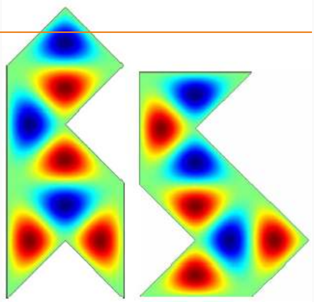
Vibrations and Fourier Analysis

Fall 2023 - R. L. Herman



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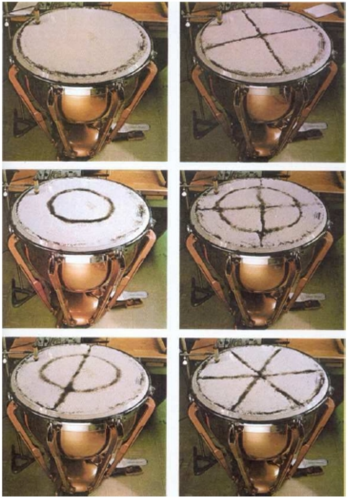
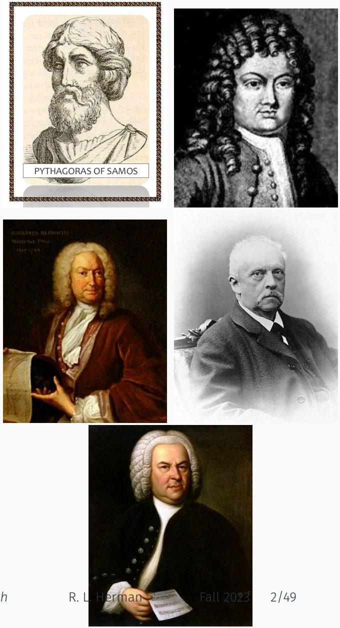


Chladni patterns on a kettledrum
from Risset, *Les instruments de l'orchestre*

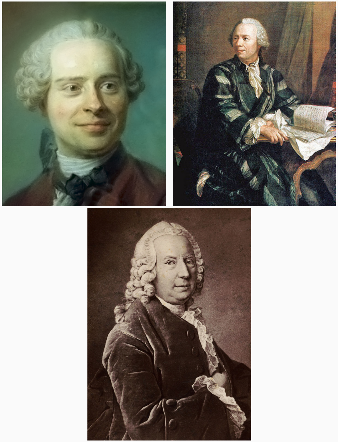
Harmonics

- Pythagoras, Ptolemy.
- Galileo and Mersenne, pitch and frequency. Strings produce several tones.
- Joseph Sauveur, 1653-1716, acoustics. Introduced nodes, “harmonic.”
- Johann Bernoulli, 1667-1748.
- Brook Taylor 1685-1731, fundamental.
- Johann Sebastian Bach Bach, 1685-1750.
- Hermann Helmholtz, 1821-1894, acoustics.



The 1700s Debate - Mathematicians vs Physicists

- Jean le Rond d'Alembet, 1717-1783.
- Vibrating string equation and general solution, $y(x,t) = f(x+t) + g(x-t)$. BCs give $g = f$.
- Leonhard Euler's papers, 1748-9. More general equation with c, and $y(x,t) = f(x+ct) + g(x-ct)$.
- Claimed - f from ICs. $y(x,t) = \frac{1}{2}\left(Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x+ct}^{x-ct} V(s)\,ds\right)$.
- Y, V are any curves *drawn by hand*.
- Daniel Bernoulli, 1709-1791, solutions are sums of harmonics, 1753:



$$y(x) = A_1 \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + A_2 \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \cdots = f(x+ct) + g(x-ct).$$

The Controversy (from Am. J. of Phys. 55, 33 (1987))

d’Alembert vs Euler

- Euler allowed corners.
- d’Alembert’s first response - f must be periodic, odd, differentiable. Introduced separation of variables.
- 1761 - the attack! Use of physical arguments is prohibited.
- If slope discontinuous, then acceleration undefined.
- Euler responded 1762, 1765. For small displacement, the function at corner is infinitesimally close to differentiable.

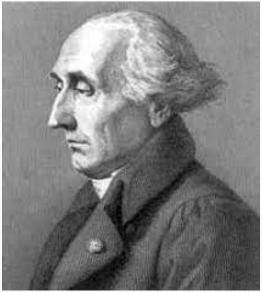
d’Alembert, Euler vs Bernoulli

- d’Alembert did not believe a sum of harmonics.
- Euler sum not general enough - snapped string.
- Bernoulli - “Listen to the string.”

They all missed general periodicity.

Joseph-Louis Lagrange

- In enters another math. physicist.
- Born Luigi de la Grange Tournier (1736-1813), in Italy.
- 1759, paper on sound propagation.
- Agreed mostly with Euler, not Bernoulli.
- Avoided wave equation. Used a discrete set of masses.



$$\begin{aligned} y(x,t) &= \frac{2}{L}\int_0^L dX\,Y(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}+\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}+\dots\right] \\ &+ \frac{2}{\pi c}\int_0^L dX\,V(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}+\frac{1}{2}\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}+\dots\right] \end{aligned}$$

He almost discovered Fourier series in 1759. [Fourier was born, 1768.]

Jean-Baptiste Joseph Fourier (1768-1830)

- French Revolution, 1789, several arrests.
- Studied under Lagrange, Laplace, Monge.
- Succeeded Lagrange, chair of analysis and mechanics, 1797.
- Joined Napoleon's invasion of Egypt, scientific adviser with Monge, Malus.
- Organizer of French retreat from Egypt.
- Produced a multi-volume work on Egyptology.
- Studied the heat equation and series solutions.
- Almost forgotten in France, not elsewhere due to P. G. J. Dirichlet who wrote on Fourier series. Open problems led Cantor to set theory.

Siméon-Denis Poisson (1781-1840)

- 1798, entered École Polytechnique.
- Studied under Laplace and Lagrange.
- Degree in mathematics two years.
- Chair of mechanics, Faculty of Sciences, 1809.
- Over 300 papers: definite integrals, Fourier analysis, applied mathematics to physics (mechanics and electrostatics), probability and statistics.
- Poisson brackets, Poisson's constant, Poisson's equation, Poisson's integral, and Poisson's spot.

See D. H. Arnold's Work.

The Heat Equation

- Controversy: Fourier vs Poisson
- Fourier 1805, 1807 - diffusion, series solutions ala D. Bernoulli.
- Examiners: Laplace, Lagrange skeptical.
- Poisson Review 1808.
- 1811 Prize problem. Fourier won, but still critics.
- Third version to be book, 1822. Timing affected by politics.
- 1815, Poisson writes his own paper, then book in 1823.
- Wm. Thomson defense of Fourier in 1845.
- The Age of the Earth and Telegraphy.

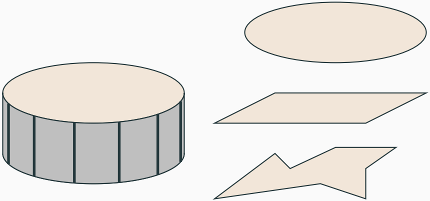
See D. H. Arnold's Work.

On to applications -

- Can you hear the shape of a drum?
- How long does it take to cook a turkey?

“Can One Hear the Shape of a Drum?”

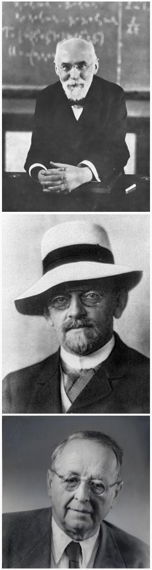
- Kac, Mark (1966). Amer. Math. Monthly. 73, Part II: 1–23.
- Title due to Lipman Bers: “If you had perfect pitch, could you hear the shape of a drum?”
- Can the frequencies (**eigenvalues**) of a resonator (**drum**) determine its shape (**geometry**)?
- Entails features of applied mathematics.
- Historical connections - from radiation theory.



Radiation Theory

- Hendrik Lorentz’s (1910) 5 lectures on old/new physics. problems
- 4th - Electromagnetic Radiation Theory.
- Compared vibrations to an organ pipe.
- The number of overtones in frequency range is independent of shape, proportional to volume.
- David Hilbert’s prediction
- Hermann Weyl - < 2 yrs

$$N(\lambda) = \sum_{\lambda_n < \lambda} \sim \frac{|\Omega|}{2\pi}\lambda.$$



What Do We Hear? Frequency, $f = \omega/2\pi$,

Seek Harmonic Solutions,
[Recall $e^{i\omega t} = \cos\omega t + i\sin\omega t$.]

$$u(\mathbf{r}, t) = U(\mathbf{r})e^{i\omega t},$$

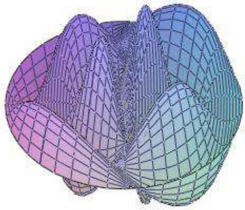
of a Wave Equation, $u(\mathbf{r}, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

Helmholtz Equation

$$\nabla^2 U = -k^2 U, \quad k^2 = \frac{\omega}{c}.$$

Eigenvalues $\sim$ frequencies



Vibrations of a String

- Ex: Violin String.
- Harmonics, $u_n(x)$.
- Wavelength, $\lambda = \frac{2L}{n}$.
- Wave Speed, $c = \sqrt{\frac{T}{\mu}}$.
- Frequency, $f = n\frac{c}{2L}$.
- A - $f = 440$ Hz, $L = 32$ cm.
  $c = 2Lf = 280$ m/s.
- Nodes, $u_n(x) = 0$

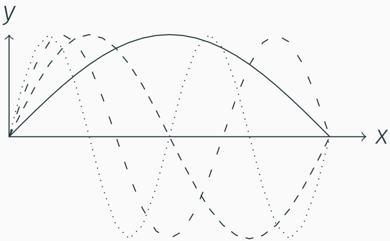


Figure 1: Plot of the eigenfunctions $u_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, 4$.

Solution of 1D Wave Equation

The one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x \leq L, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

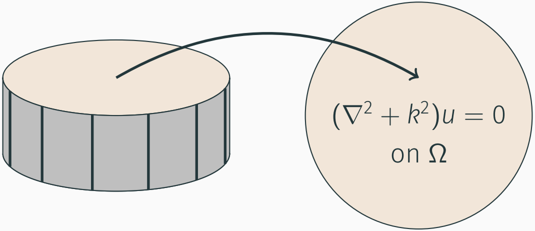
$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right] \sin \frac{n\pi x}{L}, \tag{2}$$

where $\omega_n = \frac{n\pi c}{L}$.

General 2D Membranes

- Membrane Problems.
    - Rectangular
    - Circular
    - Elliptical
    - Irregular
- Solve Helmholtz Equations
    - Normal Modes and Frequencies of Oscillation
    - Eigenvalues of Laplace Operator, $\nabla^2 u = -\lambda u$.

$$u = 0 \text{ on } \partial\Omega$$

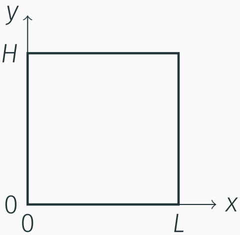


$$(\nabla^2 + k^2)u = 0$$
$$\text{on } \Omega$$

Vibrations of a Rectangular Membrane

- Harmonics
- Frequencies

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



Boundary-value problem

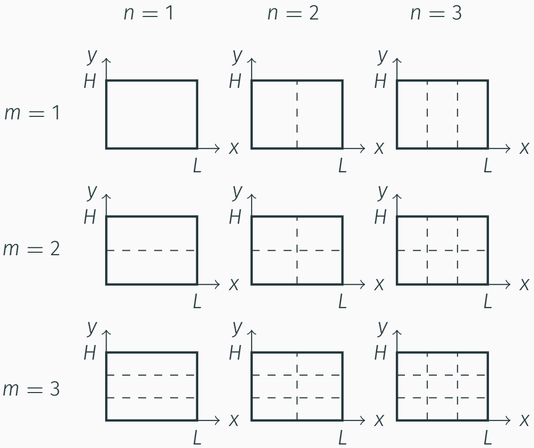
$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{3}$$

$$\begin{aligned} u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H, \\ u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned}$$

$$u(x, y, t) = \sum_{n,m} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.$$

Nodes of a Rectangular Membrane

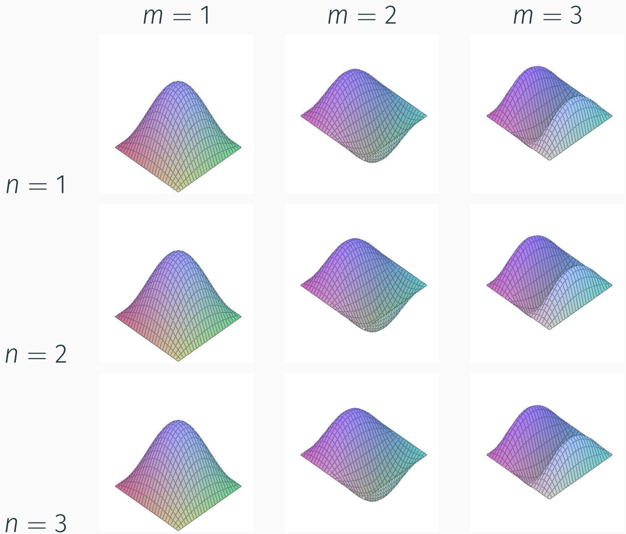
$$u_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad f = \frac{c}{2L}\sqrt{n^2+\alpha^2 m^2}, \quad \alpha = \frac{L}{H}.$$



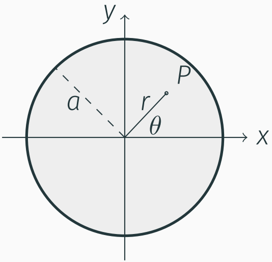
$\alpha = 1$	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

$\alpha = 2$	1	2	3
1	2.236	4.123	6.083
2	2.828	4.472	6.325
3	3.606	5.000	6.708

Vibrations of a Rectangular Membrane



Vibrations of a Circular Membrane



- Circular Symmetry.
- Harmonics
- Frequencies

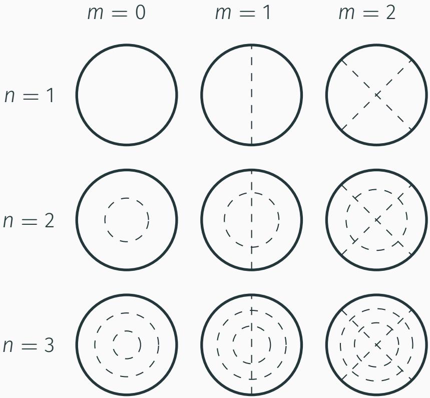
$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m(\frac{j_{mn}}{a} r). \qquad (4)$$

$$J_m(j_{mn}) = 0 \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Nodes of a Circular Membrane

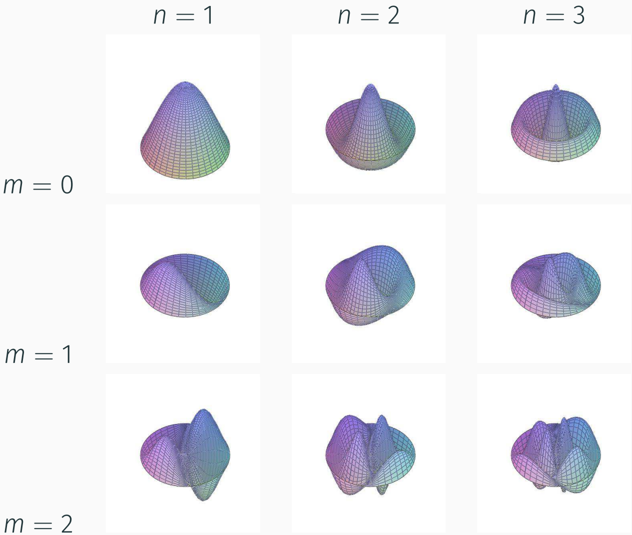
$$u_{mn}(r,\theta) = J_m\left(\frac{j_{mn}}{a}r\right)\cos m\theta, \quad f_{mn} = \frac{j_{mn}c}{2\pi a}.$$



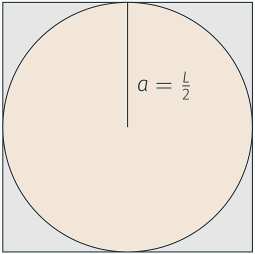
j_{mn}	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.62

f_{mn}	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Vibrations of a Circular Membrane



Rectangular and Circular Membrane Frequencies



Rectangular

	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

Circular $a = \frac{L}{2}$

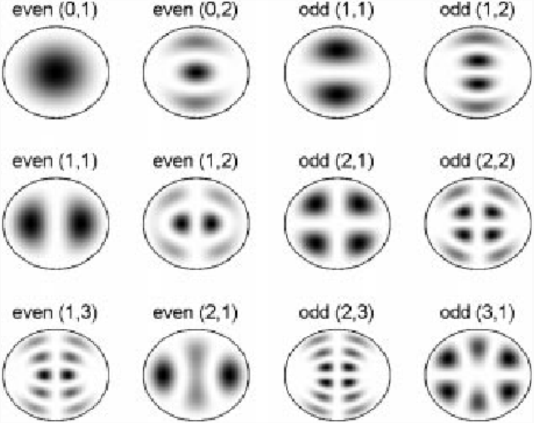
	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Circular $\pi a^2 = L^2$

	0	1	2
1	1.357	2.162	2.898
2	3.114	3.958	4.749
3	4.882	5.740	6.556

Vibrations of an Elliptical Membrane

$$\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+(kh)^2(\cosh^2\xi-\cos^2\eta)\right]u(\xi,\eta)=0.$$



Vibrations of a Balloon

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} L u, \quad \text{where} \quad L Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t \right] Y_{\ell m}(\theta, \phi),$$

where $\omega_\ell = \sqrt{\ell(\ell+1)} \dfrac{c}{R}$.

Modes for a Vibrating Spherical Membrane (Balloon?)

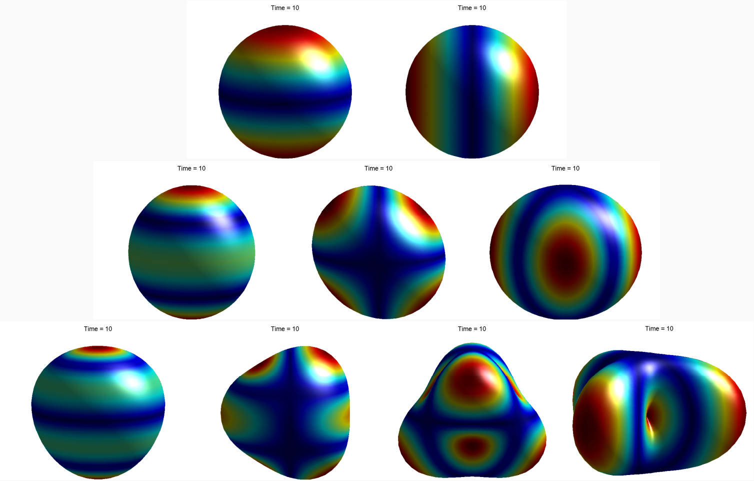
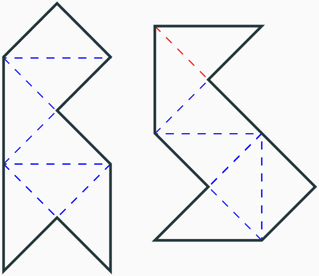
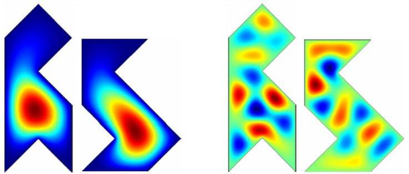


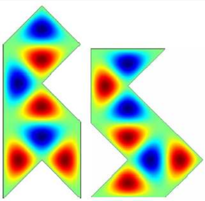
Figure 2: *`http://people.uncw.edu/hermanr/pde1/sphmem/`*
Row 1: $(1,0)$, $(1,1)$; Row 2: $(2,0)$, $(2,1)$, $(2,2)$;
Row 3 $(3,0)$, $(3,1)$, $(3,2)$, $(3,3)$.

Vibrations of a Irregular Membranes

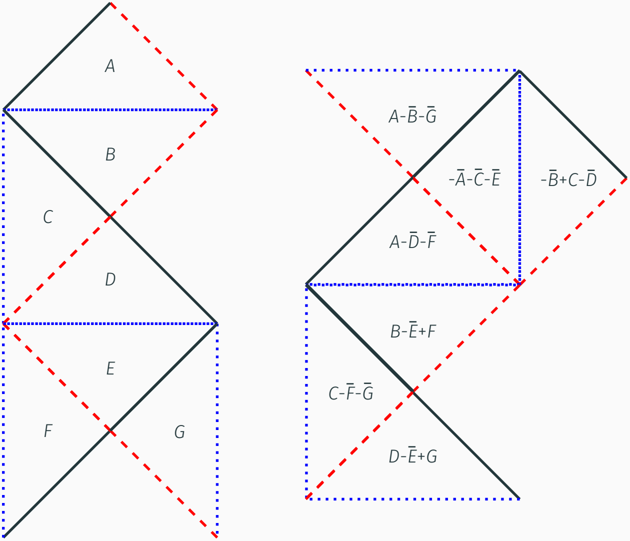
- Gordon, C., Webb, D., and Wolpert, S.(1992) - *You Cannot Hear the Shape of a Drum*
- Shapes on right have same set of frequencies - **isospectral drums.**





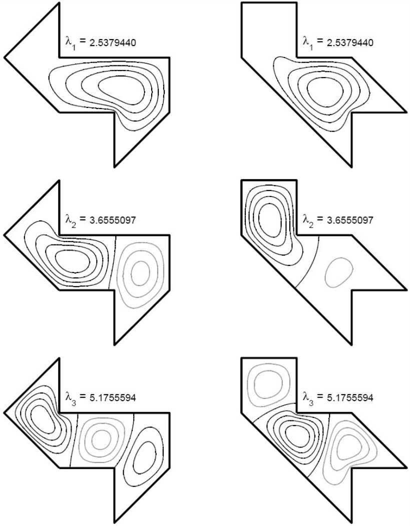


Isospectral Drums

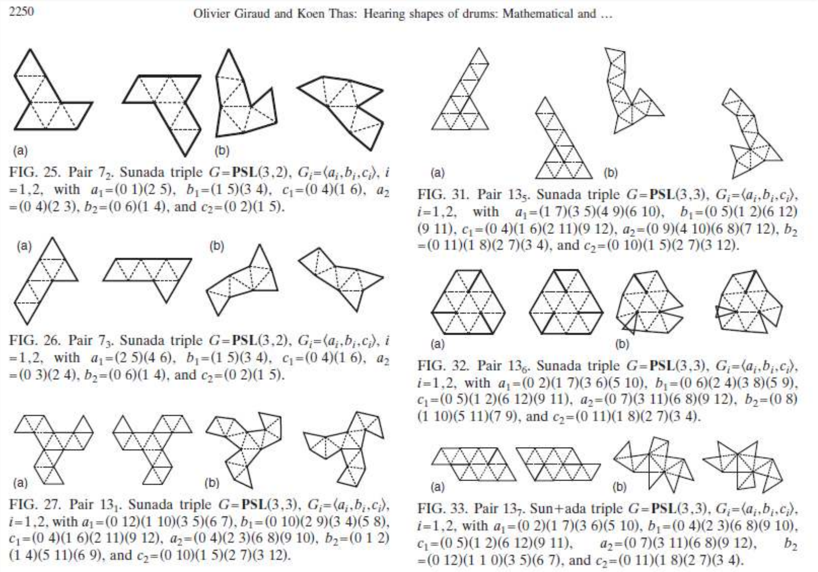


Spectra of Isospectral Drums

$\lambda = 2.5379440, 3.6555097, 5.1755594.$



Other Isospectral Drums



2250 Olivier Giraud and Koen Thas: Hearing shapes of drums: Mathematical and ...

FIG. 25. Pair 7_2. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 1)(2\ 5)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 4)(2\ 3)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

FIG. 26. Pair 7_3. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(2\ 5)(4\ 6)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 3)(2\ 4)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

FIG. 27. Pair 13_1. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 12)(1\ 10)(3\ 5)(6\ 7)$, $b_1=(0\ 10)(2\ 9)(3\ 4)(5\ 8)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $b_2=(0\ 1\ 2)(1\ 4)(5\ 11)(6\ 9)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

FIG. 31. Pair 13_5. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(1\ 7)(3\ 5)(4\ 9)(6\ 10)$, $b_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 9)(4\ 10)(6\ 8)(7\ 12)$, $b_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

FIG. 32. Pair 13_6. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 6)(2\ 4)(3\ 8)(5\ 9)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 8)(1\ 10)(5\ 11)(7\ 9)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

FIG. 33. Pair 13_7. Sun+ada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 12)(1\ 1\ 0)(3\ 5)(6\ 7)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

Can one hear the shape of a drum? -

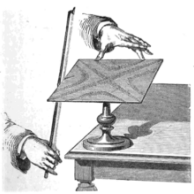
No!

Membranes - Rectangular, circular, elliptical, irregular

Never look at MATLAB logo the same way again - Why?

Chladni Plates

- Recall Sophie Germain.
- Ernst Chladni, 1756-1827, physicist and musician.
- In 1808, Chladni demonstrated vibrating plates at the Academy of Science in Paris.
- Napoleon, who attended, proposed a prize.
- Lagrange, Laplace and others – felt that it was beyond reach.
- Germain only one to try.
- 1816, two more tries, first woman awarded Grand Prize in Mathematics of the Paris Academy of Sciences.



Heat Equation vs Wave Equation

1D Wave Equation

$$u_{tt} = c^2 u_{xx}$$

1D Heat Equation

$$u_t = k u_{xx}$$

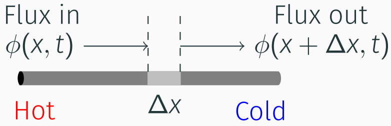
History of Heat Equation

Developed by Joseph Fourier (1768-1830)

- Discovered in early 1807 and published later in 1822

  Afterwards, diffusion processes studied outside of France.
  Lead to research in partial differential equations.
- Describes conduction and storage of heat (energy) in a body.
- Involves heat exchange with surroundings, conservation of energy.
- Leads to temperature changes inside body (diffusion).
- Uses the relation of heat energy to temperature (gradient), Fourier Law of Heat Conduction.

Heat Equation - Mathematics



Rate of change of heat energy = Flux in - Flux out

$$\frac{dQ}{dt} = \phi(x,t) - \phi(x+\Delta x,t).$$

Flux density = conductivity × temperature gradient

$$\phi = k\frac{dT}{dx}.$$

Heat energy is proportional to temperature

$$Q = mcT.$$

$q =$ Heat energy per vol, $u =$ temperature per vol

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad D = \frac{k}{mc}.$$

Thanksgiving Turkey!

- Native to North America.
- Introduced in Spain in 1500's.
- Benjamin Franklin - national bird.
- Holiday bird in Europe in 1800's
  - replacing goose.
- Turkeys mostly walk.
- Harold McGee: Breast 155-160 F, Legs 180 F.
- Cooking times

  Constant oven temp, diffusivity constant, Turkey plump
  Small - 20 min/lb + 20.
  Large - 15 min/lb + 15.
  $t \sim M^{2/3}$.

How long does it take to cook a turkey?

Example 1

If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

Figure 3: A Thanksgiving turkey - From 2015.

Panofsky Equation

- Pief Panofsky [SLAC Director Emeritus] *SLAC Today*, Nov 26, 2008 *`http://today.slac.stanford.edu/a/2008/11-26.htm`* For a stuffed turkey at 325° F

  $$t = \frac{W^{2/3}}{1.5}$$

  vs. 30 minutes/lb.
- Also, check out WolframAlpha *`http://www.wolframalpha.com/input/?i=how+long+should+you+cook+a+turkey`*
- Musings of an Energy Nerd *`http://www.greenbuildingadvisor.com/blogs/dept/musings/heat-transfer-when-roasting-turkey`*

Consider a Spherical Turkey

Figure 4: The depiction of a spherical turkey.

Scaling a Spherically Symmetric Turkey

The baking follows the heat equation.

Rescale the coordinates (r, t) to (ρ, τ) :

$$r = \beta\rho \text{ and } t = \alpha\tau.$$

Then, the heat equation rescales as

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right).$$

- Invariance of heat equation implies $\alpha = \beta^2$.
- So, if the radius increases by a factor of β, then the time to cook the turkey increases by β^2.

Problem Solution

Example 1

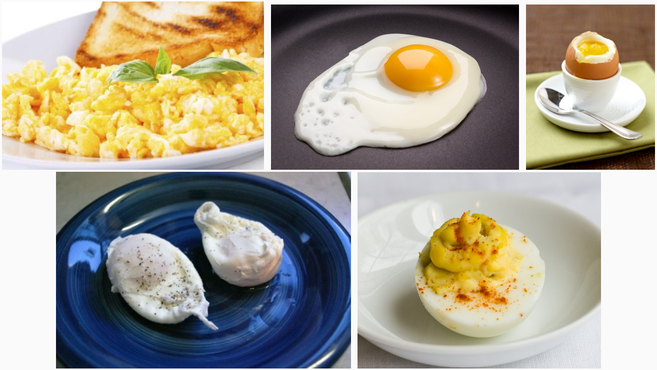
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

- The weight doubles $\Rightarrow$ the volume doubles.
  (if density $=$ constant).
- $V \propto r^3 \Rightarrow r$ increases by factor: $2^{1/3}$.
- Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$.
- If 4 lb turkey takes 4 hrs, then a 20 lb turkey takes

$$t = 4(2^{2/3}) = 2^{8/3} \approx 6.35 \text{ hours.}$$

- In general, if the weight increases by a factor of x, then the time increases by $x^{2/3}$.

Eggs



Omelettes

Egg Protein

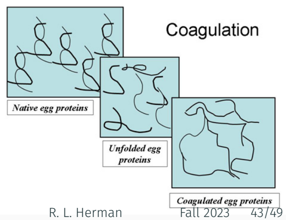
Proteins in eggs can be used

- to help food set (e.g. egg custards),
- as a foam to add air and volume (e.g. sponge cakes),
- as an emulsifier (e.g. mayonnaise).

Two different major proteins, egg white (albumin) and egg yolk,

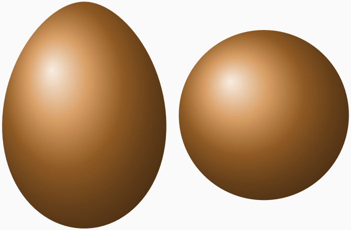
- Albumin starts coagulating at 63^oC
- Yolks start at 70^oC

Coagulation - protein unfolds, denaturation.
As heat increases the proteins rearrange and coagulate.
Egg albumin turns from clear to cloudy white.



Egg Cooking Time

Peter Barnham, *The Science of Cooking* & Dr. Charles Williams of Exeter:



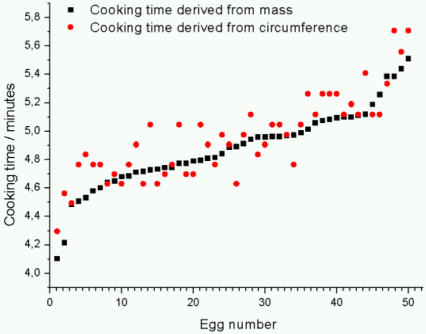
$$t = 0.0152 d^2 \log\left[2 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$

$$t = 0.451 M^{2/3} \log\left[0.76 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$

for t min, diameter d cm, M g, and temperatures in ^{o}C.

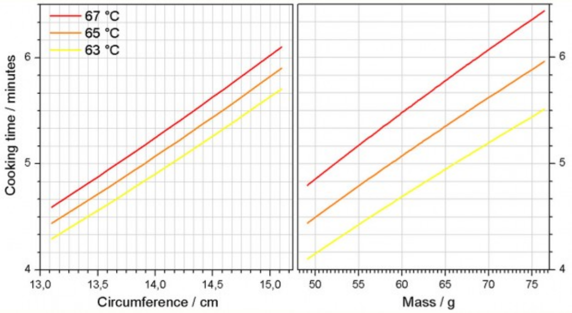
Egg Cooking Time - Data

From Khymos *Towards the perfect soft boiled egg* by Martin Lersch, April 9th, 2009. See also University of Oslo Applet



50 eggs with $T_{yolk} = 63^oC$, $T_{water} = 100^oC$ and $T_{egg} = 4^oC$.

Egg Cooking Time - Formula



Given circumference or mass to reach to reach 63, 65 and 67^o C, respectively, at the yolk-white boundary with $T_{water} = 100^o$ C and $T_{egg} = 4^o$ C.

Egg Consistency

Temp	White	Yolk
62	Begins to set, runny	Liquid
64	Partly set, runny	Begins to set
66	Largely set, still runny	Soft solid
70	Tender solid	Soft solid, waxy
80	Firm	Firm
90	Rubbery solid	Crumbly texture

At sea level, boiling water is 100^o C. At higher altitudes, the boiling temperature of water is lowered 0.3^o C for each additional 100 m above sea level.

Fast Fourier Transform - FFT

- One of top algorithms of 20th Century.
- Developed by Cooley and Tukey, 1965, to compute DFT (Discrete fourier transform)
- Some traced the ideas back to Gauss.
- Limit of Fourier series = Fourier Transform.
- Related to Laplace transform.

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx, \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk. \\ F(s) &= \int_{0}^{\infty} f(t)e^{-st}\,dt. \end{aligned} \tag{5}$$

Left for another course!

References for Drums

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