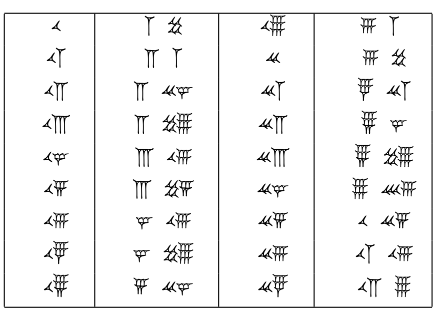
Another method for multiplication relied on using tables of squares such as shown in Table 3. How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \tag{1}$$

Problem 3 Show that Equation (1) is true.

Let's find the product 11×14. Using Table 3, the formula gives

$$\begin{aligned} 11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\ &= \frac{1}{4}(25^2 - 3^2) \\ &= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\ &= \frac{1}{4}(10/16) \text{ (base 60)} \\ &= \frac{1}{4}(10(60)+16) = \frac{616}{4} = 154. \end{aligned} \tag{2}$$



10	1/40	19	6/1
11	2/1	20	6/40
12	2/24	21	7/21
13	2/49	22	8/4
14	3/16	23	8/49
15	3/45	24	9/36
16	4/16	25	10/25
17	4/49	26	11/16
18	5/24	27	12/9

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

Problem 4

Use Equation (1) to find the products:

1. 10×12.
2. 9×13.

4 Reciprocals

Babylonians did not have division, per se. Instead they used tables of reciprocals. We think of the reciprocal $\bar{x}$ of x as that number such that $x\bar{x} = 1$. Since the Babylonians did not have zeros or use decimal points, then $1 =$ 𒁹 , $60 =$ 𒁹 , and $60^2 =$ 𒁹 would all appear the same on a clay tablet. So, a reciprocal would actually satisfy $x\bar{x} = 60^n$ since 60^n looks like 1. In Table 4 we list a few reciprocals.

Table of

x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$
2	0/30	8	7/30	16	3/45	30	2
3	0/20	9	6/40	18	3/20	32	1/52/30
4	0/15	10	6	20	3	36	1/40
5	0/12	12	5	24	2/30	40	1/30
6	0/10	15	4	25	2/24	45	1/20

Table 4: A Table of Reciprocals

If you wanted to divide 8 by 2, you could multiply by $\frac{1}{2}$ instead of dividing. In the same way, one can you would look up the reciprocal to 2 in Table 4, 0/30, and multiply 8 by it. So, $8(0/30) = 8 \times \frac{30}{60} = \frac{240}{60} = 4$.

There are several missing reciprocals: $\frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots$. These do not have a finite sexigesimal representation. For example, we know $\frac{1}{7} = .\overline{142857}$. We convert it to base 60 using repeated multiplications by 60 and extracting the integer parts of the results.

$$\begin{aligned} 60(.\overline{142857}) &= 8.\overline{5714285} \\ 60(.\overline{5714285}) &= 34.\overline{285714} \\ 60(.\overline{285714}) &= 17..\overline{142857} \end{aligned} \tag{3}$$

After a few steps, we see the pattern begin to repeat. In more modern base 60 notation, $\frac{1}{7} = \overline{..83417}_{60}$. Then, we have

$$\frac{1}{7} = \frac{8}{60} + \frac{34}{60^2} + \frac{17}{60^3} + \frac{8}{60^4} + \cdots .$$

So, just using the first four terms, the representation differs from $\frac{1}{7}$ after the sixth decimal place.

Problem 5

1. Divide 37 by 6.
2. What is the base 60 representation of $\frac{1}{13}$.

5 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, C, and one leg, B, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple (D, B, C) is called a Pythagorean triple.

We know from Euclid that Pythagorean triples are parametrized by the pair (a, b) as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

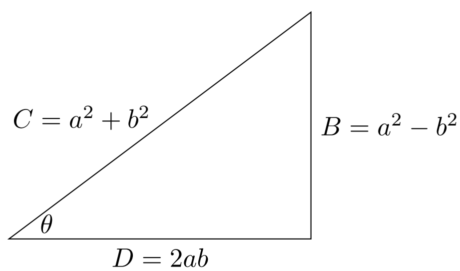


Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples are known to have a parametrization denoted (a, b).

Some effort has gone into figuring out the entries on the tablet and making corrections. Buck describes some of this in his paper[1]. As seen in Figure 4, there are also some missing entries. A cleaner transcription is shown in Table 5. Buck discusses the gaps and notes a few errors. Is Figure 5 easier to read? Other versions are seen in Figures 3 and 6.

In Table 5 we show Buck's corrected values for the Plimpton 322 Tablet [1]. Column A gives the base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$. The last two columns give the parameters[2] (a, b).

Problem 6: Verify that row 7 of the Plimpton 322 Tablet gives a Pythagorean triple.

Buck then goes on to give an explanation of Column A. He suggests that this column is $\left(\frac{B}{D}\right)^2$. Since $B < D$, these numbers are less than one. Others suggest that Column A was $\left(\frac{C}{D}\right)^2$, and that left missing left part of the stone had 1's. Since

$$\left(\frac{C}{D}\right)^2 = 1 + \left(\frac{B}{D}\right)^2,$$

it does not matter.

[1] There have been several attempts to determine the purpose of this tablet. See Wikipedia.
[2] In row 11 there appears to be a problem which Buck explains using base 60. This is the familiar $3, 4, 5$ triangle.

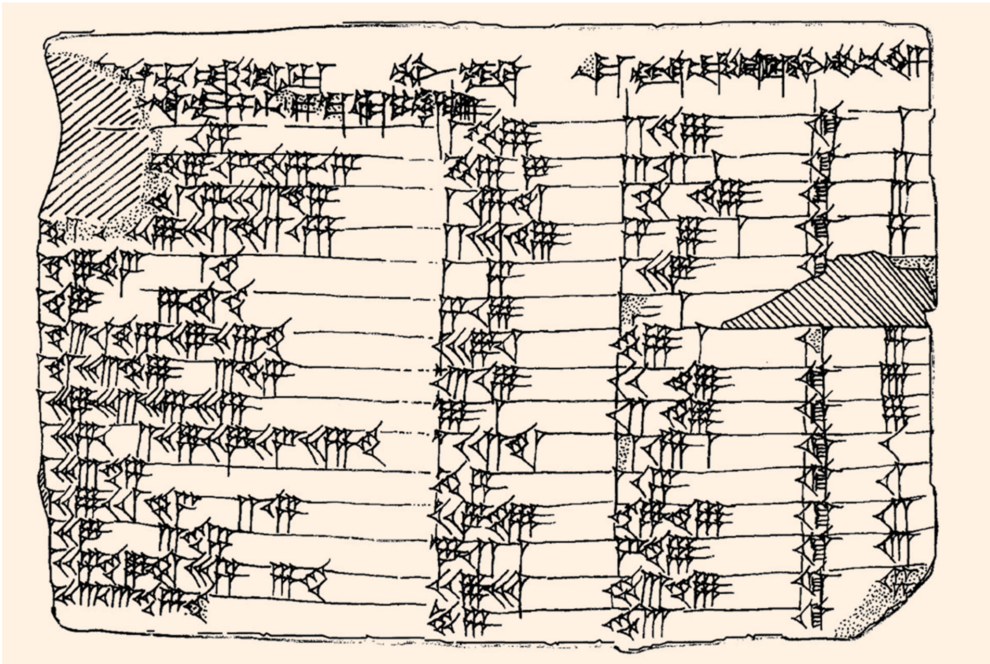


Figure 2: A sketch of the Plimpton 322 Tablet.

[59]/0/15	1/59	2/49	ki	1
[56/56]/58/14/50/6/15	56/7	1/20/25	ki	2
[55/7]/41/15/33/45	1/16/41	1/50/49	ki	3
53/10/29/32/52/16	3/31/49	5/9/1	ki	4
48/54/1/40	1/5	1/37	ki	[5]
47/6/41/40	5/19	8/1	[ki]	[6]
43/11/56/28/26/40	38/11	59/1	ki	7
41/33/45/14/3/45	13/19	20/49	ki	8
38/33/36/36	8/1	12/49	ki	9
35/10/2/28/27/24/26/40	1/22/41	2/16/1	ki	10
33/45	45	1/15	ki	11
29/21/54/2/15	27/59	48/49	ki	12
27/0/3/45	2/41	4/49	ki	13
25/48/51/35/6/40	29/31	53/49	ki	14
23/13/46/40	56	53	ki	[15]

Table 5: A transcription of the tablet. Brackets indicate guesses.

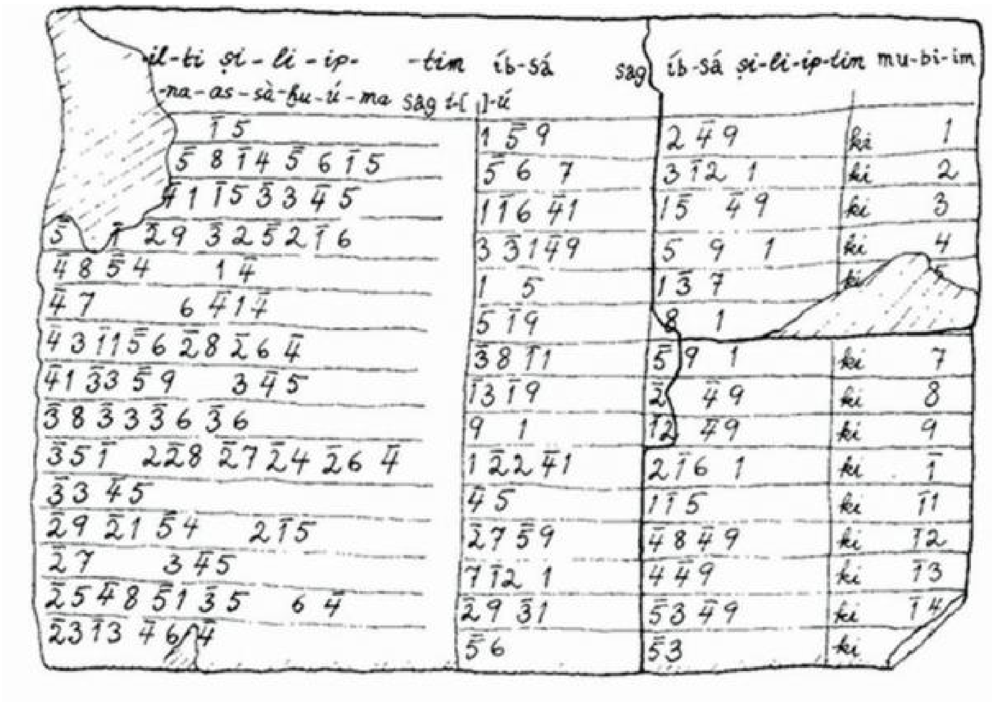


Figure 3: A sketch of the Plimpton 322 Tablet with arabic numerals base 60. The bars designate place holders.

#	A	B	C	a	b
1	59/0/15	119	169	12	5
2	56/56/58/14/50/6/15	3367	4825	64	27
3	55/7/41/15/33/45	4601	6649	75	32
4	53/10/29/32/52/16	12709	18541	125	54
5	48/54/1/40	65	97	9	4
6	47/6/41/40	319	481	20	9
7	43/11/56/28/26/40	2291	3541	54	25
8	41/33/45/14/3/45	799	1249	32	15
9	38/33/36/36	481	769	25	12
10	35/10/2/28/27/24/26/40	4961	8161	81	40
11	33/45	45	75	1	0.5 = 30
12	29/21/54/2/15	1679	2929	48	25
13	27/0/3/45	161	289	15	8
14	25/48/51/35/6/40	1771	3229	50	27
15	23/13/46/40	56	106	9	5

Table 6: Buck's corrected values for the Plimpton 322 Tablet. The second column gives the base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

Figure 4: Plimpton 322 Tablet.

A	B	C
59 15	1 59	2 49
56 56 58 14 50 6 15	56 7	1 20 25
55 7 41 15 33 45	1 16 41	1 50 49
53 10 29 32 52 16	3 31 49	5 9 1
48 54 1 40	1 5	1 37
47 6 41 40	5 19	8 1
43 11 56 28 26 40	38 11	59 1
41 33 45 14 3 45	13 19	20 49
38 33 36 36	8 1	12 49
35 10 2 28 27 24 26 40	1 22 41	2 16 1
33 45	45	1 15
29 21 54 2 15	27 59	48 49
27 3 45	2 41	4 49
25 48 51 35 6 40	29 31	53 49
23 13 46 40	56	1 46

Figure 5: Buck's corrected values for the Plimpton 322 Tablet using Babylonian numerals [1].

For example, 59/0/15 represents a number less than one. In base ten, we have

$$59/0/15 = \frac{59}{60} + \frac{0}{60^2} + \frac{15}{60^3} = \frac{14161}{14400} \approx .9834027778.$$

In Table 7 we show the decimal equivalents for column one of Table 5 and the values of $\left(\frac{B}{D}\right)^2 = \tan^2\theta$.

Note how good the approximations are. This suggests also that computing the exact fractions will give exactly $\left(\frac{B}{D}\right)^2$. Again, from row 1 we have $B = 119$ and $C = 169$. This gives $B^2 = 119^2 = 14161$ and $D^2 = 169^2 - 119^2 = 14400$. So, we have found that $\left(\frac{B}{D}\right)^2 == \frac{14161}{14400}$, which in base 60 is 59/0/15.

Problem 7: Verify row 6 in Table 7.

#	A	Decimal Value	$(B/D)^2$
1	59/0/15	0.983402777777778	0.983402777777778
2	56/56/58/14/50/6/15	0.949158552088692	0.949158552088692
3	55/7/41/15/33/45	0.918802126736111	0.918802126736111
4	53/10/29/32/52/16	0.886247906721536	0.886247906721536
5	48/54/1/40	0.815007716049383	0.815007716049383
6	47/6/41/40	0.785192901234568	0.785192901234568
7	43/11/56/28/26/40	0.719983676268862	0.719983676268862
8	41/33/45/14/3/45	0.692709418402778	0.692709418402778
9	38/33/36/36	0.642669444444444	0.642669444444444
10	35/10/2/28/27/24/26/40	0.586122566110349	0.586122566110349
11	33/45	0.562500000000000	0.562500000000000
12	29/21/54/2/15	0.489416840277778	0.489416840277778
13	27/0/3/45	0.450017361111111	0.450017361111111
14	25/48/51/35/6/40	0.430238820301783	0.430238820301783
15	23/13/46/40	0.387160493827161	0.387160493827161

Table 7: Decimal equivalents for column one of Table 5 and $(B/D)^2$.

6 The Diagonal of a Square

Another tablet that was found seems to be that of a student's computation of the diagonal of a square dating from 1800–1600 BCE. It is labeled as YBC7289 as shown on the left in Figure 6. Reading the etched numbers, we have a square with sides of length 30. There are two rows of numbers along the diagonal as shown on the right of Figure 6: In Buck's notation, we have the numbers 1/24/51/10 and 42/25/35.

Problem 8: Show that

- The first set gives and approximation to $\sqrt{2}$.
- The second set gives the length of the diagonal.

Figure 6: Tablet YBC7289 and it's interpretation.

References

[1] Buck, R. C. Sherlock Holmes in Babylon, *Amer. Math. Monthly* 87 (1980), 335-345.

[2] Friberg, J. Methods and traditions of Babylonian mathematics: Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations, *Historia Mathematica*, Volume 8, Issue 3,(1981), 277-318.

[3] Mansfield, D. F. Wildberger, N. J. Plimpton 322 is Babylonian exact sexagesimal trigonometry, *Historia Mathematica*, Volume 44, Issue 4,(2017), 395-419.

[4] Neugebauer, O. and Sachs, A. J. *Mathematical Cuneiform Texts*, American Oriental Series, vol. 49, American Oriental Society, American Schools of Oriental Research (1945).

[5] Robson, E. Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 323 *Hist. Math.*, 28 (2001), 167-206.

[6] Robson, E. Words and pictures: New light on Plimpton 322. *Amer. Math. Monthly*, 109 (2002) 105–120.